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# A COVARIANT APPROACH TO GENERALIZED ROBERTSON-WALKER AND TWISTED SPACE-TIMES

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# Contents

# Abstract

One of the most interesting generalizations of the notion of direct product of two manifolds is the notion of twisted product and its subclass of warped product. In particular, if the spacetime $(M, g)$ is the twisted or the warped product between a time interval and a spatial submanifold, $M$ is named respectively twisted spacetime or generalized Robertson-Walker spacetime (GRW). From a physical point of view, the importance of these spaces lies in the fact that they are a straightforward generalization of Robertson-Walker spacetimes (RW). RW metrics are the exact solutions of Einstein's field equations that describe a spatially homogeneous, isotropic, expanding or contracting universe. When the hypothesis of homogeneity and isotropy are relaxed they can give rise to GRW spacetimes or to twisted spacetimes, if the scale function also depends on the spatial coordinates.

The aim of this work is to describe and investigate the broad classes of twisted and GRW spacetimes through a covariant approach. We adopt the characterization given by C. A. Mantica and L. G. Molinari through the existence of a timelike unit *torse-forming* vector field, i.e., a velocity field $u$ ($g_{ij}u^i u^j = -1$) such that its components satisfy $\nabla_i u_j = \varphi(g_{ij} + u_i u_j)$. Important results concerning twisted and GRW spacetimes have already been published in [28][29][30][31].

The first chapter provides a covariant description of fluids in General Relativity [15]. In particular, we show how the covariant derivative of a velocity vector field and the stress-energy tensor can be decomposed in a natural way. We also report the Einstein's field equations and the propagation and constraint equations given by the integrability conditions and by the Bianchi identities.

In the second chapter we characterize a twisted spacetime through the existence of a timelike unit torse-forming vector. In this way, the spacetime admits a totally umbilical foliation orthogonal to a totally geodesic one, so that the metric takes the local form

$$ds^2 = -dt^2 + f(t, \vec{x})^2 g^*_{\mu\nu} dx^\mu dx^\nu,$$

without restrictions for the spatial submanifold $(M^*, g^*)$. With the additional condition that the torse-forming vector is eigenvector of the Ricci tensor, the scalar function $f$ depends only on time and the metric describes a GRW spacetime. The general expression of the Ricci tensor is obtained in both cases and the properties for the Weyl tensor are listed. In particular, we discuss perfect

fluid GRW spacetimes, i.e., GRW spacetimes such that the Ricci tensor has a perfect fluid form. Finally, RW spacetimes are characterized as GRW spacetimes such that the Weyl tensor is zero. This additional condition is equivalent to the requirement that the spatial submanifold is a space of constant curvature.

In the last chapter the possible existence of a second timelike unit torse-forming vector is investigated, excluding the trivial antiparallel vector. On a twisted spacetime there can exist at most two distinct timelike unit torse-forming vectors connected by a hyperbolic rotation. Moreover, since the Ricci tensor assumes two equivalent expressions, we obtain a restriction for the Weyl tensor. The existence of a second torse-forming vector provides restrictions for the spatial submanifold, in fact it can be expressed as a doubly twisted manifold. The 4-dimensional twisted spacetime is a very particular case since if it admits two distinct torse-forming vectors, then the complete tensorial structure of the Weyl tensor is determined. In particular, using the Einstein's field equation the associated stress-energy tensor can be expressed as a mixture of two perfect fluids. Finally, we discuss the unicity on GRW spacetimes. In the most cases, the additional condition on the first torse-forming vector prevents the existence of a second one. However, if the eigenvalue of the Ricci tensor associated to the torse-forming vector is a constant, a second torse-forming vector can exist and the submanifold takes the form of a doubly warped manifold.

### Notation

For the totally symmetric and totally antisymmetric parts of tensors of type $(0,\ell)$, we use the notation

$$T_{(i_1\dots i_\ell)} = \frac{1}{\ell!}\sum_{\pi} T_{i_{\pi(1)}\cdots i_{\pi(\ell)}},$$
$$T_{[i_1\dots i_\ell]} = \frac{1}{\ell!}\sum_{\pi} \delta_\pi T_{i_{\pi(1)}\cdots i_{\pi(\ell)}},$$

where the sum is taken over all permutations, $\pi$, of $1,\dots,n$ and $\delta_\pi$ is $+1$ for even permutations and $-1$ for odd permutations. Symmetric or antisymmetric tensors for a group of indices are defined in a similar way.

Given a tensor of type $(k,\ell)$, the symbol ';' indicates the components of the covariant derivative as

$$T^{i_1\dots i_k}{}_{j_1\dots j_\ell;m} = \nabla_m T^{i_1\dots i_k}{}_{j_1\dots j_\ell}\,,$$

while the symbol ',' indicates the ordinary derivative as

$$T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell , m} = \partial_m T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} \, .$$

We adopt the sign convention $(- + + \dots +)$ for the metric because it is generally much more convenient than the alternative choice $(+ - - \dots -)$. In fact, the first one induces a positive definite metric (rather than a negative definite one) on spacelike hypersurfaces. We use latin indices for space or time components of a tensor ($i = 0, 1, \dots, n-1$), 0 or $t$ for time components and greek indices for purely spatial components ($\mu = 1, 2, \dots, n-1$). In the last chapter, the indices $A, B, \dots$ are used for the spatial components of the ($n$-2)-dimensional spatial submanifold ($A = 2, 3, \dots, n-1$).

Finally, we use the Planck units, such that the gravitational constant $G$ and the speed of light $c$ are set equal to one.

# Chapter 1

# Fluids in General Relativity

In General Relativity the relation between the spacetime geometry and the matter distribution is described by Einstein's field equations (EFE)

$$R_{ij} - \frac{1}{2}Rg_{ij} = \kappa T_{ij}. \tag{1.0.1}$$

where $R_{ij}$ is the Ricci tensor, $R = R^i{}_i$ is the scalar curvature, $g_{ij}$ is the metric tensor and $T_{ij}$ is the stress-energy tensor describing the matter distribution such that it satisfies the conservation law $\nabla^i T_{ij} = 0$. For large scales the hypothesis of homogeneity and isotropy hold and, on considering a perfect fluid stress-energy tensor, an expanding or contracting universe is found. In general, if the scale is not sufficiently large, the previous hypothesis are no longer satisfied and a new description is necessary. Following G. Ellis and H. van Elst [15] we provide a covariant description of fluids in General Relativity.

## 1.1 Velocity vector fields

Given a manifold $M$ with dimension $n$, we define a *congruence* as a family of curves, not necessarily geodesics, such that through each point there passes precisely one curve in this family. The tangents to a congruence yield a vector field, and, conversely, every continuous vector field generates a congruence of curves. Without loss of generality, we may assume that the congruences are parametrized by the proper time $\tau$, i.e., $u^i = dx^i/d\tau$, so that the vector field $u^i$ is normalized to unit length, $u^i u_i = -1$. In General Relativity, particle motion is represented by one of these curves, i.e., through the tangent vectors to the curve. Then, a congruence can be interpreted as a collection of particles having timelike velocity field $u$.

The components $u^i$ define unique projection tensors

$$U^i{}_j = -u^i u_j,$$
$$h_{ij} = g_{ij} + u_i u_j.$$

The first one projects along the velocity vector $u^i$, while $h_{ij}$ projects on the perpendicular direction to $u^i$. Then, two derivatives are defined: the derivative along the curves of the congruence, denoted by a dot

$$\dot{T}^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} = u^m \nabla_m T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell}$$

and the fully orthogonally projected covariant derivative $\tilde{\nabla}$ such that

$$\tilde{\nabla}_m T^{i_1 \dots i_k}{}_{j_1 \dots j_\ell} = h_{r_1}{}^{i_1} \dots h_{j_\ell}{}^{s_\ell} h_m{}^t \nabla_t T^{r_1 \dots r_k}{}_{s_1 \dots s_\ell}.$$

In this way, the derivative of the vector field $u$ can be decomposed into its irreducible parts, defined by their symmetry properties,

$$\begin{aligned} \nabla_i u_j &= \tilde{\nabla}_i u_j - u_i a_j \\ &= \frac{1}{n-1} \theta h_{ij} + \sigma_{ij} + \omega_{ij} - u_i a_j \end{aligned} \tag{1.1.1}$$

where $\theta = \nabla_k u^k = u^k{}_{;k}$, named *expansion scalar*, represents the rate of change of space volume for unit volume, $d\dot{V}/dV$. The term $a_i = \dot{u}_i = u_j \nabla^j u_i$ is the acceleration, with $u_k a^k = 0$. The tensor $\sigma_{ij}$ is the symmetric traceless *shear tensor* such that $\sigma_{ij} u^i = 0$, related to the distortion in shape of the fluid without change of its volume (by the property $\sigma^i{}_i = 0$):

$$\sigma_{ij} = \frac{1}{2} \left( h_i{}^k \nabla_k u_j + h_j{}^k \nabla_k u_i - \frac{2}{n-1} \theta h_{ij} \right) = u_{(j;i)} - \frac{1}{n-1} \theta h_{ij} + u_{(i} a_{j)}.$$

Finally, the antisymmetric tensor $\omega_{ij}$ represents the *vorticity tensor*, describing the rotation of the matter relative to a non-rotating frame

$$\omega_{ij} = \frac{1}{2} \left( h_i{}^k \nabla_k u_j - h_j{}^k \nabla_k u_i \right) = u_{[j;i]} + u_{[i} a_{j]}.$$

Note that, since any second rank tensor could be written in terms of its symmetric and antisymmetric parts and hence the terms involving $h_{ij}$ and $a_i$ cancel out in (1.1.1). They are however required for the definitions of shear and vorticity tensors in view of their being orthogonal to $u_i$ and traceless.

## 1.2 The stress-energy tensor

Physically, a velocity vector field can be associated to a continuous matter distribution. In General Relativity, the energy density, the flux of energy and the momentum of a matter distribution are described by a symmetric tensor $T_{ij}$ called the *stress-energy-momentum tensor*, (often abbreviated as *stress-energy tensor*), that satisfies the equations of motion

$$\nabla_i T^i{}_j = 0. \tag{1.2.1}$$

Using the EFE, the previous equations corresponds to the *twice-contracted Bianchi identities* (A.6.2). To classify stress-energy tensors we use the following theorem concerning the natural decomposition of a general symmetric tensor:

**Theorem 1.2.1.** *Let $S_{ij}$ be a symmetric tensor and $u_i$ a timelike unit vector, $u^i u_i = -1$, then $S_{ij}$ can be written as follows:*

$$S_{ij} = (A+B)u_i u_j + B g_{ij} - (S_i u_j + S_j u_i) + \sigma_{ij},$$

*where $A$ and $B$ are scalar fields, $S_i$ is a spacelike vector ($S^i u_i = 0$), and $\sigma_{ij}$ is a symmetric tensor such that $\sigma_{ij} u^j = 0$ and $\sigma^k{}_k = 0$. If $u_i$ is also an eigenvector of $S_{ij}$, then $S_i = 0$.*

*Proof.* Let us expand the identity $S_{ij} = S^{mn}(h_{mi} - u_m u_i)(h_{nj} - u_n u_j)$:

$$\begin{aligned} S_{ij} &= (S^{mn} u_m u_n) u_i u_j - (S^{mn} u_n h_{mi}) u_j - (S^{mn} u_m h_{nj}) u_i + S^{mn} h_{mi} h_{nj} = \\ &= A u_i u_j - S_i u_j - S_j u_i + B h_{ij} + (S^{mn} - B h^{mn}) h_{mi} h_{nj}, \end{aligned}$$

with $S^i u_i = 0$ and $\sigma_{ij} u^j = 0$. The scalar field $B$ is chosen such that the last tensor $\sigma_{ij}$ is traceless, i.e., $S^k{}_k = -A + B(n-1)$. The identity $S_{ij} u^j = -(A+B)u_i + B u_i + S_i$ shows that if $u_i$ is an eigenvector of $S_{ij}$, then $S_i$ must be equal to zero. □

**Remark 1.2.1.** *Note that $S^i S_i = S^{mn} S^{ab} h_{ma} u_n u_b = (S^2)_{mn} u^m u^n + A^2$. The orthogonality $S^i u_i = 0$ implies that $S^0 = 0$ in the rest frame, i.e. $S_i$ is a spacelike vector, $S^i S_i > 0$.*

### 1.2.1 Perfect fluid

A *perfect fluid* is defined to be a continuous distribution of matter with stress-energy tensor

$$T_{ij} = (\mu + p) u_i u_j + p g_{ij}, \tag{1.2.2}$$

where $u^i$ is a unit timelike vector field representing the velocity of the fluid, the functions $\mu$ and $p$ are respectively the mass-energy density and pressure of the fluid as measured in its rest frame, with a suitable equation of state relating $\mu$ and $p$. In simple case the energy density and the pressure are related by a barotropic relation $p = p(\mu)$. For example, baryons satisfy $p_b = 0$, while radiation $p_r = \mu_r/3$.

The fluid is called *perfect* because of the absence of heat conduction terms and stress terms, corresponding to viscosity. The equation (1.2.1) for a perfect fluid yields

$$\begin{aligned}
&\dot{\mu} + (\mu + p)\theta = 0, \\
&(p + \mu)a_i + h_{ij}\nabla^j p = 0.
\end{aligned}$$

For a flat spacetime, i.e., $g_{ij} = (-1, 1, \ldots, 1)$, in the non-relativistic limits ($p \ll \mu$, $u^i = (1, \vec{v})$ and $v\, dP/dt \ll |\vec{\nabla} p|$), the previous equations correspond respectively to the conservation of mass and Euler's equations, i.e., the Navier-Stokes equations with zero viscosity and zero heat flux,

$$\begin{aligned}
&\frac{\partial \mu}{\partial t} + \vec{\nabla} \cdot (\mu \vec{v}) = 0, \\
&\mu \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \right) = -\vec{\nabla} p.
\end{aligned}$$

### 1.2.2 Imperfect fluids

If the viscosity terms are non-zero, the fluid is named *imperfect fluid* and the stress-energy tensor has the general form:

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij} + (q_i u_j + q_j u_i) + \pi_{ij}. \tag{1.2.3}$$

The quantities $\mu$, $p$, $q_i$ and $\pi_{ij}$ are given by:

$$\begin{aligned}
\mu =& T_{ij} u^i u^j, \qquad p = \frac{1}{n-1} h^{ij} T_{ij}, \\
q_i =& - {h_i}^j T_{jk} u^k, \\
\pi_{ij} =& {h_i}^m {h_j}^n T_{mn} - \frac{1}{n-1}(h^{mn} T_{mn}) h_{ij}.
\end{aligned}$$

The viscosity terms drastically increase the complexity of the equations of motion (1.2.1) and play an important role in the dispersion of energy of a system. On a 4-dimensional spacetime

$$\begin{aligned}
&\dot{\mu} + \tilde{\nabla}_i q^i = -\theta(\mu + p) - 2a_i q^i - \sigma_{ij}\pi^{ij}, \\
&{h_j}^i \dot{q}^j + \tilde{\nabla}^i p + \tilde{\nabla}_j \pi^{ij} = -\frac{4}{3}\theta q^i - \sigma^{ij} q_j - (\mu + p)a^i - a_j \pi^{ij} - \eta^{ijk}\omega_j q_k,
\end{aligned}$$

where $\eta_{ijk} = \eta_{\ell ijk}u^\ell$ is the *volume element* for the rest-spaces ($\eta_{ijk\ell} = \eta_{[ijk\ell]}$ is the 4-dimensional volume element, $\eta_{0123} = \sqrt{-\det g}$) and $\omega^i = \frac{1}{2}\eta^{ijk}\omega_{jk}$ is the *vorticity vector*.

There exist two more important set of equations: the Ricci and the Bianchi identities. In the following we consider a 4-dimensional spacetime and the physical quantities $\mu$, $p$, $q_i$ and $\pi_{ij}$ as in [15] such that the EFE are

$$R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}.$$

**Ricci identities**

The second set of equations is given by the integrability conditions. The definition of the Riemann tensor gives

$$R_{jk\ell m}u^m = \nabla_j\nabla_k u_\ell - \nabla_k\nabla_j u_\ell. \tag{1.2.4}$$

Using the EFE and the decomposition of $\nabla_i u_j$, separating out the parallel and orthogonally projected parts into a trace, symmetric trace-free and anti-symmetric part, we obtain three propagation equations and three constraint equations.

The propagation equation are:

i. the Raychauduri equation

$$\dot{\theta} + \frac{1}{3}\theta^2 = 2(\omega^2 - \sigma^2) + \nabla_k a^k - R_{ij}u^i u^j,$$

where $2\sigma^2 = \sigma_{ij}\sigma^{ij}$ and $2\omega^2 = \omega_{ij}\omega^{ij}$;

ii. the vorticity propagation equation

$$h^i{}_j\dot{\omega}^j - \frac{1}{2}\eta^{ijk}\tilde{\nabla}_j a_k = -\frac{2}{3}\theta\omega^i + \sigma^i{}_j\omega^j$$

shows that the vorticity vanishes if and only if $\omega = 0$ and the fluid flow is hypersurface orthogonal;

iii. the shear propagation equation

$$\dot{\sigma}^{\langle ij\rangle} - \tilde{\nabla}^{\langle i}a^{j\rangle} = -\frac{2}{3}\theta\sigma^{ij} + a^{\langle i}a^{j\rangle} - \sigma^{\langle a}{}_k\sigma^{j\rangle k} - \left(E^{ij} - \frac{1}{2}\pi^{ij}\right),$$

where $E_{ij} = C_{irjs}u^r u^s$ corresponds to the electric part relative to $u^i$ of the Weyl tensor and the angle brackets denote the orthogonally projected symmetric trace-free part of tensors, e.g.,

$$A^{\langle ij\rangle} = \left(h^{(i}{}_k h^{j)}{}_\ell - \frac{1}{3}h^{ij}h_{k\ell}\right)A^{k\ell}.$$

Moreover, eq. (1.2.4) also gives the three constraint equations:

i. the $(0\alpha)$-*equation*

$$\tilde{\nabla}_j \sigma^{ij} - \frac{2}{3}\tilde{\nabla}^i \theta + \eta^{ijk}(\tilde{\nabla}_j \omega_k + 2a_j \omega_k) + q^i = 0,$$

that shows how the momentum flux $q_i$ relates to the spatial inhomogeneity of the expansion;

ii. the *vorticity divergence identity*

$$\tilde{\nabla}_i \omega^i - a_i \omega^i = 0;$$

iii. the $H_{ij}$-*equation*

$$H^{ij} + 2a^{\langle i}\omega^{j\rangle} + \tilde{\nabla}^{\langle i}\omega^{j\rangle} - (\operatorname{curl}\sigma)^{ij} = 0,$$

that gives the magnetic part of the Weyl tensor $H_{ij} = \frac{1}{2}\eta_{ik\ell}C^{k\ell}{}_{jm}u^m$ from the covariant derivative of the vorticity and the curl of the shear, $(\operatorname{curl}\sigma)^{ij} = \eta^{k\ell\langle i}\tilde{\nabla}_k \sigma^{j\rangle}{}_\ell$.

### Bianchi identities

The last important set of equations arise from the Bianchi identities

$$\nabla_{[i}R_{jk]\ell}{}^m = 0,$$

using the EFE and the definition of the Weyl tensor in terms of the Riemann and the Ricci tensors. The *once-contracted Bianchi identities* give two further propagation equations and two further constraint equations.

The propagation equations are

i. the $\dot{E}$-*equation*

$$\begin{aligned}\dot{E}^{\langle ij\rangle} + \frac{1}{2}\dot{\pi}^{\langle ij\rangle} =& (\operatorname{curl}H)^{ij} - \frac{1}{2}\tilde{\nabla}^{\langle i}q^{j\rangle} - \frac{1}{2}(\mu+p)\sigma^{ij} - \theta\left(E^{ij} + \frac{1}{6}\pi^{ij}\right)\\ &+ 3\sigma^{\langle i}{}_k\left(E^{j\rangle k} - \frac{1}{6}\pi^{j\rangle k}\right) - a^{\langle i}q^{j\rangle}\\ &+ \eta^{k\ell\langle i}\left[2a_k H^{j\rangle}{}_\ell + \omega_k\left(E^{j\rangle}{}_\ell + \frac{1}{2}\pi^{j\rangle}{}_\ell\right)\right];\end{aligned}$$

ii. the $\dot{H}$-*equation*

$$\begin{aligned}\dot{H}^{\langle ij\rangle} =& -(\operatorname{curl}E)^{ij} + \frac{1}{2}(\operatorname{curl}\pi)^{ij} - \theta H^{ij} + 3\sigma^{\langle i}{}_k H^{j\rangle k} + \frac{3}{2}\omega^{\langle i}q^{j\rangle}\\ &- \eta^{k\ell\langle i}\left[2a_k E^{j\rangle}{}_\ell - \omega_k H^{j\rangle}{}_\ell - \frac{1}{2}\sigma^{j\rangle}{}_k q_\ell\right],\end{aligned}$$

where (curl $H$), (curl $E$) and (curl $\pi$) are defined in the obvious way (see the expression of (curl $\sigma$)). In analogy to the electromagnetic case, these equations yield to the gravitational radiation.

Finally, the constraint equations are

i. the (div $E$)-*equation*

$$\begin{aligned}\tilde{\nabla}_j E^{ij} + \frac{1}{2}\tilde{\nabla}_j \pi^{ij} =& \frac{1}{3}\tilde{\nabla}^i \mu - \frac{1}{3}\theta q^i + \frac{1}{2}\sigma^i{}_j q^j + 3\omega_j H^{ij} \\ & + \eta^{ijk}\left(\sigma_{j\ell} H^\ell{}_k - \frac{3}{2}\omega_j q_k\right);\end{aligned}$$

ii. the (div $H$)-*equation*

$$\begin{aligned}\tilde{\nabla}_j H^{ij} =& -(\mu + p)\omega^i - 3\omega_j\left(E^{ij} - \frac{1}{6}\pi^{ij}\right) \\ & - \eta^{ijk}\left(\sigma_{j\ell}E^\ell{}_k + \frac{1}{2}\tilde{\nabla}_j q_k + \frac{1}{2}\sigma_{j\ell}\pi^\ell{}_k\right).\end{aligned}$$

### 1.2.3 Perfect fluid mixture

Several times in General Relativity, the matter is represented by a mixture of two or more fluids. In particular, some astrophysical and cosmological situations need to be described by a stress-energy tensor, made up of the sum of two or more perfect fluids. Let us consider two perfect fluids with energy densities $\mu_1$, $\mu_2$, pressures $p_1$, $p_2$ and velocity vector fields $u_i$, $z_i$:

$$\begin{aligned}T^{(1)}_{ij} &= (\mu_1 + p_1)u_i u_j + p_1 g_{ij}, \\ T^{(2)}_{ij} &= (\mu_2 + p_2)z_i z_j + p_2 g_{ij}.\end{aligned}$$

The total stress-energy tensor is

$$T_{ij} = T^{(1)}_{ij} + T^{(2)}_{ij} = (\mu_1 + p_1)u_i u_j + (p_1 + p_2)g_{ij} + (\mu_2 + p_2)z_i z_j\,, \qquad (1.2.5)$$

where $z_i$ is such that $z_i \neq u_i$, else it would formally be that of a single fluid with energy density $\mu_1 + \mu_2$ and pressure $p_1 + p_2$. The expression (1.2.5) is equivalent to a stress-energy tensor of a single fluid with a non-zero heat flux and a non-zero anisotropic stress tensor (1.2.3). Using the decomposition

$$z_i = u_i \cosh\psi + b_i \sinh\psi \qquad \psi \neq 0,$$

where $\psi$ is called the tilt angle and $b^i u_i = 0$, we can deduce the equivalent pressure and energy density of the mixture as

$$\mu = T_{ij}u^i u^j = \mu_1 + (\mu_2 + p_2)\cosh^2\psi - p_2, \qquad (1.2.6)$$

$$p = \frac{1}{n-1}h^{ij}T_{ij} = p_1 + p_2 + \frac{1}{n-1}(\mu_2 + p_2)\sinh^2\psi. \qquad (1.2.7)$$

The heat flux $q$ and the anisotropic stress tensor $\pi$ are given respectively by

$$q_i = -h_i{}^j T_{jk} u^k = q b_i,$$
$$\pi_{ij} = h_i{}^m h_j{}^n T_{mn} - \frac{1}{n-1}(h^{mn}T_{mn})h_{ij} = \pi\left[q_i q_j - \frac{1}{n-1}q^2 h_{ij}\right],$$

with the scalar quantities $q$ and $\pi$ given by

$$q = (\mu_2 + p_2)\cosh\psi\sinh\psi, \tag{1.2.8}$$
$$\pi = \frac{1}{\cosh^2\psi(\mu_2 + p_2)}. \tag{1.2.9}$$

The two perfect fluids mixture is widely studied in literature. In [23], Letelier examined the case in which the stress-energy tensor consists of a mixture of two perfect fluids or a mixture of one perfect fluid and a null fluid. He studied the two perfect fluids model in the instance in which both the velocities were irrotational. The algebraic properties of the stress-energy tensor are studied by J. J. Ferrando et al. in [16]. A. A. Coley and D. J. McManus in [13] and [12] studied the special case in which the first fluid forms a shear-free, irrotational and geodesic timelike congruence and the second is taken to be pure radiation or a perfect fluid with a general velocity vector field non-collinear with the first one. Moreover, they investigated the case of a single perfect fluid tilting with respect to a shear-free, irrotational and geodesic timelike congruence.

## 1.3 Energy conditions

For any observer with velocity $v^i$, the quantity $T_{ij}v^iv^j$ is interpreted as the energy density, i.e., the mass-energy per unit volume, as measured by this observer. For normal matter, the energy density must be non-negative, i.e.,

$$T_{ij}v^iv^j \geq 0. \tag{1.3.1}$$

Such property is also named *weak energy condition.* Writing the stress-energy tensor in the diagonal form

$$T_{ij} = \mu t_i t_j + \sum_{A=1}^{n-1} p_A x_i^A x_j^A, \tag{1.3.2}$$

where $(t_i, x_i^A)$ is an orthonormal basis with a timelike vector $t_i$ and $p_A$ are called *principal pressures*, the weak energy condition is satisfied if and only if

$$\mu \geq 0 \quad \text{and} \quad \mu + p_A \geq 0, \quad \text{with} \quad A = 1, \dots n-1.$$

If $x^i$ is orthogonal to $v^i$, the component $-T_{ij}v^i x^j$ is interpreted as the momentum density of the matter in the $x^i$-direction. *The dominant energy condition* stipulates that, in addition to the weak energy condition, for any observer with velocity field $v^i$, the vector field $-T^i{}_j v^j$ must be a future-pointing causal vector, i.e., a timelike or at most a null vector field. Such condition corresponds to the request that the matter can never be observed with a velocity higher than light. For the stress-energy tensor (1.3.2) the dominant energy condition takes the form

$$\mu \geq |p_A| \quad \text{with} \quad A = 1, \dots n-1.$$

Using the EFE, we can write an energy condition involving the Ricci tensor, named *strong energy condition*

$$R_{ij}v^i v^j = \kappa \left( T_{ij} - \frac{1}{2} T g_{ij} \right) v^i v^j \geq 0. \tag{1.3.3}$$

Such condition does not imply the weak energy condition, in fact, we refer to it as strong only because it is more difficult to satisfy the strong energy condition. In terms of the energy density and the principal pressures, (1.3.3) becomes

$$\mu + \sum_{A=1}^{n-1} p_A \geq 0 \quad \text{and} \quad \mu + p_A \geq 0, \quad \text{with} \quad A = 1, \dots n-1.$$

The strong energy condition can be physically interpreted as a manifestation of the attractiveness of gravity. Using the Raychaudhuri equation, eq. (1.3.3) reduce to a condition about the expansion parameter $\theta$. Since $\sigma_{ij}v^j = 0$, we have $\sigma^2 \geq 0$ and if the congruence is hypersurface orthogonal, then the vorticity tensor $\omega_{ij}$ is null. Under these assumptions, if the strong condition is satisfied, we obtain (in a 4-dimentional spacetime)

$$\frac{d\theta}{d\tau} + \frac{1}{3}\theta^2 \leq 0 \qquad \text{i.e.} \qquad \frac{d}{d\tau}(\theta^{-1}) \geq \frac{1}{3},$$

then

$$\theta^{-1}(\tau) \geq \theta_0^{-1} + \frac{1}{3}\tau, \tag{1.3.4}$$

where $\theta_0$ is the initial value of $\theta$. If the congruence is initially contracting with $\theta_0 < 0$, $\theta^{-1}$ will pass through zero, then $\theta$ will diverge ($\theta \to -\infty$) in a finite proper time $\tau \leq 3/|\theta_0|$. In general, the divergence of the expansion parameter does not imply a singularity of spacetime, but represents only a singularity in the congruence.

Finally, the *null energy condition* requires that for any future-pointing null vector field $k^i$

$$T_{ij}k^i k^j \geq 0 \tag{1.3.5}$$

or equivalently

$$\mu + p_A \geq 0, \quad \text{with} \quad A = 1, \dots n-1.$$

# Chapter 2

# The hierarchy of twisted spacetimes

In this chapter we shall discuss the hierarchy of the large class of twisted spacetimes, which comes to life starting from the Robertson-Walker (RW) spacetimes with a gradual relaxation of the constrains. RW metric is an exact solution of the EFE of General Relativity that describes a homogeneous, isotropic, expanding (or contracting) universe. The general form of the metric follows from the geometric properties of homogeneity and isotropy and the EFE are only needed to derive the scale factor of the universe as a function of time.

Small deformations of the metric on the fiber of classical RW spacetimes fit into the class of generalized Robertson-Walker (GRW) spacetimes, introduced by L. J. Alías, A. Romero and M. Sánchez in 1995 [1]. GRW spacetimes are the first wide generalization of the classical RW spacetime obtained by relaxing the spatial homogeneity, that is reasonable as a first approximation of the large scale structure of the universe, but it is not appropriate when we consider a more accurate scale. GRW spacetimes include many interesting spaces, e.g., the Einstein-de Sitter spacetime, the Friedmann cosmological models, the static Einstein spacetime and the de Sitter spacetime. Over the years, several characterizations of GRW spacetimes have been obtained in terms of vectors satisfying peculiar properties. In particular, B.-Y. Chen in 2014 [8] established a simple characterization in terms of a timelike concircular vector field. Recently, C. A. Mantica and L. G. Molinari in 2016 [28] introduced a new covariant formalism to describe GRW spacetimes, with the help of a timelike unit torse-forming vector field.

A further generalization is given by twisted spacetimes, such that the scale factor depends on both time and position. They were introduced by B.-Y. Chen in 1979 [7] as the natural generalization of warped manifolds that avoids the constancy of mean curvature of slices. In [10] B.-Y. Chen gave a simple characterization in terms of a timelike torqued vector and C. A. Mantica and L. G. Molinari in 2017 [30] extended the characterization through the existence of a timelike unit torse-forming vector from GRW spacetimes to twisted spacetimes.

We start the discussion from the broader class of twisted spacetimes through the torse-forming vector characterization. A first result is the equivalence between Chen's and our characterization. The tensorial structure of the Ricci tensor is completely determined in terms of the torse-forming vector, the metric, the covariant derivative of the mean curvature and the electric components of the Weyl tensor. Using the EFE, we will show how a twisted spacetime can be associated to an imperfect fluid. We also describe GRW spacetimes with the additional condition that the torse-forming vector is an eigenvector of the Ricci tensor. The Ricci tensor is simpler than the previous case and some new properties are obtained. In particular, we discuss GRW perfect fluid spacetimes using results of literature. Finally, we characterize RW spacetimes through the covariant description in terms of the torse-forming vector as conformally flat GRW spacetimes.

## 2.1 Twisted spacetimes

Let $B$ and $F$ be two pseudo-Riemannian manifolds equipped with pseudo-Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive smooth function on M, named *twisting function*:

**Definition 2.1.1.** *The twisted product* $M = B \times_f F$ *is the manifold* $B \times F$ *with the pseudo-Riemannian metric* $g = g_B + f^2 g_F$ *and dimension* $\dim(B) + \dim(F)$.

When $f$ depends only on $B$ the twisted product is reduced to a warped product. $B$ is called the *base* and $F$ the *fiber* of the twisted product $B \times_f F$. Both the leaves $B \times \{q\}$ $(q \in F)$ and the fibers $\{p\} \times F$ $(p \in B)$ are pseudo-Riemannian submanifolds of $B \times_f F$.

An interesting case of twisted spacetime is the Lorentzian manifold $\mathscr{L}_n$ written as a twisted product between a temporal one-dimensional interval $I$ and a spatial Riemannian submanifold $(M^*, g^*)$. In privileged coordinates the manifold $M$ gains the metric structure:

$$ds^2 = -dt^2 + f(t, \vec{x})^2 g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu, \tag{2.1.1}$$

where $f$ represents the scale factor and $g^*_{\mu\nu}$ is the metric tensor of the submanifold $M^*$ with dimension $n-1$. The scale factor depends on time, otherwise the manifold is a product of disjoint manifolds.

Twisted spacetimes were introduced by B.-Y. Chen in 1979 [7] as the generalization of warped manifolds that avoid the constancy of mean curvature of fibers $\{t\} \times M^*$. In [10] B.-Y. Chen gave a simple characterization:

**Theorem 2.1.1.** *A Lorentzian manifold $\mathscr{L}_n$ admits a timelike torqued vector field $\tau$, i.e.,*

$$\tau_i \tau^i < 0, \qquad \nabla_i \tau_j = \rho\, g_{ij} + \alpha_i \tau_j, \qquad \alpha_i \tau^i = 0 \tag{2.1.2}$$

*and $\rho$ is a function on $\mathscr{L}_n$ if and only if it is locally a twisted product $I \times_f M^*$, where $I$ is an open interval, $M^*$ is a Riemannian (n-1)-manifold.*

**Remark 2.1.1.** *Given a timelike torqued vector $\tau$ on a Lorentzian manifold $\mathscr{L}_n$, for each scalar function $\lambda$ such that $\tau^i \nabla_i \lambda = 0$, $\lambda\tau$ is a timelike torqued vector. In fact $(\lambda\tau)^2 < 0$ and*

$$\nabla_i(\lambda\tau_j) = \tau_j \nabla_i \lambda + \lambda(\rho g_{ij} + \alpha_i \tau_j) = \psi g_{ij} + \beta_i \tau_j, \tag{2.1.3}$$

*where $\psi = \lambda\rho$ and $\beta_i = (\lambda\alpha_i + \nabla_i \lambda)$. Since the condition $\tau^i \nabla_i \lambda = 0$ is verified, $\beta_i \tau^i = 0$ and $\lambda\tau$ is a timelike torqued vector.*

### 2.1.1 Distributions and foliations

Let $M$ and $TM$ denote respectively an $n$-dimensional smooth manifold and its tangent bundle defined as $TM = \bigcup_{p \in M} T_p M$. We introduce the notion of distribution as:

**Definition 2.1.2.** *A $m$-dimensional (tangent) distribution on $M$ is a assignment of a linear subspace $D_p \subset T_p M$ at each point $p \in M$. We will denote this by $D$, where*

$$D = \left( \bigcup_{p \in M} D_p \right) \subset TM.$$

Locally, one can say that a $m$-distribution is generated by a set of $m$ linearly independent vector fields if and only if in every point $p$ their values span the $m$-dimensional subspace $D_p$ , i.e., $D_p = \mathrm{Span}\{X_1(p), \ldots, X_m(p)\}$.

**Definition 2.1.3.** *An immersed submanifold $\Sigma$ is an integral manifold of the distribution $D$ if $T_p\Sigma = D_p$ for any $p \in \Sigma$.*

*The distribution $D$ is integrable if each point of $M$ is contained in an integral manifold of $D$.*

There are distributions for which no integral manifolds exist. The reason relies on the following definition:

**Definition 2.1.4.** *The Lie bracket of the vector fields $u$ and $v$ is*

$$[u, v] = uv - vu.$$

The espression $[u, v]$ gives a new vector field. Let be $u_p, v_p \in D_p$, if $[u_p, v_p] \in D_p$ holds for any $p$ the distribution $D$ is called *involutive*. It's easy to show that

**Lemma 2.1.1.** *If $D$ is an integrable distribution, then $D$ is necessarily involutive.*

Obviously, every smooth 1-dimensional distribution is integrable. The integrability of a distribution is closely related to the notion of foliation, defined as:

**Definition 2.1.5.** *A $m$-dimensional $L$ foliation of an $n$-dimensional manifold $M$ is a decomposition of $M$ into a union of disjoint connected submanifolds $L_\alpha$, called the leaves of the foliation, with the following property: $\forall p \in M$ exists a neighborhood $U$ and a system of local coordinates $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$ such that for each leaf $L_\alpha$, the components of $U \cap L_\alpha$ are described by the equations $x^k = const$ for $k = m + 1, \ldots, n$.*

We have the following theorem by Frobenius [4]:

**Theorem 2.1.2.** *If $D$ is an involutive distribution on $M$, then the collection of all maximal connected integral manifolds of $D$ forms a foliation of $M$.*

**Lemma 2.1.2.** *If $L$ is a $m$-dimensional foliation of $M$, then the collection of tangent spaces to the leaves of $L$ forms an involutive distribution.*

In physics, with a $m$-dimensional foliation we mean that the manifold is decomposed into hypersurfaces of dimension $m$ and there exists a smooth scalar field such that each hypersurface is a level surface. The case of our interest is a $n-1$ dimensional spatial foliation of the Lorentzian manifold $M$. Thus, let us consider the spacelike hypersurface $\Sigma \in M$ that is a smooth immersion of a domain of dimension $n-1$ in $M$, with a Riemannian induced metric. At any point $p \in \Sigma$, there is a timelike unit normal vector $N_p$, $g_{ij} N_p^i N_p^j = -1$, called the *future pointing Gauss map* of the hypersurface, with the orientation of $\partial_t|_p$ (hereafter we omit to specify $p$). The normal vector and the tangent space of $\Sigma$ at $p$ provide the natural decomposition $\partial_t = \alpha N + Y$, where $\alpha > 0$ and $g_{ij}N^iY^j = 0$. From $-1 = g(\partial_t, \partial_t) = -\alpha^2 + g_{ij}Y^iY^j$ it follows that $\alpha \geq 1$; the value $\alpha = -g(N, \partial_t) = \cosh\theta$ defines the normal hyperbolic angle $\theta$ of the hypersurface at $p$. The tangential component $Y$ of the decomposition introduces the *height function*, $h(p)$, of the hypersurface through the relation $Y = \nabla h$. It is $|\nabla h|^2 = \sinh^2\theta$.

The hypersurface $\Sigma$ is rapresented parametrically by $x^i = x^i(\vec{q})$. Let us suppose be maximal the rank of the matrix of first derivatives $B^i{}_\mu = \partial x^i/\partial q^\mu$, i.e., $n-1$. Relative to the coordinate trasformation $x'^j = x'^j(q^i)$, the $B^i{}_\mu$ behave as components of a type (1,0) tensor, while relative to the parameter trasformantion $q'^\mu = q'^\mu(q^\nu)$ the $B^i{}_\mu$ behave as components of a type (0,1). The $n-1$ 1-forms $dq^1, \ldots, dq^{n-1}$ at a point $p \in \Sigma$ determine $n$ 1-forms $dx^1, \ldots, dx^n$, given by $dx^i = B^i{}_\alpha dq^\alpha$, which are interpreted as the components of a displacement in $M$, tangential to $\Sigma$ at P. More generally, given the components $Q^\alpha$ of an element of the tangent space $T_p\Sigma$, the associate tangent vector in $T_pM$ has components $v^j = B^j{}_\alpha Q^\alpha$.

The induced metric, defined by

$$g^*_{\mu\nu} = g_{ij}B^i{}_\mu B^j{}_\nu \tag{2.1.4}$$

is also named *first fundamental form*. The quantities $B^\mu{}_i = g^{*\mu\nu}g_{ij}B^j{}_\nu$ satisfy $B^\mu{}_iB^j{}_\nu = \delta^\mu{}_\nu$, but $B^\mu{}_iB^j{}_\mu \neq \delta^j{}_i$. Moreover, the normal vector $N$ is such that

$$g_{ij}N^iB^j{}_\mu = 0. \tag{2.1.5}$$

The *mixed covariant derivative* ([24] Section 5.7) of a field $X^i{}_\nu$ is defined as

$$X^i{}_{\nu||\mu} = \frac{\partial X^i{}_\nu}{\partial q^\mu} - \Gamma^{*\lambda}{}_{\mu\nu}X^i{}_\lambda + \Gamma^i{}_{hk}X^k{}_\nu B^h{}_\mu,$$

where the $\Gamma$ and $\Gamma^*$ are the Christoffel symbols of the connections respectively on $M$ and $\Sigma$. $X^i{}_{\nu||\mu}$ represents the components of a tensor field wich is of type (1,0) relative to the coordinate trasformation and of type (0,2) relative to the parameter trasformation. For $B^i{}_\nu$

$$B^i{}_{\nu||\mu} = \frac{\partial B^i{}_\nu}{\partial q^\mu} - \Gamma^{*\lambda}{}_{\mu\nu}B^i{}_\lambda + \Gamma^i{}_{hk}B^k{}_\nu B^h{}_\mu,$$

Because of the symmetry of $\Gamma$ and $\Gamma^*$, $B^i{}_{\nu||\mu} = B^i{}_{\mu||\nu}$ holds. The relation

$$B^i{}_{\mu||\nu} = -N^i\Omega_{\mu\nu},$$

defines the *second fundamental form* of the immersion with components $\Omega_{\mu\nu}$. The eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ of the symmetric matrix $\Omega_{\mu\nu}$ are the *principal* *curvatures* of $\Sigma$ at $p$. The *fundamental invariants* of $\Sigma$, named mean curvature, second mean curvature, etc., are defined to be the $n-1$ elementary symmetric function of $\lambda_1, \dots, \lambda_{n-1}$:

$$\begin{aligned} H &= \frac{1}{n-1}\sum_{j=1}^{n-1}\lambda_j, \quad H_2 = \frac{1}{(n-1)(n-2)}\sum_{i<j}^{n-1}\lambda_i\lambda_j, \quad \dots \\ H_{n-1} &= \frac{1}{(n-1)!}\lambda_1\cdot\ldots\cdot\lambda_{n-1}. \end{aligned}$$

In particular, for the mean curvature

$$H = \frac{1}{n-1}g^{*\mu\nu}\Omega_{\nu\mu},$$

while

$$H_{n-1} = \frac{(-1)^{n-1}}{(n-1)!}\det(\Omega^{\mu}{}_{\nu}) = \frac{(-1)^{n-1}}{(n-1)!}\frac{\det(\Omega_{\mu\nu})}{\det(g^*_{\mu\nu})}.$$

A further property for the coefficients of the second fundamental form is given by the expression of the covariant derivative of the unit normal vector $N^j$ in terms of them. Using

$$\begin{aligned} g_{ij||\beta} &= \frac{\partial g_{ij}}{\partial q^\beta} - g_{kj}\Gamma^k{}_{i\ell}B^\ell{}_\beta - g_{ki}\Gamma^k{}_{j\ell}B^\ell{}_\beta \\ &= \left(\frac{\partial g_{ij}}{\partial x^\ell} - g_{kj}\Gamma^k{}_{i\ell} - g_{ki}\Gamma^k{}_{j\ell}\right)B^\ell{}_\beta = (\nabla_\ell g_{ij})B^\ell{}_\beta = 0, \end{aligned}$$

and the definition of $\Omega_{\mu\nu}$, the covariant derivative of (2.1.5) respect to $q^\beta$ gives

$$\Omega_{\mu\nu} = -g_{ij}B^i{}_\mu N^j{}_{||\nu}, \tag{2.1.6}$$

while the differentiation of $g_{ij}N^iN^j = -1$ yields

$$g_{ij}N^iN^j{}_{||\beta} = 0,$$

which implies that $N^j{}_{||\beta}$ is tangential to $\Sigma$. Thus, there exist coefficients $C^\alpha{}_\beta$ such that $N^j_{||\beta} = B^j{}_\alpha C^\alpha{}_\beta$. Replacing this decomposition in (2.1.6), we obtain the expression of the coefficient $C^\beta{}_\nu$ as $\Omega_{\mu\nu} = -g^*_{\mu\beta}C^\beta{}_\nu$, where we used eq. (2.1.4). Then, the covariant derivative of the (timelike) unit normal vector $N^j$ is given by

$$N^j{}_{||\beta} = -B^j{}_\alpha\Omega^\alpha{}_\beta. \tag{2.1.7}$$

where we put $\Omega^\alpha{}_\beta = g^{*\alpha\mu}\Omega_{\mu\beta}$.

Let's introduce the important notion of totally geodesic and umbilical foliation that will be useful in view of the following results:

**Definition 2.1.6.** *Let $\Sigma \in M$ be a spacelike hypersurface. A point $p \in \Sigma$ is called umbilical if $\Omega|_p$ is proportional to the metric tensor $g^*|_p$. The hypersurface is totally umbilical if every point of $\Sigma$ is umbilical.*

**Definition 2.1.7.** *A submanifold $(M^*, g^*)$ of a manifold $(M, g)$ is called totally geodesic if any geodesic on $M^*$ is also a geodesic on $M$, i.e., for any vector field $v$ on $M^*$ such that $\tan \nabla_w w = 0$, where $w|_{M^*} = v$ is an extension of $v$ to $M$, $\nabla_w w = 0$ holds.*

The notions of foliations or hypersurfaces of a Lorentzian manifold hold for Riemannian manifolds as well, with the appropriate sign changes: at any point $p \in \Sigma$, the vector $N_p$ is such that $g_{ij}N_p^i N_p^j = 1$ and the decomposition of a general vector follows. Moreover, the second fundamental form of the immersion is defined by $\nabla_\nu B^i_\mu = N^i \Omega_{\mu\nu}$.

### 2.1.2 The torse-forming characterization

We are ready to characterize twisted spacetimes through the existence of a torse-forming timelike unit vector field $u$, i.e., a vector such that

$$\nabla_i u_j = \varphi(g_{ij} + u_i u_j) = \varphi h_{ij}, \qquad u_i u^i = -1. \tag{2.1.8}$$

By comparing (2.1.8) with (1.1.1), the vector field $u$ can be viewed as a shear-free, vorticity-free and acceleration-free velocity field. This characterization is shown by C. A. Mantica and L. G. Molinari in [30] and relies on the following theorem by R. Ponge and H. Reckziegel in [33].

**Theorem 2.1.3.** *Let $(M, g)$ be a pseudo-Riemannian space with $M = B \times F$ and assume that the canonical foliations $L_B$ and $L_F$ intersect perpendicularly everywhere. Then $g$ is the metric tensor of a twisted product $B \times_f F$ if and only if $L_B$ is a totally geodesic foliation and $L_F$ is a totally umbilic foliation.*

**Theorem 2.1.4.** *A Lorentzian manifold $\mathscr{L}_n$ is twisted if and only if it admits a torse-forming timelike unit vector field.*

*Proof.* Let $\mathscr{L}_n$ be a Lorentzian twisted manifold, then there is a frame where the metric has the form (2.1.1). The timelike unit vector field with components $u_0 = -1$, $u_\mu = 0$ identically solves the equation $\nabla_i u_j = \varphi(g_{ij} + u_i u_j)$, which gives the non-trivial equation $-\Gamma^0_{\mu\nu} u_0 = \varphi f^2 g^*_{\mu\nu}$, so $\varphi = \dot{f}/f$.

Conversely, suppose that a Lorentzian manifold is endowed with a torse-forming timelike unit vector field $u = \partial_0$, with components $u^i = \delta^i{}_0$: $\nabla_i u_j =$

$\varphi h_{ij}$, $g_{ij}u^i u^j = -1$. It is

$$\begin{aligned}\nabla_u u = \nabla_{\partial_0}\partial_0 =& u^k(\nabla_k u^j)\partial_0 = 0\\ =&\Gamma^k{}_{00}\partial_k,\end{aligned}$$

then

$$\tan \nabla_u u = 0,$$

i.e., $u$ is geodesic on $M$ and its restriction on the one-dimensional submanifold. If we put $D = \mathrm{Span}\{\partial_0\}$, $D$ is a totally geodesic foliation, i.e., $D$ is an integrable distribution whose leaves are totally geodesic in $\mathscr{L}_n$. Moreover, $\nabla_i u_j = \nabla_j u_i$ ($u$ is closed), then, being $u^i = \nabla^i\theta$, it is the unit normal vector field for the surfaces $\theta = const.$

Any vector $v_p \in T_p\mathscr{L}_n$ is decomposable into a normal and a tangent component to the hypersurface: $v^i = -(u^k v_k)u^i + \tilde{v}^i$, where the components $\tilde{v}^i$ are given by $\tilde{v}^i = B^i{}_\mu v^\mu$, where $B^0{}_\mu = 0$ and $B^\nu{}_\mu = \delta^\nu{}_\mu$ are the components of the immersion matrix (abuse of notation, but we use the natural parametrization $x^0 = t$ and $x^\nu = \delta^\nu{}_\mu x^\mu$). Since $v^0 = 0$ and $v^\nu = B^\nu{}_\mu v^\mu$, we can write $\tilde{v}^i = h^i{}_j v^j$. The induced metric is given by (2.1.4) and for the second fundamental of the hypersurface eq. (2.1.7) holds, where the normal vector $N$ corresponds to $u$ (except for the sign). The expression of $u^j{}_{||\beta}$ is evaluated as

$$u^j{}_{||\beta} = u^j{}_{,\beta} + \Gamma^j{}_{hk}B^h{}_\beta u^k = u^j{}_{,\beta} + \Gamma^j{}^k_{\beta k} = \nabla_\beta u^j = \varphi h_\beta{}^j,$$

then

$$\varphi h_\beta{}^j = B^j{}_\alpha \Omega^\alpha{}_\beta.$$

For $j = \nu$, $h_\beta{}^\nu = \delta_\beta{}^\nu$ and $B^\nu{}_\alpha\Omega^\alpha{}_\beta = \Omega^\nu{}_\beta$, then

$$\Omega_{\mu\nu} = \varphi g^*_{\mu\nu}. \tag{2.1.9}$$

The previous equation implies that $D^\perp = \mathrm{Span}\{\partial_1, \dots, \partial_{n-1}\}$ is an integrable distribution whose leaves are totally umbilical hypersurfaces of $\mathscr{L}_n$.

Since the manifold decomposes into a totally geodesic foliation orthogonal to a totally umbilical foliation, according to the theorem 2.1.3, the metric has the twisted form.

□

Equation (2.1.9) implies that the function $\varphi$ is equal to the mean scalar curvature, $H = \dot{f}/f$, where the dot represents the total derivative respect to the proper time: $\dot{f} = \frac{df}{d\tau} = \frac{dx^i}{d\tau}\nabla_i f = u^i\nabla_i f$. The covariant expression for the scalar function is $\varphi = u^i\nabla_i \log f$.

The property that the space $\mathscr{L}_n$ admits a timelike unit torse-forming vector $u$ is strongly related to the existence of the torqued vector $\tau$ by the following theorem:

**Theorem 2.1.5.** *A Lorentzian manifold $\mathscr{L}_n$ admits a timelike torqued vector $\tau_i$ if and only if $u_i = \tau_i/\sqrt{-\tau^2}$ is a torse-forming timelike unit vector field.*

*Proof.* Let $\tau$ be a timelike torqued-vector on $\mathscr{L}_n$: $\tau_i\tau^i < 0$, $\nabla_i\tau_j = \rho\, g_{ij} + \alpha_i\tau_j$, $\alpha_i\tau^i = 0$. The derivative of $u_i = \tau_i/\sqrt{-\tau^2}$ gives

$$\nabla_i u_j = \frac{1}{\sqrt{-\tau^2}}(\rho\, g_{ij} + \alpha_i\tau_j) + \frac{\tau_j\tau^\ell}{(-\tau^2)^{3/2}}(\rho\, g_{il} + \alpha_i\tau_\ell) = \varphi(g_{ij} + u_i u_j),$$

where $\varphi = \rho/\sqrt{-\tau^2}$, and $u^2 = -1$.

Conversely, given a vector $u_i$ such that $u^2 = -1$ and $\nabla_j u_k = \varphi(g_{jk} + u_j u_k)$, we can define $X_i = e^{-\sigma}u_i$, where $\sigma$ is a scalar function. This allows to evaluate

$$\begin{aligned}\nabla_i\tau_j &= \nabla_i(e^{-\sigma}u_j) = e^{-\sigma}[-u_j\nabla_i\sigma + \varphi(g_{ij} + u_i u_j)]\\ &= \rho g_{ij} + (u_i\varphi - \nabla_i\sigma)\tau_j = \rho g_{ij} + \alpha_i\tau_j,\end{aligned}$$

with $\rho = e^{-\sigma}\varphi$ and $\alpha_i = (u_i\varphi - \nabla_i\sigma)$. The vector $\tau_i$ satisfies $\tau^2 < 0$ and the condition $\alpha_i\tau^i = 0$ provides $\varphi = -\dot\sigma$. In the frame (2.1.1) the solution is $\sigma = -\int dt\varphi + c$, where $c$ is such that $\dot c = 0$. □

**Remark 2.1.2.** *The spatial function $c$ can be chosen arbitrarily: by (2.1.3) given a timelike torqued vector $\tau_i$, the vector $e^c\tau_i$ with $\dot c = 0$ is torqued too.*

Let us introduce the orthogonal decomposition $\nabla_i\varphi = v_i - u_i u^k\nabla_k\varphi$, where $v_i = h_i{}^k\nabla_k\varphi$. In the frame (2.1.1) $v$ is a spacelike vector: $v_0 = 0$ and $v_\mu = \partial_\mu\varphi$. The expression of the Ricci tensor in terms of $u_i$ and $v_i$ is given by:

**Proposition 2.1.1.** *The Ricci tensor on a twisted Lorentzian spacetime has the form:*

$$R_{k\ell} = \frac{R - n\xi}{n-1}u_k u_\ell + \frac{R-\xi}{n-1}g_{k\ell} + (n-2)(u_k v_\ell + u_\ell v_k - u^r u^s C_{rk\ell s}), \quad (2.1.10)$$

*where $R$ is the scalar curvature, $C_{jk\ell m}$ is the Weyl tensor, $v_k = h_k{}^m\nabla_m\varphi$ and $\xi = (n-1)(\varphi^2 + \dot\varphi)$.*

*Proof.* The Weyl tensor contracted with $u_m$ leads to:

$$\begin{aligned}C_{jk\ell}{}^m u_m = R_{jk\ell}{}^m u_m &+ \frac{1}{n-2}(u_j R_{k\ell} - u_k R_{j\ell} + g_{k\ell}R_{jm}u^m - g_{j\ell}R_{km}u^m)\\ &- R\,\frac{u_j g_{k\ell} - g_{j\ell}u_k}{(n-1)(n-2)}.\end{aligned} \quad (2.1.11)$$

The evaluation of $\nabla_j(\nabla_k u_\ell) = \nabla_j(\varphi h_{k\ell})$ and the subtraction of the expression with $jk$ exchanged give the expression of $R_{jk\ell}{}^m u_m$, that contracted with $g^{k\ell}$ provides $R_j{}^m u_m$:

$$\begin{aligned} R_{jk\ell}{}^m u_m &= h_{k\ell}\nabla_j\varphi - h_{j\ell}\nabla_k\varphi + \varphi^2(u_k g_{j\ell} - u_j g_{k\ell}). \\ R_j{}^m u_m &= -(n-2)v_j + \xi u_j. \end{aligned} \tag{2.1.12}$$

Another contraction with $u_j$ of (2.1.11), using the two previous expressions found for $R_{jk\ell}{}^m u_m$ and $R_j{}^m u_m$, verifies (2.1.10). □

The multiplication of (2.1.11) and (2.1.12) by $u_i$ and the summation on cyclic permutation of indices $ijk$, after some algebra, show that the symmetric tensor $u_i u_m$ is Weyl-compatible [27][25], i.e.,

$$u_i u^m C_{jk\ell m} + u_j u^m C_{ki\ell m} + u_k u^m C_{ij\ell m} = 0. \tag{2.1.13}$$

As shown in [20], the property (2.1.13) classifies the Weyl tensor as purely electric with respect to $u_j$. The contraction with $u^i$ gives

$$u^m C_{jk\ell m} = u_k C_{j\ell} - u_j C_{k\ell}, \tag{2.1.14}$$

where $C_{k\ell} = u^i u^m C_{ik\ell m}$. It follows that $C_{k\ell} = 0$ if and only if $u^m C_{ik\ell m} = 0$. In [32] C. A. Mantica and L. G. Molinari obtain important results concerning the Weyl tensor, in particular:

**Proposition 2.1.2.** *On a twisted spacetime, if $\nabla^m C_{jk\ell m} = 0$ then*

$$\nabla^m C_{mk} = 0 \quad \textit{and} \quad u^p \nabla_p C_{km} = 0. \tag{2.1.15}$$

Thanks to (2.1.15), the main theorem is easily demonstrated:

**Theorem 2.1.6.** *On a twisted spacetime of dimension $n > 3$:*

*i.* $\quad u^m C_{jk\ell m} = 0 \quad \Longrightarrow \quad \nabla^m C_{jk\ell m} = 0,$

*ii.* $\quad \nabla^m C_{jk\ell m} = 0 \quad \Longrightarrow \quad u^p \nabla_p (u^m C_{jk\ell m}) = -\varphi(n-1) u^m C_{jk\ell m}.$

Thus, on a twisted spacetime the divergence free of Weyl tensor is weaker than the property $u^m C_{jk\ell m} = 0$.

Using the expression (2.1.10) for the Ricci tensor and the EFE we obtain an imperfect fluid stress-energy tensor (1.2.3) with

$$\mu = -\frac{\xi}{\kappa} + \frac{R}{2\kappa}, \tag{2.1.16}$$

$$p = -\frac{1}{n-1}\frac{\xi}{\kappa} - \frac{n-3}{2(n-1)}\frac{R}{\kappa}, \tag{2.1.17}$$

$$q_j = \frac{n-2}{\kappa} v_j, \tag{2.1.18}$$

$$\pi_{ij} = -\frac{n-2}{\kappa} C_{ij}. \tag{2.1.19}$$

The weak, dominant and strong energy conditions respect to $u^i$ imply respectively

$$R/2 - \xi \geq 0, \quad 4v^2 - (R/2-\xi)^2 \leq 0 \quad \text{and} \quad \xi \leq 0.$$

The vector $u^i$ is a velocity field perpendicular to the energy flux $q_i$, the anisotropic stress tensor $\pi_{ij}$ is given by the Weyl tensor (that satisfies the properties $\pi_{ij}u^j = 0$, $\pi^j{}_j = 0$), $p$ and $\mu$ are respectively the effective pressure and the energy density. Spacetimes admitting a torse-forming timelike vector field $u_i$ are largely studied in literature, in particular by A. A. Coley and D. J. McManus in [13], [12].

## 2.2 Generalized Robertson-Walker spacetimes

A subclass of twisted spacetimes are warped spacetimes, defined by R. L. Bishop and B. O'Neill in 1964 [2], but introduced before by Kruchkovich in 1957 [22] as *semi-reducible spaces.*

Let $B$ and $F$ be two pseudo-Riemannian manifolds equipped with pseudo-Riemannian metrics $g_B$ and $g_F$, respectively, and let $f$ be a positive smooth function on B, named *warping function*:

**Definition 2.2.1.** *The warped product $M = B \times_f F$ is the manifold $B \times F$ equipped with the pseudo-Riemannian metric $g = g_B + f^2 g_F$.*

If $B$ is a temporal interval $I$ and $(F, g_F)$ a Riemannian manifold $(M^*, g^*)$, the warped product $B \times_f F$ defines the interesting class of generalized Robertson-Walker (GRW) spacetimes, obtained from the twisted product with metric structure given by (2.1.1) and the function $f$ depends only on the time, $f(t, \vec{x}) = f(t)$. Then, the GRW metric has the local shape

$$ds^2 = -dt^2 + f(t)^2 g^*_{\mu\nu}(\vec{x}) dx^\mu dx^\nu. \tag{2.2.1}$$

If $g^*_{\mu\nu}$ has dimension 3 and $M^*$ has constant curvature, the manifold $M$ corresponds to the ordinary Robertson-Walker spacetime.

**Remark 2.2.1.** *The mean curvature $H = \varphi = \dot{f}/f$ corresponds to the Hubble parameter in standard cosmology.*

GRW spacetimes were characterized in 2014 by B.-Y Chen through the existence of a vector that satisfies concircularity property [8]:

**Theorem 2.2.1.** *A Lorentzian manifold $\mathscr{L}_n$ of dimension $n \geqslant 3$ is a GRW spacetime if and only if it admits a timelike concircular vector field $X$, named Chen's vector, i.e.,*

$$X^i X_i < 0, \quad \nabla_i X_j = \rho g_{ij}, \tag{2.2.2}$$

*where $\rho$ is a function on $\mathscr{L}_n$.*

**Remark 2.2.2.** *In view of expression* (2.1.2)*, a concircular vector corresponds to the special torqued vector with $\alpha_i = 0$. Moreover, given a vector $X$ that satisfies* (2.2.2)*, if the scalar $\lambda$ is a constant, $\lambda X$ is a timelike concircular vector field:*

$$(\lambda X)^2 < 0, \quad \nabla_i(\lambda X_j) = (\lambda\rho) g_{ij}. \tag{2.2.3}$$

*As shown in [9], up to constants, there exists at most one concircular vector field associated with a warped product $I \times_f M^*$.*

In [28] C. A. Mantica and L. G. Molinari obtained important results for the Ricci tensor of a GRW spacetime using the Chen's vector $X_i$. Given the Chen's vector $X_i$, the vector $u_i = X_i/\sqrt{-X^2}$ is clearly torse-forming, but the opposite way it's not necessarily true. The torse-forming property is weaker than concircularity, but we can give a characterization of GRW spacetimes through the existence of $u_i$ with an additional condition:

**Theorem 2.2.2.** *A Lorentzian manifold $\mathscr{L}_n$ of dimension $n \geqslant 3$ admits a timelike concircular vector $X_i$ if and only if $u_i = X_i/\sqrt{-X^2}$ is a torse-forming timelike unit vector, i.e., $\nabla_i u_j = \varphi(g_{ij} + u_i u_j)$, and $\varphi$ is such that $\nabla_i \varphi = -\dot{\varphi} u_i$.*

*Proof.* If $\mathscr{L}_n$ admits a timelike concircular vector $X_i$, then $u_i = X_i/\sqrt{-X^2}$ is such that $u^2 = -1$ and $\nabla_i u_j = \varphi(g_{ij} + u_i u_j)$, where $\varphi = \rho/\sqrt{-X^2}$, and $u^2 = -1$. The integrability condition of $\nabla_k X_j = \rho g_{jk}$ gives $R_{jm}X^m = -(n-1)\nabla_j \rho$, then $(n-1)\nabla_j \rho = -\xi X_j$. Replacing $X$ and $\rho$ in terms of $u$ and $\varphi$ we get $\nabla_i \varphi = -\dot{\varphi} u_i$.

Conversely, given a vector $u^i$ such that $u^2 = -1$ and $\nabla_j u_k = \varphi(g_{jk} + u_j u_k)$, we can define $X_i = e^{-\sigma} u_i$, where $\sigma$ is a scalar function. This allows to evaluate $\nabla_i X_j$ as in theorem 2.1.5, but for $X^i$ being a concircular vector $\alpha_i = (u_i\varphi - \nabla_i\sigma)$ has to vanish, then $\varphi u_i = \nabla_i\sigma$. From $\nabla_i\varphi = -\dot\varphi u_i$ follows that $\nabla_i(\varphi u_j) - \nabla_j(\varphi u_i) = 0$ and $\varphi u_i$ is locally a gradient.

□

The following results hold for the Ricci tensor:

**Proposition 2.2.1.** *On a GRW spacetime the torse-forming vector $u_i$ is an eigenvector of the Ricci tensor:*

$$R_{jm}u^m = \xi u_j \tag{2.2.4}$$

*and the eigenvalue satifies*

$$\xi = (n-1)(\dot\varphi + \varphi^2), \qquad \nabla_i\xi = -\dot\xi u_i. \tag{2.2.5}$$

*Proof.* The integrability condition of $\nabla_k u_j = \varphi(g_{jk} + u_j u_k)$ is

$$\begin{aligned} R_{jk\ell}{}^m u_m &= [\nabla_j, \nabla_k]u_\ell = \nabla_j(\varphi h_{k\ell}) - \nabla_k(\varphi h_{j\ell}) \\ &= u_k u_\ell \nabla_j\varphi - u_j u_\ell \nabla_k\varphi + \varphi^2(u_k g_{j\ell} - u_j g_{k\ell}) + g_{k\ell}\nabla_j\varphi - g_{j\ell}\nabla_k\varphi \\ &= \varphi^2(u_k g_{j\ell} - u_j g_{k\ell}) - g_{k\ell}u_j\dot\varphi + g_{j\ell}u_k\dot\varphi, \end{aligned}$$

where we have used the property $\nabla_i\varphi = -\dot\varphi u_i$. The contraction with $g^{k\ell}$ shows that $u^i$ is an eigenvector of the Ricci tensor with eigenvalue $\xi$ given by (2.2.5). Moreover

$$R_{jk\ell}{}^m u_m = -\frac{\xi}{n-1}(u_j g_{k\ell} - u_k g_{j\ell}), \tag{2.2.6}$$

whose covariant derivative is

$$u^m\nabla_s R_{jk\ell m} + \varphi R_{jk\ell m}h^m{}_s = -\frac{\nabla_s\xi}{n-1}(u_j g_{k\ell} - u_k g_{j\ell}) - \frac{\varphi\xi}{n-1}(h_{js}g_{k\ell} - h_{ks}g_{j\ell})$$

The sum on cyclic permutations of indices $sjk$ and the Bianchi identities give

$$0 = g_{k\ell}(u_j\nabla_s\xi - u_s\nabla_j\xi) + g_{j\ell}(u_s\nabla_k\xi - u_k\nabla_s\xi) + g_{s\ell}(u_k\nabla_j\xi - u_j\nabla_k\xi).$$

The contraction with $g^{s\ell}$ finally provides $u_j\nabla_k\xi - u_k\nabla_j\xi = 0$, with solution (2.2.5). □

**Theorem 2.2.3.** *On a GRW spacetime the Ricci tensor can be expressed in terms of the Weyl tensor, the curvature scalar $R$, the eigenvalue $\xi$ and the torse-forming vector $u^i$ as*

$$R_{k\ell} = \frac{R - n\xi}{n-1}u_k u_\ell + \frac{R-\xi}{n-1}g_{k\ell} - (n-2)C_{jk\ell m}u^j u^m. \tag{2.2.7}$$

*Proof.* The expression of the Ricci tensor is given by the contraction of Weyl tensor with $u^m u^j$ and using the eigenvalue equations and (2.2.6), i.e.,

$$\begin{aligned} C_{jk\ell m}u^m u^j &= \left[\frac{\xi - R}{(n-1)(n-2)}(u_j g_{k\ell} - u_k g_{j\ell}) + \frac{1}{n-2}(u_j R_{k\ell} - u_k R_{j\ell})\right] u^j \\ &= -\frac{1}{n-2}\left[\frac{\xi - R}{n-1}h_{k\ell} + \xi u_k u_\ell + R_{k\ell}\right]. \end{aligned}$$

□

**Remark 2.2.3.** *From the integrability condition of $\nabla_k u_j$,*

$$R_{jm}u^m = -\nabla_j\varphi - u_j\dot{\varphi} - \varphi^2(n-1)u_j + n\nabla_j\varphi - \nabla_j\varphi,$$

*the condition $\nabla_i\varphi = -\dot{\varphi}u_i$ is equivalent to requiring that $u_i$ is an eigenvector of the Ricci tensor. In the comoving frame, the property $\nabla_i\varphi = -\dot{\varphi}u_i$ corresponds to $\varphi(\vec{x}, t) = \varphi(t)$, i.e., $f(\vec{x}, t) = f(t)$ or alternately, for geometers, the spatial submanifold $M^*$ is a spherical foliation.*

The multiplication of (2.2.6) by $u_\ell$ and the summation on cyclic permutations of $ijl$ show that $u_i u_m$, is Riemann-compatible [26]:

$$u_i u^m R_{j\ell km} + u_j u^m R_{\ell ikm} + u_\ell u^m R_{ijkm} = 0,$$

that implies Weyl compatibility:

$$u_i u^m C_{j\ell km} + u_j u^m C_{\ell ikm} + u_\ell u^m C_{ijkm} = 0. \tag{2.2.8}$$

### 2.2.1 Perfect fluid GRW spacetimes

In this subsection we discuss necessary and sufficient conditions for GRW spacetimes to be perfect fluid spacetimes, more often named quasi-Einstein manifolds [7], and for a perfect fluid to be a GRW spacetime. Let's start on defining a perfect fluid spacetime as:

**Definition 2.2.2.** *A Lorentzian manifold $\mathscr{L}_n$ is named perfect fluid spacetime if the Ricci tensor has the form*

$$R_{ij} = \alpha g_{ij} + \beta v_i v_j, \tag{2.2.9}$$

*where $\alpha$ and $\beta$ are scalar fields and $v^2 = -1$.*

As $v_i$ is an eigenvector of the Ricci tensor, the latter can be parameterized in terms of the scalar curvature $R$ and the eigenvalue $\eta$ as

$$R_{k\ell} = \frac{R - n\eta}{n-1}v_k v_\ell + \frac{R-\eta}{n-1}g_{k\ell}. \tag{2.2.10}$$

**Remark 2.2.4.** *Suppose that the GRW spacetime is also perfect fluid. Thus there exists a vector $v_i$ such that the Ricci tensor has the form (2.2.10) and a torse-forming timelike unit vector $u_i$ such that $R_{ij}u^j = \xi u_i$. Then*

$$\left(\xi - \frac{R-\eta}{n-1}\right) u_k = \frac{R-n\eta}{n-1}(u^\ell v_\ell)v_k$$

*Since both $u_k$ and $v_k$ are timelike vectors, it cannot be $u^k v_k = 0$. Unless $R = n\eta$, it must be $v_k = \pm u_k$ and $\xi = \eta$. Instead, if $R = n\eta$ the spacetime is Einstein, i.e., $R_{ij} = \eta g_{ij}$, and $u_i$ is not necessarily equal to $v_i$.*

M. Sánchez in [34] and A. Gębarowski in [19], [18] respectively proved the following theorems:

**Theorem 2.2.4.** *A GRW spacetime $M$ is perfect fluid if and only if the submanifold $M^*$ is an Einstein manifold, i.e., $R^*_{\mu\nu} = \frac{R^*}{n-1}\, g^*_{\mu\nu}$.*

**Theorem 2.2.5.** *On a GRW spacetime $M$ the submanifold $M^*$ is Einstein if and only if $\nabla^m C_{jk\ell m} = 0$.*

Theorem 2.2.4 can be nimbly demonstrated as follows:
On a GRW spacetime the spatial components of the Ricci tensor are listed in appendix B.2, but are also given by (2.2.7):

$$R_{\mu\nu} = \frac{R-\xi}{n-1} g^*_{\mu\nu} f^2 - (n-2)C_{0\mu\nu 0}.$$

Using the expression of $\xi$ in terms of $f$ and the expression of scalar curvature listed in appendix B, the comparison of the two equations for $R_{\mu\nu}$ gives

$$\begin{aligned} R^*_{\mu\nu} &= \frac{R-\xi}{n-1} g^*_{\mu\nu} f^2 - g^*_{\mu\nu}[(n-2)\dot{f}^2 + \ddot{f} f] - (n-2)C_{0\mu\nu 0} \\ &= \frac{R^*}{n-1} g^*_{\mu\nu} - (n-2)C_{0\mu\nu 0}, \end{aligned} \tag{2.2.11}$$

that shows the condition $R^*_{\mu\nu} = \frac{R^*}{n-1} g^*_{\mu\nu}$ is equivalent to require that $M$ is a perfect fluid spacetime.

The two theorems together imply that a GRW spacetime is perfect fluid if and only if $\nabla^m C_{jk\ell m} = 0$. It's easily to check also:

**Proposition 2.2.2.** *On a GRW spacetime with torse-forming timelike unit vector $u_i$, $C_{jk\ell m}u^m = 0$ if and only if $R_{jk} = \alpha g_{jk} + \beta u_j u_k$, for suitable scalars $\alpha$ and $\beta$.*

*Proof.* If $C_{jk\ell m}u^m = 0$, eq. (2.2.7) gives to $R_{jk}$ the perfect fluid form.

Conversely, the Weyl tensor contracted with $u^m$ is

$$C_{jk\ell m}u^m = \frac{\xi - R}{(n-1)(n-2)}(u_j g_{k\ell} - u_k g_{j\ell}) + \frac{1}{n-2}(u_j R_{k\ell} - u_k R_{j\ell}).$$

If $R_{jk} = \alpha g_{jk} + \beta u_j u_k$, the trace and the eigenvalue equations respectively give $R = n\alpha - \beta$ and $\xi = \alpha - \beta$, then $C_{jk\ell m}u^m = 0$. □

Therefore, by the previous propositions there exists an algebraic equivalence between the condition of harmonic Weyl tensor and $C_{jk\ell m}u^m = 0$. Such equivalence was found by C. A Mantica and L. G. Molinari in [28]:

**Theorem 2.2.6.** *On a GRW spacetime with torse-forming timelike unit vector $u_i$, $C_{jk\ell m}u^m = 0$ if and only if $\nabla^m C_{jk\ell m} = 0$.*

Thanks to this important theorem, we can prove theorem 2.2.5 in a straightforward way. Since $\nabla^m C_{jk\ell m} = 0$ corresponds to $C_{jk\ell m}u^m = 0$, $C_{jk\ell 0} = 0$ and equation (2.2.11) shows that the submanifold $M^*$ is Einstein. Conversely, if $R^*_{\mu\nu} = \frac{R^*}{n-1} g^*_{\mu\nu}$, we have $C_{j\mu\nu m}u^j u^m = C_{\mu\nu} = 0$ that, with the use of (2.1.14), corresponds to $C_{j\mu\nu m}u^m = 0$. Then $C_{jk\ell m}u^m = 0$ and $\nabla^m C_{jk\ell m} = 0$.

We can summarize all the equivalent conditions:

**Theorem 2.2.7.** *A GRW spacetime $M$ is perfect fluid if and only if one of the following statements is satisfied:*

*i.* $R^*_{\mu\nu} = \dfrac{R^*}{n-1}\, g^*_{\mu\nu}$,

*ii.* $\nabla^m C_{jk\ell m} = 0$,

*iii.* $u^m C_{jk\ell m} = 0$.

Conversely, C. A. Mantica, L. G. Molinari et al. in [31] found sufficient conditions for a perfect fluid spacetime to be a GRW spacetime:

**Theorem 2.2.8.** *A perfect fluid spacetime with Ricci tensor*

$$R_{k\ell} = \frac{R - n\xi}{n-1} u_k u_\ell + \frac{R - \xi}{n-1} g_{k\ell}$$

*is GRW if the vector field $u^i$ is geodesic and such that $u^j \nabla^m C_{jk\ell m} = 0$.*

**Einstein's equation in perfect fluid GRW spacetimes**

Let us consider a GRW spacetime such that the Ricci tensor has the perfect fluid form. Friedmann equations can be recovered in a straightforward way, using the characterization of the space through the torse-forming vector. The covariant divergence of the eigenvalue equations (2.2.4) gives the important scalar relation

$$\frac{1}{2}\dot{R} = n\varphi\xi - \varphi R + \dot{\xi},$$

where we used the torse-forming property, the relation $\nabla^k R_{kj} = \frac{1}{2}\nabla_j R$ and eq. (2.2.5). The solution is

$$R = \frac{R^*}{f^2} + 2\xi + (n-1)(n-2)\varphi^2,$$

where $R^*$ is the scalar curvature of the spacelike submanifold.

The perfect fluid form of the Ricci tensor (2.2.9) makes these spaces solutions of the EFE for perfect fluids: $G_{k\ell} = \kappa T_{k\ell}$ where $T_{k\ell}$ has the form (1.2.2). The trace and the eigenvalue equations respectively give

$$\begin{cases} (n-2)R = -2\kappa p(n-1) + 2\kappa\mu \\ 2\xi - R = -2\kappa\mu \end{cases}$$

and the substitution of the expression of $R$ and of $\xi$ in terms of the scalar function $f$ and its derivatives provides

$$\begin{cases} \kappa\left(p + \mu\dfrac{n-3}{n-1}\right) = -(n-2)\dfrac{\ddot{f}}{f} \\ \kappa\mu = \dfrac{R^*}{2f^2} + \dfrac{1}{2}(n-1)(n-2)\dfrac{\dot{f}^2}{f^2} \end{cases}$$

If $n = 4$, $R^* = const$ and defining $f(t) = a(t)$, the scale factor of the universe, the previous equations correspond to the usual Friedmann equations, i.e.,

$$\begin{cases} \dfrac{3\ddot{a}}{a} = -\dfrac{\kappa}{2}(3p + \mu), \\ \dfrac{3\dot{a}^2}{a^2} = \kappa\mu - \dfrac{R^*}{2a^2}. \end{cases}$$

### 2.2.2 Conformal trasformations

Given a GRW spacetime $(M, g)$, if $\sigma$ is a smooth function we can replace the metric tensor with a locally rescaled one through a *conformal transformation*

$$\bar{g}_{ij}(x) = e^{2\sigma(x)} g_{ij}(x).$$

A timelike, null or spacelike vector $v^i$ with respect to the metric $g_{ij}$ has the same property with respect to $\bar{g}_{ij}$. Conversely, if the light cones of two Lorentz metrics $g_{ij}$ and $\bar{g}_{ij}$ coincide at a point $p \in M$, then $\bar{g}_{ij}$ must be a multiple of $g_{ij}$ at $p$.

Under conformal transformation the Weyl tensor is invariant, i.e., $\bar{C}_{jk\ell}{}^m = C_{jk\ell}{}^m$, while $\bar{C}_{jk\ell m} = e^{2\sigma} C_{jk\ell m}$. The Christoffel symbols, the Ricci tensor and the divergence of the Weyl tensor transform as [35]

$$\begin{aligned}
&\bar{\Gamma}^k_{ij} = \Gamma^k{}_{ij} + 2\delta^k{}_{[j}\nabla_{i]}\sigma - g_{ij}\nabla^k\sigma, \\
&\bar{R}_{ij} = R_{ij} - (n-2)[\nabla_i\nabla_j\sigma - (\nabla_i\sigma)(\nabla_j\sigma) + g_{ij}(\nabla^k\sigma)(\nabla_k\sigma)] - g_{ij}\nabla^2\sigma, \\
&\bar{\nabla}_m\bar{C}_{jk\ell}{}^m = \nabla_\ell C_{jk\ell}{}^m + (n-3)C_{jk\ell}{}^m\nabla_m\sigma.
\end{aligned} \tag{2.2.12}$$

In [5], S. Capozziello et al. show that any conformal transformation

$$\bar{g}_{ij} = e^{2\sigma} g_{ij} \qquad \nabla_i\sigma = -u_i\dot{\sigma}, \tag{2.2.13}$$

maps a GRW spacetime $(M, g)$ to a GRW spacetime $(M, \bar{g})$. In particular, $\bar{u}^i = e^{-\sigma}u^i$ is a torse-forming vector in $(M, \bar{g})$ with

$$\bar{\nabla}_i\bar{u}_j = e^{-\sigma}(\varphi + \dot{\sigma})(\bar{g}_{ij} + \bar{u}_i\bar{u}_j)$$

and the Ricci tensor $\bar{R}_{ij}$ reduces to

$$\bar{R}_{ij} = R_{ij} - (n-2)[\varphi\dot{\sigma} - \dot{\sigma}^2 - \ddot{\sigma}]u_iu_j + [(2n-3)\varphi\dot{\sigma} + (n-2)\dot{\sigma}^2 + \ddot{\sigma}]g_{ij}.$$

Since $u^i$ is an eigenvector of $R_{ij}$, the rescaled vector $\bar{u}^i$ is an eigenvector of $\bar{R}_{ij}$ with eigenvalue $\bar{\xi}$ and scalar curvature $\bar{R}$ given by

$$\begin{aligned}
&\bar{\xi} = e^{-2\sigma}[\xi + (n-1)(\varphi\dot{\sigma} + \ddot{\sigma})], \\
&\bar{R} = e^{-2\sigma}[R + 2(n-1)^2\varphi\dot{\sigma} + (n-1)(n-2)\dot{\sigma}^2 + 2(n-1)\ddot{\sigma}].
\end{aligned}$$

On a GRW spacetime with harmonic Weyl tensor, the conformal transformation (2.2.13) guarantees $\bar{\nabla}_m\bar{C}_{jk\ell}{}^m = 0$. In fact, from theorem 2.2.6, $u_m C_{jk\ell}{}^m = 0$ holds and eq. (2.2.12) gives the result. Then, under the conformal map (2.2.13) a perfect fluid spacetime GRW $(M, g)$ provides another perfect fluid spacetime GRW $(M, \bar{g})$, with Ricci tensor given by

$$\bar{R}_{ij} = \frac{\bar{R} - n\bar{\xi}}{n-1}\bar{u}_i\bar{u}_j + \frac{\bar{R} - \bar{\xi}}{n-1}\bar{g}_{ij}.$$

### 2.2.3 A recent result for $f(R)$ gravity in GRW

General Relativity can be formulated in a very useful way through the Lagrangian formalism. The Einstein-Hilbert action is defined as

$$S_{EH} = \frac{1}{2\kappa}\int d^n x\sqrt{-g}R,$$

where $g = \det(g_{ij})$. The full action of the theory is given by $S_{EH}$ and a matter term

$$S = S_{EH} + S_m, \quad \text{where} \quad S_m = \int d^n x\sqrt{-g}\mathcal{L}_m.$$

The principle of least action provides $\delta S = 0$, i.e. the variation of $S$ with respect to the inverse metric is zero, yielding the usual EFE with stress-energy tensor given by

$$T_{ij} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{ij}}.$$

When the cosmological constant $\Lambda$ is included in the total action as

$$S = \int d^n x\sqrt{-g}\left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_m\right],$$

the field equations take the well-known form

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = \kappa T_{ij}$$

These are the EFE used in the current standard model of cosmology known as the $\Lambda$CDM model. The cosmological constant has the same effect as an intrinsic energy density of the vacuum and represents the simplest possible explanation for dark energy, that satisfies the equation of state $\mu_\Lambda = -p_\Lambda$.

A generalization of Einstein's theory are the so called $f(R)$ theories of gravitation. They were introduced by H. S. Buchdahl in 1970 and gained popularity with the works by A. Starobinsky on cosmic inflation. The task of these theories would be to explain the problem of *dark side* (dark energy + dark matter) in a purely geometric way. Instead of searching for new material ingredients in the universe, the dark side problem could be formally solved by replacing the scalar curvature in the Einstein-Hilbert action with a smooth function of it, so that the total action becomes

$$S = \frac{1}{2\kappa}\int d^n x\sqrt{-g}f(R) + S_m$$

Thanks to the principle of least action the field equations are

$$f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} + [g_{ij}\nabla^2 - \nabla_i\nabla_j]f'(R) = \kappa T_{ij} \tag{2.2.14}$$

where a prime denotes derivative with respect to $R$. It is easy to check that the property $\nabla_i T^i{}_j = 0$ is preserved for any differentiable $f(R)$. If $R_{ij}$ has the perfect fluid form, the presence of the terms $\nabla_i \nabla_j R$ and $(\nabla_i R)(\nabla_j R)$, prevents $T_{ij}$ to describe a perfect fluid, but if in addition there exists a timelike torse-forming vector field $u^i$ such that $R_{ij}u^i = \xi u_j$ (i.e., if the spacetime is a GRW spacetime) and if $\nabla_i R = -u_i \dot{R}$, the lhs of (2.2.14) has the perfect fluid form, so that the field equations for $f(R)$ gravity give a perfect fluid stress-tensor. In [28] L. G. Molinari and C. A. Mantica show that $\nabla_i R = -u_i \dot{R}$ holds on a GRW spacetime with harmonic Weyl tensor. Then, S. Capozziello et al. in [5] show that:

**Theorem 2.2.9.** *On a GRW spacetime with $\nabla_m C_{jk\ell}{}^m = 0$, the stress-energy tensor is a perfect fluid in any $f(R)$ theory of gravity.*

Another important result is proved considering quadratic gravity, that corresponds to Einstein-Hilbert action corrected with quadratic combinations of curvature invariants

$$S = \int d^n x \sqrt{-g} \left[ \frac{R - 2\Lambda}{\kappa} + \alpha R^2 + \beta R_{ij} R^{ij} + \gamma (R_{jk\ell m} R^{jk\ell m} - 4 R_{ij} R^{ij} + R^2) \right] + S_m,$$

where the term $\mathcal{G} = R_{ijk\ell} R^{ijk\ell} - 4 R_{ij} R^{ij} + R^2$ is the Gauss-Bonnet topological invariant. On a GRW spacetime such that the Weyl tensor is zero (i.e., on a Robertson-Walker spacetime) the stress-energy tensor has a perfect fluid in any quadratic theory of gravity, i.e., the quadratic gravity contributions have the perfect fluid form. Very recently the same authors in [6] show that, with the same hypothesis, a general smooth function $f(R, \mathcal{G})$ gives again a perfect fluid stress-energy tensor.

## 2.3 Robertson-Walker spacetimes

Robertson-Walker (RW) spacetimes are an important subclass of GRW spacetimes that share the property of being conformally flat, i.e., $C_{jk\ell m} = 0$. The Ricci tensor assumes the perfect fluid form

$$R_{k\ell} = \frac{R - n\xi}{n-1} u_k u_\ell + \frac{R - \xi}{n-1} g_{k\ell}$$

and the stress-energy tensor is determined by the EFE,

$$T_{k\ell} = (p + \mu) u_k u_\ell + p g_{k\ell},$$

with $p$ and $\mu$ respectively given by (2.1.17) and (2.1.16).

RW spacetimes are usually characterized by the properties of the spatial submanifold $M^*$. By the homogeneity and isotropy hypothesis, the Riemann tensor on $M^*$ must be (see [35])

$$R^*_{\mu\nu\rho\sigma} = k(g^*_{\nu\sigma}g^*_{\mu\rho} - g^*_{\mu\sigma}g^*_{\nu\rho}), \tag{2.3.1}$$

where $k$ is a constant: $k = \frac{R^*}{(n-1)(n-2)}$. A space that satisfies (2.3.1) is named space of constant curvature. In [14] is shown that any two spaces of constant curvature of the same dimension and metric signature which have equal values of $k$ must be (locally) isometric.

**Remark 2.3.1.** *The constant curvature property implies an Einstein manifold, but the opposite way is not necessarily true.*

In $n = 4$, the RW spacetime metric can be written as:

$$ds^2 = -dt^2 + f(t)^2 \left[ \frac{d\psi^2}{1 - k\psi^2} + \psi^2(d\theta^2 + \sin^2\theta d\phi^2) \right],$$

where $\psi, \theta, \phi$ are the space coordinates. Through an appropriate rescaling of the coordinate $\psi$ and the scale function $f(t)$, the constant $k$ takes three possible values $\pm 1, 0$. For $k = 1$, after the substitution $\psi = \sin r$, the induced metric on $M^*$ in spherical coordinates represents the surface of a 3-sphere, i.e.,

$$dr^2 + \sin^2 r(d\theta^2 + \sin^2\theta d\phi^2).$$

The value $k = 0$ gives the ordinary 3-dimensional flat space. In spherical coordinates, the metric is

$$d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Finally, for $k = -1$ the submanifold $M^*$ is a 3-dimensional hyperboloids. In hyperbolic coordinates, with $\psi = \sinh r$, the metric takes the form

$$dr^2 + \sinh^2 r(d\theta^2 + \sin^2\theta d\phi^2).$$

In [3], Brozos-Vázquez et al. studied conformally flat spacetimes and proved that the two characterization are equivalent. We report the proof in terms of the torse-forming vector:

**Theorem 2.3.1.** *A GRW spacetime $M$ is conformally flat if and only if the submanifold $M^*$ in the warped product $M = I \times_f M^*$ is a space of constant curvature.*

*Proof.* With $C_{jk\ell m} = 0$ the Ricci tensor has the perfect fluid form and, in view of proposition 2.2.2, the Riemann tensor is largely determined:

$$\begin{aligned} R_{jk\ell m} =& \frac{2\xi - R}{(n-1)(n-2)}(g_{k\ell}g_{jm} - g_{km}g_{j\ell}) \\ &+ \frac{R - n\xi}{(n-1)(n-2)}\left[g_{km}u_j u_\ell - g_{jm}u_k u_\ell + g_{j\ell}u_k u_m - g_{k\ell}u_j u_m\right] \end{aligned} \tag{2.3.2}$$

The spatial components of the Riemann tensor are

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \frac{2\xi - R}{(n-1)(n-2)} f^2 (g^*_{\nu\rho}g^*_{\mu\sigma} - g^*_{\nu\sigma}g^*_{\mu\rho}) \\ &= -\left[\frac{R^*}{(n-1)(n-2)} + \dot{f}^2\right](g^*_{\nu\rho}g^*_{\mu\sigma} - g^*_{\nu\sigma}g^*_{\mu\rho}) \end{aligned}$$

From appendix B.2, $R_{\mu\nu\rho\sigma} = R^*_{\mu\nu\rho\sigma} - \dot{f}^2(g^*_{\nu\rho}g^*_{\mu\sigma} - g^*_{\nu\sigma}g^*_{\mu\rho})$. Comparing the two expressions for $R_{\mu\nu\rho\sigma}$, the tensor $R^*_{\mu\nu\rho\sigma}$ satisfies (2.3.1).

Conversely, the general expression of the Riemann tensor is given by

$$\begin{aligned} R_{jk\ell m} =& C_{jk\ell m} + \frac{2\xi - R}{(n-1)(n-2)}(g_{k\ell}g_{jm} - g_{km}g_{j\ell}) \\ &+ \frac{R - n\xi}{(n-1)(n-2)}\left[g_{km}u_j u_\ell - g_{jm}u_k u_\ell + g_{j\ell}u_k u_m - g_{k\ell}u_j u_m\right], \end{aligned}$$

then

$$R^*_{\mu\nu\rho\sigma} = \frac{R^*}{(n-1)(n-2)}(g^*_{\nu\sigma}g^*_{\mu\rho} - g^*_{\nu\rho}g^*_{\mu\sigma}) + C_{\mu\nu\rho\sigma}.$$

If the submanifold $M^*$ is a space of constant curvature it must hold $C_{\mu\nu\rho\sigma} = 0$. Since $M^*$ is also an Einstein manifold, from (2.2.11) we obtain $C_{0\mu\nu 0} = 0$ and (2.1.14) gives $C_{\sigma\mu\nu 0} = 0$. Finally we have $C_{jk\ell m} = 0$. □

The expression (2.3.2) corresponds exactly to the general form of the Riemann tensor in a RW spacetime and characterizes quasi-constant curvature manifolds, introduced by B.-Y. Chen and K. Yano in 1972, [11], such that the submanifold $M^*$ is a space of constant curvature.

In $n = 4$, the Weyl tensor of pseudo-Riemannian manifolds presents the special algebraic identity reported by D. Lovelock and H. Rund in [24]

$$\begin{aligned} 0 =& g_{in}C_{jk\ell m} + g_{jn}C_{ki\ell m} + g_{kn}C_{ij\ell m} \\ &+ g_{im}C_{jkn\ell} + g_{jm}C_{kin\ell} + g_{km}C_{ijn\ell} \\ &+ g_{i\ell}C_{jkmn} + g_{j\ell}C_{kimn} + g_{k\ell}C_{ijmn}. \end{aligned} \tag{2.3.3}$$

If $C_{jk\ell m}u^m = 0$, where $u$ is a generic vector such that $u^2 \neq 0$, the contraction with $u^i$ gives the important identity

$$u_n C_{jk\ell m} + u_m C_{jkn\ell} + u_\ell C_{jkmn} = 0$$

and another contraction with $u^n$ results in $C_{jk\ell m} = 0$. As a consequence we have:

**Proposition 2.3.1.** *In $n = 4$, a GRW manifold with $\nabla^m C_{jk\ell m} = 0$ is a Robertson-Walker spacetime.*

*Proof.* In a GRW the condition $\nabla^m C_{jk\ell m} = 0$ is equivalent to $C_{jk\ell m}u^m = 0$, with $u_i$ the torse-forming vector. Then, in $n = 4$, it is $C_{jk\ell m} = 0$. □

# Chapter 3

# Unicity of the torse-forming vector

In the previous chapter we reported the characterization of the large class of twisted spacetimes as in [30]. In particular, we discussed how the existence of a torse-forming vector field gives a peculiar metric structure. Then, it's natural to think about the existence of a second torse-forming vector and its implication, excluding the trivial twin antiparallel vector. The possible existence of a second torse-forming vector would rise strong restrictions on the spatial submanifold $M^*$.

First of all we report a preliminary analysis in the most general case of twisted spacetimes that admit two torse-forming vector fields. Such request implies a doubly twisted metric for the spatial submanifold. Moreover, since the Ricci tensor can be expressed in two different forms through the two torse-forming vectors, we find an interesting property that the Weyl tensor must satisfy. The 4-dimensional case is very particular since the existence of a second torse-forming vector provides the complete tensorial structure of the Weyl and, thus, Riemann tensors. In this way, using the EFE, also the stress-energy tensorial structure is completely determined and corresponds to a two perfect fluid mixture discussed in the first chapter. Before the discussion of unicity in GRW case, we examine how, in [13], A. A. Coley and D. J. McManus reach the same result in a 4-dimensional twisted spacetime with the aid of the EFE.

Finally, we handle GRW spacetimes specializing the results already obtained for twisted spacetimes. The unicity is guaranteed by the property $\nabla_i \varphi = -\dot{\varphi} u_i$ only if the eigenvalue of the Ricci tensor associated to $u_i$ is not constant. Otherwise, we can prove the possible existence of a second non-trivial torse-forming vector and how the spatial submanifold takes the form of a warped manifold.

## 3.1 Duality in twisted spacetimes

Let us consider a twisted spacetime endowed with a torse-forming timelike unit vector field $u_i$, i.e.,

$$\nabla_i u_j = \varphi h_{ij},$$
$$\nabla_i \varphi = -u_i \dot{\varphi} + v b_i,$$

where $b^k u_k = 0$ ($b_k$ is spacelike), $b^k b_k = 1$ and $v = b^i \nabla_i \varphi \neq 0$. Besides $u_k$ with scalar field $\varphi$, let us suppose the existence of a second timelike unit torse-forming vector field $w_i$, not collinear with $u_i$, with scalar field $\lambda$ such that

$$\nabla_i w_j = \lambda \tilde{h}_{ij}, \tag{3.1.1}$$

where $\tilde{h}_{ij} = g_{ij} + w_i w_j$.

**Remark 3.1.1.** *The vector $w_k$ is such that $u^k w_k \neq 0$, otherwise $w_k$ would be spacelike. Furthermore, we are assuming that $u^k w_k \neq \pm 1$, otherwise the two vectors would be collinear, i.e., $w_i = \pm u_i$ with $\lambda = \pm\varphi$.*

We prove the following main theorem about the possible existence of two distinct torse-forming vectors:

**Theorem 3.1.1.** *In a twisted spacetime, for another non-collinear timelike unit torse-forming vector field to exist, it is necessary that*

$$\nabla_i b_j = \varphi b_i u_j + (h_{ij} - b_i b_j)\frac{\nabla_k b^k}{n-2}, \tag{3.1.2}$$
$$2|R_{ij} u^i b^j| \leq |R_{ij}(u^i u^j + b^i b^j)|, \tag{3.1.3}$$

*If it exists, it is $w_i = u_i \cosh\alpha + b_i \sinh\alpha$, with*

$$\tanh\alpha = -\frac{2R_{ij}u^i b^j}{R_{ij}(u^i u^j + b^i b^j)}$$

*and $\nabla_i w_j = \lambda \tilde{h}_{ij}$, where $\lambda = \varphi \cosh\alpha + \frac{\nabla_k b^k}{n-2}$ and $\tilde{h}_{ij} = (g_{ij} + w_i w_j)$.*

The following identities with the Riemann tensor, $[\nabla_i, \nabla_j]u_k = R_{ijkm}u^m$ and $[\nabla_i, \nabla_j]w_k = R_{ijkm}w^m$, are evaluated with the torse-forming conditions for $u$ and $w$:

$$R_{ijkm}u^m = h_{jk}\nabla_i\varphi - h_{ik}\nabla_j\varphi + \varphi^2(u_j g_{ik} - u_i g_{jk}),$$
$$R_{ijkm}w^m = \tilde{h}_{jk}\nabla_i\lambda - \tilde{h}_{ik}\nabla_j\lambda + \lambda^2(w_j g_{ik} - w_i g_{jk}).$$

The trace of both equations on $i$ and $k$ gives identities with the Ricci tensor

$$R_{jm}u^m = h_{jk}\nabla^k\varphi + (n-1)(\varphi^2 u_j - \nabla_j\varphi), \tag{3.1.4}$$
$$R_{jm}w^m = \tilde{h}_{jk}\nabla^k\lambda + (n-1)(\lambda^2 w_j - \nabla_j\lambda), \tag{3.1.5}$$

while the contractions with $w^i$ or with $u^i$ respectively are

$$R_{ijkm}w^i u^m = h_{jk}\varphi' - (w^i u_i u_k + w_k)\nabla_j\varphi + \varphi^2(u_j w_k - w^i u_i g_{jk}), \tag{3.1.6}$$
$$R_{ijkm}u^m w^i = \tilde{h}_{jk}\dot{\lambda} - (u^i w_i w_j + u_j)\nabla_k\lambda + \lambda^2(w_k u_j - u^i w_i g_{jk}), \tag{3.1.7}$$

where, for the second equation we have exchanged $k$ with $j$, after renaming $i$ with $m$ and vice versa. In (3.1.6), $\varphi'$ denotes the derivative of the scalar function along $w_k$: $\varphi' = w^k\nabla_k\varphi$. Subtracting one equation to the other,

$$\begin{aligned} g_{jk}[\varphi' - (\varphi^2 - \lambda^2)w^i u_i - \dot{\lambda}] + u_j u_k\varphi' - (w^i u_i u_k + w_k)\nabla_j\varphi \\ +(\varphi^2 - \lambda^2)u_j w_k - w_j w_k\dot{\lambda} + (u^i w_i w_j + u_j)\nabla_k\lambda = 0. \end{aligned}$$

The vectors $u_j$, $w_j$ and $\nabla_j\varphi$ span, at most, a 3-dimensional space, then there exist, at least, $n-3$ orthogonal vectors to them. The contraction with one of these non-zero vectors gives

$$\varphi' - \dot{\lambda} = (u^i w_i)(\varphi^2 - \lambda^2) \tag{3.1.8}$$

and, using this result in the starting equation,

$$\begin{aligned} u_j u_k\varphi' - (w^i u_i u_k + w_k)\nabla_j\varphi + (\varphi^2 - \lambda^2)u_j w_k \\ -w_j w_k\dot{\lambda} + (u^i w_i w_j + u_j)\nabla_k\lambda = 0. \end{aligned} \tag{3.1.9}$$

The trace of the latter is: $(w^i u_i)(\varphi^2 - \lambda^2 + \lambda' - \dot{\varphi}) + 2(\dot{\lambda} - \varphi') = 0$. With the aid of eq. (3.1.8) we obtain, after cancelling $u^j w_j \neq 0$,

$$\dot{\varphi} + \varphi^2 = \lambda' + \lambda^2. \tag{3.1.10}$$

Hereafter, we denote $\xi = (n-1)(\dot{\varphi} + \varphi^2)$. The contraction of (3.1.9) with $u^j$ or with $w^k$ and the use of (3.1.10) give

$$u_k[\varphi' + w^i u_i\dot{\varphi}] + w_k[\lambda' + u^i w_i\dot{\lambda}] = [(u^i w_i)^2 - 1]\nabla_k\lambda, \tag{3.1.11}$$
$$u_j[\dot{\varphi} + w^k u_k\varphi'] + w_j[\dot{\lambda} + u^i w_i\lambda'] = [(u^i w_i)^2 - 1]\nabla_j\varphi. \tag{3.1.12}$$

If $\nabla_k\varphi$ is not collinear with $u_k$, the coefficient of $w_k$ in (3.1.12) cannot be zero, and the same equation shows that $\nabla_k\varphi$ is a linear combination of $u_k$ and $w_k$. Eq. (3.1.11) shows that $\nabla_k\lambda$ is a linear combination of $u_k$ and $w_k$. Since $w_k$ is spanned by the vectors $u_k$ and $b_k$, it is convenient to introduce the hyperbolic rotation of the orthogonal pair $(u, b)$ to the orthogonal pair $(w, c)$:

$$\begin{cases} w_i = u_i \cosh\alpha + b_i \sinh\alpha \\ c_i = u_i \sinh\alpha + b_i \cosh\alpha \end{cases} \qquad \alpha \neq 0. \tag{3.1.13}$$

Then: $w^2 = -1$, $c_k w^k = 0$, $c^k c_k = 1$, $u_i u_j - b_i b_j = w_i w_j - c_i c_j$. The vector $w$ must have a component parallel to $u$, otherwise $w$ would be spacelike. If $w$ exists, also $-w$ is a timelike unit torse-forming with scalar field $-\lambda$. The following proposition shows that at most two different torse-forming vectors can exist:

**Proposition 3.1.1.** *The only possible hyperbolic rotation is:*

$$\tanh\alpha = -\frac{2R_{ij}u^i b^j}{R_{ij}(u^i u^j + b^i b^j)}. \tag{3.1.14}$$

*Proof.* The contraction of (3.1.4) with $u^i$ and of (3.1.5) with $w^i$ give the same result: $R_{ij}u^i u^j = R_{ij}w^i w^j = -\xi$. Then

$$0 = R_{ij}(w^i w^j - u^i u^j) = \sinh\alpha[R_{ij}(u^i u^j + b^i b^j)\sinh\alpha + 2(R_{ij}u^i b^j)\cosh\alpha].$$

If $\alpha \neq 0$, the result is obtained. □

**Remark 3.1.2.** *The property* $|\tanh\alpha| < 1$ *poses the condition* (3.1.3) *on the Ricci tensor of the twisted spacetime for a second vector to exist. The evaluation of* (3.1.14) *gives*

$$\left(\frac{R - n\xi}{n-1} - (n-2)C_{ij}b^i b^j\right)\sinh\alpha = 2v(n-2)\cosh\alpha. \tag{3.1.15}$$

**Proposition 3.1.2.** *The vector field $w_i$ is associated to the scalar field*

$$\lambda = \varphi\cosh\alpha + \frac{\nabla_k b^k}{n-2}\sinh\alpha. \tag{3.1.16}$$

*Proof.* The torse-forming conditions for $u_i$ and $w_i$ give the identity

$$\nabla_i(u^k w_k) = (u^k w_k)(\varphi u_i + \lambda w_i) + \varphi w_i + \lambda u_i,$$

which becomes

$$\nabla_i\alpha = (\lambda\cosh\alpha - \varphi)b_i + \lambda\sinh\alpha u_i, \quad \textit{i.e.}, \quad \nabla_i\alpha = \lambda c_i - \varphi b_i. \tag{3.1.17}$$

The relations $\nabla_k w^k = \nabla_k(u^k \cosh\alpha + b^k \sinh\alpha) = \lambda(n-1)$ and $\nabla_k u^k = \varphi(n-1)$ give:

$$(\lambda - \varphi\cosh\alpha)(n-1) = \sinh\alpha u^k \nabla_k \alpha + \cosh\alpha b^k \nabla_k \alpha + \sinh\alpha \nabla_k b^k.$$

The terms $u^k\nabla_k\alpha$ and $b^k\nabla_k\alpha$ are obtained from (3.1.17) and the previous equation gives the function $\lambda$. □

**Remark 3.1.3.** *Since* $u^i u^j - b^i b^j = w^i w^j - c^i c^j$ *and* $R_{ij}u^i u^j = R_{ij}w^i w^j = -\xi$, *it follows that* $R_{ij}b^i b^j = R_{ij}c^i c^j$. *Therefore, the angle* $\alpha$ *is also given by*

$$\tanh\alpha = \frac{2R_{ij}w^i c^j}{R_{ij}(w^i w^j + c^i c^j)}. \tag{3.1.18}$$

*Eq.* (3.1.18)*, together with* (3.1.14)*, implies*

$$v = -c^i \nabla_i \lambda, \tag{3.1.19}$$

*that corresponds to* $R_{ij}u^i b^j = -R_{ij}w^i c^j$.

Eq. (3.1.11) in terms of $u_k$ and $b_k$ becomes, after the cancellation of a factor $\sinh\alpha$,

$$u_k[(b^i\nabla_i\varphi) + \cosh\alpha(b^i\nabla_i\lambda)] + b_k \sinh\alpha\,(b^i\nabla_i\lambda) = \sinh\alpha\nabla_k\lambda.$$

The contraction with $b^k$ gives nothing, while the contraction with $u^k$ and the use of $b^i\nabla_i\varphi = v$ give again the (3.1.19). After some simplifications, we obtain that the derivative of $\lambda$ is decomposed in a parallel and an orthogonal part to $w$, as

$$\nabla_k\lambda = -w_k\lambda' - vc_k. \tag{3.1.20}$$

Let us finally write the condition (3.1.1) for $w_j$ to be torse-forming, in terms of its hyperbolic components

$$\begin{aligned}
\nabla_i w_j &= \nabla_i(\cosh\alpha u_j + \sinh\alpha b_j)\\
&= \varphi h_{ij}\cosh\alpha + (u_j\sinh\alpha + b_j\cosh\alpha)\nabla_i\alpha + \sinh\alpha(\nabla_i b_j)\\
&= \lambda(g_{ij} + w_i w_j)\\
&= \lambda[g_{ij} + u_i u_j\cosh^2\alpha + (u_i b_j + u_j b_i)\cosh\alpha\sinh\alpha + b_i b_j\sinh^2\alpha].
\end{aligned}$$

By means of (3.1.17), the elimination of $\nabla_i\alpha$ gives

$$(h_{ij} - b_i b_j)(\varphi\cosh\alpha - \lambda) = \sinh\alpha(\varphi b_i u_j - \nabla_i b_j).$$

The elimination of $\lambda$ by (3.1.16) and the hypothesis $\sinh\alpha \neq 0$ give the condition (3.1.2). If not fulfilled, the non-collinear vector $w_j$ does not exist.

**Proposition 3.1.3.** *Condition* (3.1.2) *for* $\nabla_i b_j$ *is the most general combination of the kind* $\nabla_i b_j = A g_{ij} + B u_i u_j + C b_i b_j + D b_i u_j + E b_j u_i$.

*Proof.* The relation $b^j \nabla_i b_j = 0$ gives $A + C = 0$ and $E = 0$, while $u^j \nabla_i b_j = -\varphi b_i$ provides $D = \varphi$ and $A - B = 0$. Then: $\nabla_i b_j = A(g_{ij} + u_i u_j - b_i b_j) + \varphi b_i u_j$. Finally: $\nabla_k b^k = (n-2)A$, and (3.1.2) is obtained. □

**Proposition 3.1.4.** *If there exist two different timelike unit torse-forming vectors* $u_i$ *and* $w_i$*, the Weyl tensor satisfies*

$$C_{ij} = (u_i u_j + b_i b_j) - C_{rijs} b^s b^r, \qquad \textit{i.e.,} \qquad C_{rijs}(u^s u^r + b^s b^r) = (u_i u_j + b_i b_j). \tag{3.1.21}$$

*Proof.* Using eq. (3.1.13) and (3.1.20), the Ricci tensor can be expressed in two equivalent forms

$$\begin{aligned} R_{ij} =& \frac{R - n\xi}{n-1} u_i u_j + \frac{R-\xi}{n-1} g_{ij} + v(n-2)(b_i u_j + b_j u_i) - (n-2) C_{ij}, \\ R_{ij} =& \frac{R - n\xi}{n-1} w_i w_j + \frac{R-\xi}{n-1} g_{ij} - v(n-2)(c_i w_j + c_j w_i) - (n-2) \tilde{C}_{ij}, \end{aligned}$$

where $\tilde{C}_{ij} = C_{rijs} w^r w^s$. The subtraction of the two equations, after some algebra, gives

$$\left[\frac{R - n\xi}{n-1} \sinh\alpha - 2(n-2) v \cosh\alpha\right] [(u_i u_j + b_i b_j)\sinh\alpha + (u_i b_j + u_j b_i)\cosh\alpha] \\ + (n-2)(C_{ij} - \tilde{C}_{ij}) = 0 \ .$$

Using (3.1.15) in the previous equation, we obtain

$$\begin{aligned} C_{ij} =& [(u_i u_j + b_i b_j) + (u_i b_j + u_j b_i)\coth\alpha](C_{km} b^k b^m) \\ & - \coth\alpha C_{rijs}(u^r b^s + u^s b^r) - C_{rijs} b^s b^r. \end{aligned} \tag{3.1.22}$$

Since a torse-forming vector is Weyl-compatible, then

$$C_{rijs}(u^r b^s + u^s b^r) = C_{sjir} u^r b^s + C_{rijs} u^s b^r = (u_j C_{si} + u_i C_{sj}) b^s$$

and (3.1.22) takes the form

$$\begin{aligned} C_{ij} =& [(u_i u_j + b_i b_j) + (u_i b_j + u_j b_i)\coth\alpha](C_{km} b^k b^m) \\ & - \coth\alpha (u_j C_{si} + u_j C_{si}) b^s - C_{rijs} b^s b^r. \end{aligned} . \tag{3.1.23}$$

The contraction with $u^i$ gives

$$u_j (C_{km} b^k b^m) + b^r b^s u^i C_{rijs} = -\coth\alpha [b_j (C_{km} b^k b^m) - C_{sj} b^s].$$

The lhs of the previous equation is zero: $b^r b^s u^i C_{rijs} = b^r b^s (u_s C_{jr} - u_j C_{sr}) = -u_j (b^r b^s C_{rs})$, so that $b^j$ is an eigenvector of $C_{js}$

$$C_{js} b^s = b_j (b^r b^s C_{rs})$$

and (3.1.23) reduces to (3.1.21). □

Through the following proposition, the condition (3.1.2) can be expressed in terms of the spatial submanifold $(M^*, g^*)$:

**Proposition 3.1.5.** *Condition* (3.1.2) *is equivalent to the requirement that the spatial submanifold* $(M^*, g^*)$ *admits a unit vector* $n^*_\mu(\vec{x})$*, such that*

$$\nabla^*_\alpha n^*_\beta = \vartheta (g^*_{\alpha\beta} - n^*_\alpha n^*_\beta) + n^*_\alpha m^*_\beta, \tag{3.1.24}$$

*where*

$$\begin{aligned} &\vartheta = \frac{\nabla^{*\alpha} n^*_\alpha}{n-2}, \\ &m^*_\alpha = A_\alpha - (g^{*\gamma\delta} n^*_\gamma A_\delta) n^*_\alpha \quad \textit{with} \quad A_\alpha = \frac{f_\alpha}{f} = \nabla^*_\alpha (\ln f). \end{aligned}$$

*Proof.* Choosing $i, j = 0, \alpha$ and $i, j = \alpha, \beta$ the condition (3.1.2) gives the non-trivial equations

$$\partial_t b_\alpha = \varphi b_\alpha \tag{3.1.25}$$

$$\nabla^*_\alpha b_\beta = 2 b_{(\alpha} \nabla^*_{\beta)} \ln f - b_\rho \nabla^{*\rho} (\ln f) g^*_{\alpha\beta} + (f^2 g^*_{\alpha\beta} - b_\alpha b_\beta) \frac{\nabla_k b^k}{n-2}, \tag{3.1.26}$$

where $\nabla^*_\alpha$ is the covariant derivative with respect to the metric $g^*_{\alpha\beta}$, $\nabla^{*\alpha} \equiv g^{*\alpha\beta} \nabla^*_\beta$ and $\nabla^*_\alpha (\ln f) = \partial_\alpha (\ln f)$. The expression for $\nabla_k b^k$ is evaluated as

$$\begin{aligned} \nabla_k b^k &= f^{-2} g^{*\gamma\delta} \nabla_\gamma b_\delta = a^{-2} [\nabla^{*\gamma} b_\gamma - 2 b_\gamma \nabla^{*\gamma} (\ln f) + b_\rho \nabla^{*\rho} (\ln f) g^{*\gamma}{}_\gamma] \\ &= f^{-2} [\nabla^{*\gamma} b_\gamma + (n-3) b_\gamma \nabla^{*\gamma} \ln f] \end{aligned}$$

and $\nabla^*_\alpha b_\beta$ takes the form

$$\begin{aligned} \nabla^*_\alpha b_\beta = & 2 b_{(\alpha} \nabla^*_{\beta)} \ln f + \frac{1}{n-2} [\nabla^{*\gamma} b_\gamma - b_\gamma \nabla^{*\gamma} \ln f] g^*_{\alpha\beta} \\ & - \frac{1}{n-2} f^{-2} [\nabla^{*\gamma} b_\gamma + (n-3) b_\gamma \nabla^{*\gamma} \ln f] b_\alpha b_\beta. \end{aligned}$$

The vector $n^*_\alpha = b_\alpha / f$ is a unit vector in the spatial submanifold: $g^{*\alpha\beta} n^*_\alpha n^*_\beta = 1$. In terms of $n^*_\alpha$ the previous equation, after some algebra, gives the expression (3.1.24) and (3.1.25) reduces to $\partial_t n^*_\alpha = 0$.

□

In (3.1.24), the scalar function $\vartheta$ can be viewed as expansion parameter in the submanifold $M^*$ and the term $m^*_\beta = n^{*\alpha}\nabla^*_\alpha n^*_\beta$ is the 3-acceleration, with $n^{*\alpha}m^*_\alpha = 0$. The vector $n^*_\alpha$ is a shear-free and irrotational spacelike congruence in terms of the 3-dimensional geometry, therefore, it admits a totally umbilical foliation [21]. In fact, through the discussion in subsection 2.1.1, the normal unit vector $n^*$ is such that

$$n^{*\alpha}{}_{||B} = -B^\alpha{}_A\Omega^{*A}{}_B. \tag{3.1.27}$$

where $\alpha = 1,\ldots,n-1$, $A = 2,\ldots,n-1$, $B^\alpha{}_A$ is the matrix of the first derivative of the immersion $q^\alpha = q^\alpha(\vec{u})$ and $\Omega^*_{AB}$ is the second fundamental form of the ($n$-2)-dimensional submanifold $\tilde{M}$ with metric tensor $\tilde{g}$ given by

$$\tilde{g}_{AB} = g^*_{\alpha\beta}B^\alpha{}_A B^\beta{}_B,$$

($\Omega^{*A}{}_B = \tilde{g}^{AC}\Omega^*_{CB}$). The vector $n^{*\alpha}$ is chosen to be along the direction 1 without loss of generality. Using the natural immersion, $B^1{}_A = 0$ and $B^A{}_B = \delta^A{}_B$, the expression of $n^{*\alpha}{}_{||B}$ is given by

$$\begin{aligned} n^{*\alpha}{}_{||B} =& n^{*\alpha}{}_{,B} + \Gamma^{*\alpha}{}_{\beta\sigma}B^\beta{}_B n^{*\sigma} = n^{*\alpha}{}_{,B} + \Gamma^{*\alpha}{}_{B\sigma}n^{*\sigma} = \nabla^*_B n^{*\alpha} \\ &= \vartheta(g^*_B{}^\alpha - n^*_B n^{*\alpha}) + n^*_B m^{*\alpha} = \vartheta g^*_B{}^\alpha, \end{aligned}$$

then, for $\alpha = A$, $g^*_B{}^A = \delta_B{}^A$ and (3.1.27) reduces to

$$\Omega^*_{AB} = -\vartheta\tilde{g}_{AB},$$

i.e., the first and the second fundamental form are proportional. With the aid of the following proposition, [17]:

**Proposition 3.1.6.** *If a spacelike manifold $(M^*, g^*)$ admits an umbilical foliation, $g^*$ is the metric of the doubly twisted product $X \times_{(f_1,f_2)} Y$, i.e., there exists a coordinate system $(x, x^A)$, with $A = 2,\ldots,n-1$, such that the induced metric takes the form*

$$ds^{*2} = f_1^2 dx^2 + f_2^2\gamma_{AB}dx^A dx^B,$$

*where $f_1$ and $f_2$ are two positive functions on $M^*$ and $\gamma_{AB} = \gamma_{AB}(x^C)$ is the metric of the $n-2$ spatial submanifold.*

the vector $n^*_\alpha = (0, f_1, 0, \ldots, 0)$ and the manifold $M$ gains the metric structure

$$ds^2 = -dt^2 + f(t,\vec{x})^2[f_1(\vec{x})^2dx^2 + f_2(\vec{x})^2\gamma_{AB}(x^C)dx^Adx^B].$$

**Remark 3.1.4.** *From definition of* $v_\alpha$ *and* $n^*_\alpha$ *follows* $\partial_\alpha\dot\varphi = vfn^*_\alpha = vff_1\delta^x_\alpha$*, thus the twisting function* $f$ *has a dependence only by* $t$ *and* $x$*.*

*Moreover, if* $\partial_\alpha f$ *and* $n^*_\alpha$ *are parallel vectors, eq.* (3.1.24) *reduces to* $\nabla^*_\alpha n^*_\beta = \vartheta(g^*_{\alpha\beta} - n^*_\alpha n^*_\beta)$ *and the spatial submanifold* $(M^*, g^*)$ *is twisted, i.e., the metric has the form*

$$ds^2 = -dt^2 + f(t,x)^2[dx^2 + f_2(\vec{x})^2\gamma_{AB}(x^C)dx^A dx^B].$$

### 3.1.1 Four-dimensional twisted spacetimes

No other progress can be made without further hypothesis. For $n = 4$ the special property (2.3.3) holds and the tensorial form of the Weyl tensor is completely determined:

**Theorem 3.1.2.** *On a twisted spacetime with dimension* $n = 4$*, endowed by two different timelike unit torse-forming vector field, the electric part of the Weyl tensor has the following form*

$$2C_{ij} = 3C\left(b_i b_j - \frac{1}{3}h_{ij}\right), \tag{3.1.28}$$

*where* $C \equiv C_{k\ell}b^k b^l$*.*

*Proof.* The contraction of (2.3.3) with $u^i u^n$ gives the Weyl tensor as a Kulkarni-Nomizu product between the two symmetric tensors $(2u_i u_j + g_{ij})$ and $C_{ij}$:

$$\begin{aligned} C_{ijk\ell} =& 2(u_i u_\ell C_{jk} - u_i u_k C_{j\ell} + u_j u_k C_{i\ell} - u_j u_\ell C_{ik}) \\ &+ g_{i\ell}C_{jk} - g_{ik}C_{j\ell} + g_{jk}C_{i\ell} - g_{j\ell}C_{ik}. \end{aligned} \tag{3.1.29}$$

A further contraction with $b^i b^\ell$ gives

$$\begin{aligned} C_{ijk\ell}b^i b^\ell =& 2u_j u_k C + C_{jk} + g_{jk}C - b_k C_{j\ell}b^\ell - b_j C_{k\ell}b^\ell \\ =& 2(u_j u_k - b_j b_k)C + C_{jk} + g_{jk}C. \end{aligned}$$

Using the previous equation in (3.1.21), we obtain (3.1.28). □

**Remark 3.1.5.** *From the identification* (2.1.18)*,* $C_{ij}$ *corresponds to the anisotropic stress tensor and it is given by the heat flux* $q_i = 2vb_i$ *of the imperfect fluid as*

$$2C_{ij} = \frac{3C}{q^2}\left(q_i q_j - \frac{1}{3}q^2 h_{ij},\right) \tag{3.1.30}$$

*where* $q^2 = g_{ij}q^i q^j = 4v^2$*.*

The Ricci tensor is given by

$$R_{ij} = \frac{R-4\xi}{3}u_i u_j + \frac{R-\xi}{3}g_{ij} + 2v(u_i b_j + u_j b_j) - 3C\left(b_i b_j - \frac{1}{3}h_{ij}\right)$$

and also the Weyl and the Riemann tensors can be fully expressed in terms of vectors $u_i$ and $b_i$ and the metric $g_{ij}$

$$\begin{aligned} C_{ijk\ell} =& 6C(u_i u_{[\ell} b_{k]} b_j + u_j u_{[k} b_{\ell]} b_i) + 3C(g_{i[\ell} L_{k]j} + g_{j[k} L_{\ell]i}), \\ R_{ijk\ell} =& C_{ijk\ell} + (g_{i[k} R_{\ell]j} - g_{j[k} R_{\ell]i}) - \frac{1}{3} R g_{i[k} g_{\ell]j} \\ =& 6C(u_i u_{[\ell} b_{k]} b_j + u_j u_{[k} b_{\ell]} b_i) + g_{i[k} F_{\ell]j} + g_{j[\ell} F_{k]i}. \end{aligned}$$

where

$$\begin{aligned} L_{ij} =& b_i b_j - u_i u_j - \frac{1}{3} g_{ij}, \\ F_{ij} =& \frac{1}{3}(R - 4\xi + 12C) u_i u_j + \frac{1}{6}(R - 2\xi + 12C) g_{ij} \\ & + v(u_i b_j + u_j b_i) - 6C b_i b_j. \end{aligned}$$

### 3.1.2 Two perfect fluid picture

Using the EFE for a twisted spacetime with dimension $n = 4$ that admits two distinct timelike unit torse-forming vectors, the stress-energy tensor can be expressed in terms of geometrical quantities $R$, $C$, $\varphi$, the vectors $u_i$, $b_i$ and the metric tensor $g_{ij}$ as (in units that absorb the constant $\kappa$)

$$\begin{aligned} T_{ij} = R_{ij} - \frac{1}{2} R g_{ij} =& \frac{R-4\xi}{3} u_i u_j + \frac{-R-2\xi}{6} g_{ij} \\ & + 2v(u_i b_j + u_j b_j) - 3C\left(b_i b_j - \frac{1}{3} h_{ij}\right). \end{aligned} \tag{3.1.31}$$

The expression of $T_{ij}$ is formally equivalent to (1.2.5) for a two perfect fluid mixture. In the same way of subsection 1.2.3, let us suppose that there is a mixture of two perfect fluid. The first one has a flow that is a shear-free, irrotational and geodesic timelike congruence with velocity field $u_i$, while the second fluid has a velocity $z_i$ tilted with respect to $u_i$ (in general it is not the second torse-forming vector $w_i$). In comoving coordinates, $u_i = (-1, 0, 0, 0)$, the velocity vector field of the tilting perfect fluid and the heat flux $q_i$ are given by

$$\begin{aligned} z_i =& u_i \cosh\psi + t_i \sinh\psi = (-\cosh\psi, \sinh\psi t_\alpha), \\ q_i =& (0, q t_\alpha), \end{aligned}$$

where $t_i$ is a unit spacelike vector such that $u^i t_i = 0$ and, at the moment, $t_i \neq b_i$. The stress-energy tensor is

$$T'_{ij} = (p+\mu)u_i u_j + p g_{ij} + q(u_i t_j + u_j t_i) + \pi \left( t_i t_j - \frac{1}{3} h_{ij} \right) \tag{3.1.32}$$

and the request $T_{ij} = T'_{ij}$ implies that $b_i$ must coincide with $t_i$ and $2v = q$, since the contractions with $u^j$ and the projections of (3.1.31) and (3.1.32) on the orthogonal space of $u_i$ give ${h_k}^i(T_{ij}u^j) = -2vb_i$ and ${h_k}^i(T'_{ij}u^j) = -qt_i$.

The equations (1.2.6), (1.2.7), (1.2.8) and (1.2.9) give the five parameters $\mu_1$, $p_1$, $\mu_2$, $p_2$ and $\psi$, that can be also read in terms of $R$, $\xi$, $C$ and $v$, then

$$\begin{cases} \mu_1 + \mu_2 = \frac{R}{2} - \xi + 3C \\ p_1 + p_2 = -\frac{R}{6} - \frac{\xi}{3} + C \\ \tanh\psi = -\frac{3C}{2v} \\ \mu_2 + p_2 = -\frac{4v^2 - 9C^2}{3C} \end{cases}$$

where we used $\sinh\psi = \frac{\tanh\psi}{\sqrt{1-\tanh^2\psi}} = \frac{-3C/2v}{\sqrt{1-9C^2/4v^2}}$. With the aid of

$$\mu_1 + p_1 = (\mu_1+\mu_2) + (p_2+p_1) - (\mu_2+p_2) = \frac{R-4\xi}{3} + C + \frac{4v^2}{3C},$$

the expression for $T_{ij}$ as a sum of two stress energy perfect fluid is given:

$$T_{ij} = \left( \frac{R-4\xi}{3} + C + \frac{4v^2}{3C} \right) u_i u_j + \left( -\frac{R}{6} - \frac{\xi}{3} + C \right) g_{ij} + \frac{9C^2 - 4v^2}{3C} z_i z_j,$$

where

$$z_i = \frac{1}{\sqrt{1-9C^2/4v^2}}(u_i - \frac{3c}{2v} b_i).$$

In general $z_i \neq w_i$, but if we impose the condition $\psi = \alpha$, the expression of $\tanh\psi$ and the equation (3.1.14) give the following restriction

$$4v = -\frac{3C}{2v}\left( \frac{R-4\xi}{3} - 2C \right).$$

In terms of $p_1$, $p_2$, $\mu_1$ and $\mu_2$, it assumes the form

$$p_1 + \mu_1 = p_2 + \mu_2,$$

but the four parameters remain undefined, since the system of four equations is indeterminate. No further progress can be made until physical conditions on $\mu_1$, $\mu_2$, $p_1$ $p_2$ and $\psi$ are specified. If one such condition is specified, the remaining four physical quantities can then in principle be expressed on terms of $f$ and its derivatives. Two conditions on $\mu_1$, $\mu_2$, $p_1$ $p_2$ and $\psi$ would then give rise to a differential equation in terms of $f$ that would need to be satisfied, i.e., a further restriction on the form of the metric.

### 3.1.3 The unique torse-forming vector case

In [13], the authors show that in a twisted spacetime with dimension $n = 4$, the 3-spatial submanifold can assume a doubly twisted form using the EFE with a peculiar stress-energy tensor, i.e., a two component perfect fluids (1.2.5). The same discussion gives the same result without the use of the Einstein's equations, but imposing the form (3.1.28) for the electric part of the Weyl tensor. The relevant equations used for the proof are:

i. The projections of the Bianchi identities (conservation equations) along the vector $u$ and on the orthogonal subspace to $u$, $u^j\nabla^i G_{ij} = 0$ and $h_k{}^j\nabla^i G_{ij} = 0$:

$$\frac{\dot{R}}{2} - \dot{\xi} + (R - 4\xi)\varphi + 2\nabla^k v_k = 0,$$
$$h_i{}^j\left(-\frac{1}{6}\nabla_j R - \frac{1}{3}\nabla_j\xi - 2\nabla^k C_{jk} + 2\dot{v}_j + 8\varphi v_j\right) = 0,$$

where $\xi$ is given in terms of $\varphi$ by the Raychaudhuri equation

$$\xi = 3(\varphi^2 + \dot{\varphi}).$$

ii. A constraint and a propagation equation for $C_{ij}$, given by the Bianchi identities on expressing the Riemann tensor in terms of the Weyl and Ricci tensors

$$2\nabla^j C_{ij} = 2\varphi v_i + \frac{1}{3}h_i{}^j\nabla_j\xi - \frac{1}{6}h_i{}^j\nabla_j R, \tag{3.1.33}$$
$$u^k\nabla_k C_{ij} + 2\varphi C_{ij} = \frac{1}{2}h_i{}^k h_j{}^\ell\nabla_{(l}v_{k)} - \frac{1}{6}(\nabla^k v_k)h_{ij}. \tag{3.1.34}$$

Since $u_i$ is a timelike unit torse-forming vector, there exists a privileged coordinate system such that the metric takes the form (2.1.1) and $\varphi = \partial_t(\ln f)$. The conservation equations become

$$\frac{\dot{R}}{2} - \dot{\xi} + (R - 4\xi)\partial_t(\ln f) + 2g^{\alpha\beta}\nabla_\beta v_\alpha = 0,$$
$$2\partial_t v_\alpha + 6\partial_t(\ln f)v_\alpha - \frac{1}{3}\nabla_\alpha\xi - \frac{1}{6}\nabla_\alpha R - 2g^{\beta\gamma}\nabla_\gamma C_{\alpha\beta} = 0,$$

then, in terms of the 3-dimensional metric $g^*$

$$\frac{\dot{R}}{2} - \dot{\xi} + (R - 4\xi)\partial_t(\ln f) + 2f^{-2}[\nabla^{*\alpha}v_\alpha + v_\alpha\nabla^{*\alpha}(\ln f)] = 0, \tag{3.1.35}$$

$$\begin{aligned} 2\partial_t v_\alpha + 6\partial_t(\ln f)v_\alpha - \frac{1}{3}\nabla_\alpha\xi - \frac{1}{6}\nabla_\alpha R \\ -2f^{-2}[\nabla^{*\beta}C_{\alpha\beta} + C_{\alpha\beta}\nabla^{*\beta}(\ln f)] =0. \end{aligned} \tag{3.1.36}$$

Finally, equations (3.1.33) and (3.1.34) reduce to

$$2f^{-2}\nabla^{*\beta}C_{\alpha\beta} + 2f^{-3}(\nabla^{*\beta} f)C_{\alpha\beta} = 2\varphi v_\alpha + \frac{1}{3}\nabla^*_\alpha\xi - \frac{1}{6}\nabla^*_\alpha R, \tag{3.1.37}$$

$$2\partial_t C_{\alpha\beta} = \nabla^*_\alpha v_\beta - 2v_{(\alpha}\nabla^*_{\beta)}(\ln f) - \frac{1}{3}f^2\nabla^{*\gamma}(f^{-2}v_\gamma)g^*_{\alpha\beta}, \tag{3.1.38}$$

where we used $\nabla^*_\beta v_\alpha = \nabla^*_\beta\nabla^*_\alpha\varphi = \nabla^*_\alpha v_\beta$.

If $C_{ij}$ is given by (3.1.28), its non-zero components are the spatial components

$$2C_{\alpha\beta} = \frac{3C}{v^2}\left[v_\alpha v_\beta - \frac{1}{3}v^2 f^2 g^*_{\alpha\beta}\right], \tag{3.1.39}$$

Inserting the expression (3.1.39) into the left hand-side of (3.1.38) and then using (3.1.36) to get an expression for $\partial_t v_\alpha$, we obtain

$$\nabla^*_\beta v_\alpha = A_\alpha v_\beta + A_\beta v_\alpha + B v_\alpha v_\beta + C g^*_{\alpha\beta}$$

(the precise expressions for $A_\alpha$, $B$ and $C$ are unnecessary for the remainder of the discussion). In terms of the unit spacelike vector

$$n^*_\alpha = \frac{v_\alpha}{\sqrt{g^{*\beta\delta}v_\beta v_\delta}},$$

the expression for $\nabla^*_\beta v_\alpha$ gives condition (3.1.24). Then, we have the following theorem:

**Theorem 3.1.3.** *On a twisted spacetime with dimension $n = 4$, if* (3.1.28) *holds, the space submanifold $(M^*, g^*)$ admits a unit vector $n^*_\mu(\vec{x})$ that satisfies* (3.1.24) *and the metric assumes the form*

$$ds^2 = -dt^2 + f(t,x)^2[f^2(\vec{x})^2dx^2 + f_2(\vec{x})^2\gamma_{AB}(x^C)dx^A dx^B].$$

**Remark 3.1.6.** *From the theorem by A. Korn and L. Lichtenstein, any 2-dimensional spacetime is conformally flat, then $\gamma_{AB}(x^C)dx^A dx^B$ is proportional to $dy^2 + dz^2$ (the conformal map can be reabsorbed in $f_2$).*

The spatial components of eq. (2.1.10) give the expression for the electric part of the Weyl tensor in terms of the spatial geometry and the twisting function $f$ as

$$\begin{aligned}
-2C_{\mu\nu} =& R_{\mu\nu} - \frac{R-\xi}{3}g_{\mu\nu} \\
=& R^*_{\mu\nu} - \frac{1}{3}R^* g^*_{\mu\nu} - \nabla^*_\mu\nabla^*_\nu(\ln f) + \nabla^*_\mu(\ln f)\nabla^*_\nu(\ln f) \\
&+ \frac{1}{3}[\nabla^{*\sigma}\nabla^*_\sigma(\ln f) - \nabla^{*\sigma}(\ln f)\nabla^*_\sigma(\ln f)]g^*_{\mu\nu}.
\end{aligned}$$

Since $b_\mu = fn^*_\mu = f(0, f_1, 0, 0)$ we have $C_{xA} = 0$, then

$$\begin{aligned} R^*_{xA} =& \nabla^*_x \nabla^*_A (\ln f) = \partial_x \partial_A (\ln f) - \Gamma^{*\rho}{}_{xA} \partial_\rho (\ln f) \\ =& - \partial_x (\ln f) \partial_A (\ln f_1). \end{aligned}$$

The term $\partial_x(\ln f)$ is a function of both $t$ and $x$ (otherwise $q_\mu = \partial_\mu \partial_t (\ln f)$ is identically zero), then it must hold $R^*_{xA} = 0$ and we can take $f_1(\vec{x}) = 1$ without loss of generality. The spatial submanifold $(M^*, g^*)$ takes the form of a twisted spacetime, i.e.,

$$ds^2 = -dt^2 + f(t,x)^2 [dx^2 + f_2(\vec{x})^2 (dy^2 + dz^2)].$$

The expression of $R^*_{xA}$ for the previous metric is given by

$$0 = R^*_{xA} = f_2^{-2} \partial_x f_2 \partial_A f_2 - f_2^{-1} \partial_{xA} f_2,$$

with solution $f_2 = \chi(x)\phi(y,z)$.

## 3.2 Unicity in GRW spacetimes

On a GRW manifold, the unicity of the timelike unit torse-forming vector is guaranteed by the special property $\nabla_i \varphi = -u_i \dot{\varphi}$, but for the particular case where the eigenvalue of the Ricci tensor $\xi$ is constant. We examine in a distinct way the two different situations. As in the twisted case, let us suppose the existence of another timelike unit vector $w_k$ that satisfies the torse-forming property (3.1.1). A first answer is provided by the following theorem:

**Theorem 3.2.1.** *On a GRW spacetime with non-constant eigenvalue $\xi$, the vector field $u_k$ is unique (up to reflection).*

*Proof.* Eq. (3.1.11), with the aid of $\nabla_i \varphi = -u_i \dot{\varphi}$, simplifies to

$$w_k [\lambda' + u^i w_i \dot{\lambda}] = [(u^i w_i)^2 - 1] \nabla_k \lambda,$$

showing that $\nabla_k \lambda$ is collinear with $w_k$: $\nabla_k \lambda = -w_k \lambda'$. Equations (3.1.4) and (3.1.5) infer that $u_k$ and $w_k$ are eigenvectors of the Ricci tensor

$$\begin{aligned} R_{jm} u^m &= (n-1)(\varphi^2 u_j - \nabla_j \varphi) = (n-1)(\varphi^2 + \dot{\varphi}) u_j, \\ R_{jm} w^m &= (n-1)(\lambda^2 w_j - \nabla_j \lambda) = (n-1)(\lambda^2 + \lambda') w_j, \end{aligned}$$

with the same eigenvalue $\xi$, by means of (3.1.10). Let's evaluate:

$$\begin{aligned} \nabla_k \dot{\varphi} &= \varphi (u_k u^m + \delta_k{}^m) \nabla_m \varphi + u^m \nabla_m \nabla_k \varphi \\ &= \varphi (u_k u^m + \delta_k{}^m)(-u_m \dot{\varphi}) + u^m \nabla_m (-u_k \dot{\varphi}) \\ &= -u_k [u^m \nabla_m \dot{\varphi}] = -u_k \ddot{\varphi}, \end{aligned}$$

then the derivative of $\xi$ is parallel to $u$:

$$\nabla_k \xi = (n-1)\nabla_k(\varphi^2 + \dot{\varphi}) = (n-1)(2\varphi\nabla_k\varphi - u_k\ddot{\varphi}) = -u_k\dot{\xi},$$

where $\dot{\xi} = (n-1)(2\varphi\dot{\varphi} + \ddot{\varphi})$. A similar result holds for $w$: $\nabla_k\xi = -w_k\xi'$. From $\nabla_k\xi = -w_k w^j(-u_j\dot{\xi})$ the contraction with $u^k$ gives: $[1-(u^j w_j)^2]\dot{\xi} = 0$, i.e., $u_j$ and $w_j$ are collinear if $\xi$ is not constant. □

Let us investigate the case with $\xi = const$.

**Proposition 3.2.1.** *On a GRW spacetime, for the eigenvalue of the Ricci tensor to be degenerate, it is necessary that $\varphi(t) = ct+k$, with constants c and k. If the eigenvalue $\xi$ is constant, it must hold $\varphi = k$.*

*Proof.* The eigenvalue equations $R_{ij}w^j = \xi w_i$ in the warped frame are $R_{00}w^0 = \xi w_0$ and $R_{\mu\nu}w^\nu = \xi w_\mu$. The first equation is $-(n-1)(\ddot{f}/f)w^0 = \xi w_0$, then the (2.2.5) for any $w_0$. The second equation, by $R_{\mu\nu} = R^*_{\mu\nu} + g^*_{\mu\nu}[(n-2)\dot{f}^2 + \ddot{f}f]$, has always the solution $w^i = u^i$. Other solutions have non-zero space components $w^\mu$ solving the eigenvalue equations

$$R^*_{\mu\nu}w^\nu = \xi w_\mu - g^*_{\mu\nu}[(n-2)\dot{f}^2 + \ddot{f}f]w^\nu = (n-2)\frac{d\varphi}{dt}w_\mu,$$

where we lowered an index: $f^2 g^*_{\mu\nu}w^\nu = w_\mu$. In the warped frame, the Ricci tensor $R^*_{\mu\nu}$ of $(M^*, g^*)$ does not depend on time, and so must the eigenvalue. Then $\varphi = At + B$ where $A$ and $B$ do not depend on space coordinates, as the warping function does not.

If the eigenvalue $\xi = (n-1)[(ct+k)^2 + c]$ is a constant, it must hold $c = 0$, i.e., $\varphi = k$, with solution $f(t) = A\exp(kt)$, where $A$ is constant. □

**Theorem 3.2.2.** *On a GRW spacetime with constant $\varphi$ the torse-forming velocity is unique unless $(M^*, g^*)$ is a warped submanilfold, i.e., $(M^*, g^*)$ admits a unit vector field $n^*_\mu(\vec{x})$ such that $\nabla^*_\mu n^*_\nu = \vartheta(x)(g^*_{\mu\nu} - n^*_\mu n^*_\nu)$ with $\nabla^*_\mu\vartheta$ proportional to $n^*_\mu$.*

*Proof.* Suppose that $w_j$ exists and that it is not collinear with $u_j$. Then, there is a reference frame where $w^0 = 1$ and $w^\mu = 0$, and the scale factor $\tilde{f}(t)$, with $\lambda = \dot{\tilde{f}}/\tilde{f}$. In this frame, by the previous discussion, degeneracy of $\xi$ requires $\lambda$ to be constant. From (3.1.10), we obtain $\lambda = \varphi = k$. The torse-forming condition for $w_j$, $\nabla_i w_j = \lambda(g_{ij} + w_i w_j)$, in the warped frame may be specified by using the Christoffel symbol listed in Appendix B.2

$$\begin{aligned}
\partial_t w_0 &= k(w_0^2 - 1) \\
\partial_\mu w_0 &= k w_\mu (w_0 + 1) \\
\partial_t w_\mu &= k w_\mu (w_0 + 1) \\
\partial_\mu w_\nu - \Gamma^{*\rho}_{\mu\nu} w_\rho &= k(w_\mu w_\nu + f^2 (w_0 + 1) g^*_{\mu\nu})
\end{aligned}$$

Excluding the case $k = 0$, we can set $f(t) = \exp(kt)/k$. The first three equations are solved by

$$w_0(x,t) = \frac{1 + d^2(x)\exp(2kt)}{1 - d^2(x)\exp(2kt)}, \qquad w_\mu(x,t) = \frac{1}{k}\frac{(\partial_\mu d^2)\exp(2kt)}{1 - d^2(x)\exp(2kt)}, \quad (3.2.1)$$

where the function $d(x)$ is determined by the last differential equation

$$\partial_\mu(\partial_\nu d^2(x)) - \Gamma^{*\rho}_{\mu\nu}(\partial_\rho d^2(x)) = 2g^*_{\mu\nu}(x), \quad i.e., \quad \nabla^*_\mu(\partial_\nu d^2) = 2g^*_{\mu\nu}. \quad (3.2.2)$$

The normalization condition $-1 = -w_0^2 + f(t)^{-2} g^{*\mu\nu} w_\mu w_\nu$ gives

$$4d^2 = g^{*\mu\nu}(\partial_\mu d^2)(\partial_\nu d^2).$$

If we put $\partial_\mu d = n^*_\mu$, then $g^{*\mu\nu} n^*_\mu n^*_\nu = 1$ and the concircular condition (3.2.2) for $\partial_\mu d^2$ show that the 3-vector $n^*_\mu$ is a spacelike unit torse-forming vector on $(M^*, g^*)$, i.e.,

$$\nabla^*_\mu n^*_\nu = \frac{1}{d}(g^*_{\mu\nu} - n^*_\mu n^*_\nu), \quad (3.2.3)$$

Furthermore, since the scalar function $1/d(x)$ has derivative parallel to $n^*_\mu$: $\nabla^*_\mu(1/d) = \partial_\mu(1/d) = -n^*_\mu/d^2$, the submanifold $M^*$ can be written as a warped product, i.e., there is a choice of space coordinates $(x, x^A)$, with $A = 2, \ldots, n-1$, such that the metric takes the form

$$ds^2 = -dt^2 + \frac{1}{k^2} e^{2kt}(dx^2 + f_2^2 \gamma^*_{AB} dx^A dx^B),$$

where $f_2 = f_2(x)$ and $\gamma_{AB} = \gamma_{AB}(x^C)$ is the spatial submanifold metric with dimension $n - 2$. □

**Remark 3.2.1.** *For $d(x) = 0$ equation (3.2.1) gives $w_k = u_k$, thus the vector is unique. For $k = 0$ the warping function $f$ is constant and the manifold $(M, g)$ is a trivial disjointed product between the time interval $I$ and the spatial submanifold $M^*$, i.e., $M = I \times M^*$.*

This concludes the discussion about the unicity of the torse-forming vector on GRW spacetimes. It can be phrased as follows:

**Theorem 3.2.3.** *On a GRW spacetime the timelike unit torse-forming vector field $u^i$ is unique (up to reflection) unless the eigenvalue $\xi$ associated to the Ricci tensor is constant. Otherwise, there exists a second vector $w^i$ and the submanifold $(M^*, g^*)$ can be written as a warped product.*

# Appendix A

# A brief reminder of differential geometry

We report a short reminder of differential geometry starting from the definition of manifolds.

## A.1 Manifolds

**Definition A.1.1.** *A $n$-dimensional, $C^\infty$, real manifold $M$ is a set together with a collection of subsets $\{O_\alpha\}$ satisfying the following properties:*

*i. each $p \in M$ lies in at least one , i.e., the $\{O_\alpha\}$ covers $M$,*

*ii. $\forall\alpha$, there is a one-to-one map $\varphi_\alpha : O_\alpha \to U_\alpha$, where $U_\alpha$ is an open subset of $\mathbb{R}^n$,*

*iii. if any two sets $O_\alpha \cap O_\beta \neq \emptyset$, the maps $\varphi_\alpha$ and $\varphi_\beta$ satisfy the compatibility condition, i.e., the map $\varphi_\beta \circ \varphi_\alpha^{-1}$: $\varphi_\alpha(O_\alpha \cap O_\beta) \to \varphi_\beta(O_\alpha \cap O_\beta)$ is infinitely continuously differentiable.*

Given two manifolds $M$ and $N$ of dimension $m$ and $n$, respectively, the product space $M \times N$ consists of all pairs $(p, q)$ with $p \in M$ and $q \in N$ into an $(n+m)$-dimensional manifold as it follows. If $\varphi_\alpha : O_\alpha \to U_\alpha$ and $\psi_\beta : O'_\beta \to U'_\beta$ are charts, we define a chart $\phi_{\alpha\beta} : O_{\alpha\beta} \to U_{\alpha\beta} \subset \mathbb{R}^{n+m}$ on $M \times N$ by taking $O_{\alpha\beta} = O_\alpha \times O'_\beta$, $U_{\alpha\beta} = U_\alpha \times U'_\beta$ and setting $\phi_{\alpha\beta}(p, q) = [\varphi_\alpha(p), \psi_\beta(q)]$. It can be easily checked that the chart family $\{\phi_{\alpha\beta}\}$ satisfies (ii) and (iii).

A map $f : M \to N$, between the two manifold $M$ and $N$, is said to be $C^\infty$ if for each $\alpha$ and $\beta$, the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ taking $U_\alpha \subset \mathbb{R}^n$ into $U'_\beta \subset \mathbb{R}^m$ is

$C^\infty$. If $f : M \to N$ is $C^\infty$, one-to-one and it has $C^\infty$ inverse, $f$ is called a *diffeomorphism*.

## A.2 Tangent space

On a manifold $M$, let $\mathcal{F}$ denote the collection of $C^\infty$ functions from $M$ into $\mathbb{R}$. We define a vector on $M$ as the natural generalization of a vector on $\mathbb{R}^n$:

**Definition A.2.1.** *A tangent vector $v$ at point $p \in M$ is the map $v_p : \mathcal{F} \to \mathbb{R}$ such that:*

*i.* $v_p(af + bg) = av_p(f) + bv_p(g)$, $\forall f, g \in \mathcal{F}$, $a, b \in \mathbb{R}$,

*ii.* $v_p(fg) = f(p)v_p(g) + g(p)v_p(f)$.

The two properties of tangent vectors imply that if $f \in \mathcal{F}$ is a constant function, then $v_p(f) = 0$. The collection of tangent vectors at $p \in M$ is denoted as $T_pM$ and has the structure of a vector space under the addition law $(v_p + w_p)(f) = v_p(f) + w_p(f)$ and the scalar multiplication law $(av_p)(f) = av_p(f)$. Moreover, it can be proved that $\dim(T_pM) = \dim(M)$. Given the basis $\{e_i\}$ of $T_pM$, named *coordinate basis*, the vector $v$ is given by $v = v^i e_i$ where $v^i = v(x^i \circ \varphi)$ are the components of $v$ respect to the basis $\{e_i\}$, $\varphi$ a coordinate chart and $x^i$ the coordinates of $p$ through $\varphi$. Frequently the basis $e_i = \partial/\partial x^i$ is used.

An equivalent definition of a tangent vector $v$ is the triple $(p, \varphi, v)$, where $\varphi$ is a chart, such that $(p, \varphi, v)$ and $(p, \varphi', v')$ represent the same vector if $v'$ can be given by $v$ as

$$v'^j = \frac{\partial x'^j}{\partial x^i} v^i.$$

Finally, a tangent vector can be defined through a smooth curve $\gamma$ on a manifold $M$, that is a $C^\infty$ map of $I \subset \mathbb{R}$ into $M$. At each point $p \in \gamma$, we can associate a tangent vector $v \in T_pM$ such that, for $f \in \mathcal{F}$, we set $v(f) = d(f \circ \varphi)/dt$, where $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is evaluated at $p = \gamma(0)$. Then

$$v(f) = \frac{d}{dt}(f \circ \gamma) = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})\frac{dx^i}{dt} = \frac{dx^i}{dt} e_i(f).$$

The components $v^i$ of the tangent vector $v$ to the curve are given by

$$v^i = \frac{dx^i}{dt}.$$

A tangent field, $v$, on a manifold $M$ is an assignment of a tangent vector, $v|_p \in T_pM$, at each point $p \in M$. If $f$ is a smooth function, then at each $p \in M$, $v|_p(f)$ is a number, i.e., $v(f)$ is a function on $M$. The tangent field $v$ is said to be smooth if for each smooth function $f$, the function $v(f)$ is also smooth, i.e., if its coordinates basis components $v^i$ are smooth functions.

Let us introduce the notion of *one-parameter group of diffeomorphisms* on a manifold $M$ as:

**Definition A.2.2.** *A one-parameter group of diffeomorphisms $\phi_t : \mathbb{R} \times M \to M$ is a $C^\infty$ map such that for fixed $t \in \mathbb{R}$, $\phi_t : M \to M$ is a diffeomorphism and for $t, s \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$.*

In this way, for fixed $p \in M$, $\phi_t(p) : \mathbb{R} \to M$ is a curve, called an *orbit* of $\phi_t$, which passes through $p$ at $t = 0$ and we can define $v_p$ as the tangent vector to this curve at $t = 0$.

Conversely, given a smooth vector field, $v$, on $M$ it is possible to find integral curves of $v$ that correspond to a family of curves in $M$ having the property that one and only one curve passes through each point $p \in M$ and the tangent to this curve at $p$ is $v|_p$: if we choose a coordinate system in a neighborhood of $p$, we see that the problem of finding such curves reduces to solving the system of ordinary differential equations in $\mathbb{R}^n$

$$\frac{dx^i}{dt} = v^i(x^1, \dots, x^n),$$

where $v^i$ is the $i$-th component of $v$ in the coordinate basis $\{\partial/\partial x^i\}$. Such a system of equations has a unique solution given a starting point at $t = 0$, thus, every smooth vector field $v$ has a unique family of integral curves. Given the integral curves, for each $p \in M$ we define $\phi_t(p)$ to be the point lying at parameter $t$ along the integral curve of $v$ starting at $p$. Then, $\phi_t$ will be a one-parameter group of diffeomorphisms.

## A.3 Dual form vectors and tensors

We give now the definition of dual vector space and dual form vectors:

**Definition A.3.1.** *The dual vector space $T_p^*M$ of $T_pM$, is the collection of linear maps $\omega_p : T_pM \to \mathbb{R}$, such that $v_p \mapsto w_p(v_p) \in \mathbb{R}$. $T_p^*M$ is again a vector space, i.e., addition and scalar multiplication of such linear maps are defined in the obvious way.*

The space $T_p^*M$ is named *cotangent space* and elements of $T_p^*M$ are called *covariant vectors* or *dual form vectors*, while vectors of $T_pM$ are called *contravariant vectors*. If one defines addition and scalar multiplication of such linear maps in the obvious way, one gets a natural vector space structure on $T_p^*M$. If $\{e_i\}$ is a basis of $T_pM$, it is natural to define elements $\{\vartheta_i\}$ of $T_p^*M$ as

$$\vartheta^i(e_j) = \delta^i{}_j.$$

It follows that $\{\vartheta_i\}$ is a basis of $T_p^*M$, called *dual basis* to the basis $\{e_i\}$ of $T_pM$. In particular, given the coordinate basis $\{\partial/\partial x^i\}$ of $T_pM$, the associated dual basis of $T_p^*M$ is usually denoted as $\{dx^i\}$. This shows that $\dim(T_p^*M) = \dim(T_pM)$. Under change of basis from $\{dx^i\}$ to $\{dx'^i\}$, the components $\omega_i$ of $\omega \in T_p^*M$ become

$$\omega'_j = \frac{\partial x^i}{\partial x'^j}\omega_i.$$

After defining vectors and dual form vectors over $T_pM$, we define a general tensors as:

**Definition A.3.2.** *A tensor $T$ of type $(k, \ell)$ over $T_pM$ is the multilinear map*

$$T : \underbrace{T_p^*M \times \ldots \times T_p^*M}_{k\ times} \times \underbrace{T_pM \ldots T_pM}_{\ell\ times} \to \mathbb{R}.$$

With the obvious rules for adding and scalar multiplying maps, the space of tensors of type $(k, \ell)$, denoted as $\mathcal{F}(k, \ell)$, has the structure of a vector space with dimension $n^{k+\ell}$. Given two tensors $T$ and $T'$ of type $(k, \ell)$ and $(k', \ell')$ respectively, the *outer product* of $T$ and $T'$, denoted by $T \otimes T'$, is a tensor of type $(k + k', \ell + \ell')$ defined as

$$\begin{aligned}(T \otimes T')(\omega^{(1)}, \ldots, v_{(\ell+\ell')}) =& T(\omega^{(1)}, \ldots, \omega^{(k)}, v_{(1)}, \ldots, v_{(\ell)}) \\ &\cdot T'(\omega^{(k+1)}, \ldots, \omega^{(k+k')}, v_{(\ell+1)}, \ldots, v_{(\ell+\ell')}),\end{aligned}$$

where $\{\omega^{(i)}\}$ are $k+k'$ dual vectors $\in T_p^*M$ and $\{v_{(i)}\}$ are $\ell+\ell'$ vectors $\in T_pM$. If $\{e_i\}$ is a basis of $T_pM$ and $\{\vartheta_i\}$ is its dual basis, it is easy to show that the $n^{k+\ell}$ simple tensors $\{e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes \vartheta^{j_1} \otimes \ldots \otimes \vartheta^{j_\ell}\}$ yield a basis of $\mathcal{F}(k, \ell)$. A tensor $T$ of type $(k, \ell)$ can be expressed as sum of simple tensors

$$T = T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell} e_1 \otimes \ldots e_k \otimes \vartheta^1 \otimes \ldots \otimes \vartheta^\ell,$$

with basis expansion components $T^{i_1 \ldots i_k}{}_{j_1 \ldots j_\ell}$, named components of the tensor $T$ with respect to the basis $\{e_i\}$.

Another important operation on tensors is the contraction $C : \mathcal{F}(k,\ell) \to \mathcal{F}(k-1,\ell-1)$ defined as

$$CT = T(\dots,\vartheta^i,\dots,e_i,\dots),$$

where $\{e_i\}$ is a basis of $T_pM$ and $\{\vartheta^i\}$ is its dual basis. Given a tensor $T \in \mathcal{F}(k,\ell)$ with components $T^{i_1\dots i_k}{}_{j_1\dots j_\ell}$, the contraction $CT$ has components

$$(CT)^{i_1\dots i_{k-1}}{}_{j_1\dots j_{\ell-1}} = T^{i_1\dots m\dots i_{k-1}}{}_{j_1\dots m\dots j_{\ell-1}}.$$

If $T' \in \mathcal{F}(k',\ell')$ has components $T^{i_1\dots i_{k'}}{}_{j_1\dots j_{\ell'}}$, the components of $P = T \otimes T'$ are given by

$$P^{i_1\dots i_{k+k'}}{}_{j_i\dots j_{\ell+\ell'}} = T^{i_1\dots i_k}{}_{j_1\dots j_\ell} T^{i_k\dots i_{k+k'}}{}_{j_\ell\dots j_{\ell+\ell'}}.$$

Finally, given the basis $\{\partial/\partial x^i\}$ and its dual basis $\{dx_i\}$ of $T_pM$ and $T_p^*M$ respectively, the components $T^{i_1\dots i_k}{}_{j_1\dots j_\ell}$ of a tensor $T \in \mathcal{F}(k,\ell)$ transform as

$$T'^{i'_1\dots i'_k}{}_{j'_1\dots j'_\ell} = T^{i_1\dots i_k}{}_{j_1\dots j_\ell} \frac{\partial x'^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{i'_k}}{\partial x^{i_k}} \frac{\partial x^{j_1}}{\partial x'^{j'_1}} \cdots \frac{\partial x^{j_\ell}}{\partial x'^{j'_\ell}},$$

that defines the *tensor transformation law.*

## A.4 Pseudo-Riemannian manifolds

**Definition A.4.1.** *A pseudo-Riemannian manifold* $(M,g)$ *is a differentiable manifold* $M$ *equipped with an everywhere non-degenerate, smooth, symmetric tensorial field* $g$ *of type* $(0,2)$*, named metric tensor:*

*i.* $g_{ij} = g_{ji}$,

*ii. for* $p \in M$ *if* $g_p(v,w) = 0$ $\forall v \in T_pM$ *then* $w = 0$.

In particular, if $g_p(v,v) > 0$ $\forall v \in T_pM$ the manifold is named Riemaniann manifold, otherwise pseudo-Riemannian manifold. Given a metric tensor $g$ on an $n$-dimensional manifold $M$, the quadratic form $q(v) = g_p(v,v)$ applied to each vector $v$ of any orthogonal basis produces $n$ real values. The signature of the metric tensor $(p,q,r)$ is the number of each positive, negative and zero values produced, that are invariants of the metric tensor, independent of the choice of the orthogonal basis. A non-degenerate metric tensor has $r = 0$ and the signature is denoted by $(p,q)$.

**Definition A.4.2.** *A Lorentzian manifold* $(M,g)$ *is a pseudo-Riemannian manifold with signature of the metric tensor* $(n-1,1)$.

Given the basis $\{dx^i\}$ of $T_p^*M$, we can define

$$ds^2 \equiv g = g_{ij}dx^i \otimes dx^j \equiv g_{ij}dx^i dx^j,$$

where $dx^i dx^j$ is the symmetrized product. The metric tensor $g$ establishes a scalar product in $T_pM$ as $(v|w) = g_p(v,w)$ for $v, w \in T_pM$. If $e_i$ is the dual base of $\{\vartheta^i\}$, the previous expression becomes

$$g_p(v,w) = g_p(v^i e_i, w^j e_j) = g_{ij}v^i w^j,$$

where $g_{ij}$ denoted $g_p(e_i, e_j)$, since a metric $g$ is a tensor of type (0,2). If we apply the metric to the vector $v^i$, we get the dual vector $g_{ij}v^j$. It is convenient to denote this vector as simply $v_i$, thus making notationally explicit the isomorphism between $T_pM$ and $T_p^*M$ defined by $g_{ij}$. The inverse of $g_{ij}$ is a tensor of type (2,0) and is denoted as $g_{ij}$. If we apply the inverse metric to a dual vector $\omega_i$, we denote the resultant vector $g^{ij}\omega_i$ as simply $\omega^j$. In general, raised or lowered indices on a general tensor denote application of the metric or inverse metric.

## A.5 Derivative operators

**Definition A.5.1.** *A covariant derivative $\nabla$ on a manifold $M$ is a map which takes each smooth tensor field of type $(k,\ell)$ to a smooth tensor field of type $(k, \ell+1)$ such that:*

*i. linearity: $\forall A, B$ of type $(k,\ell)$ and $\alpha, \beta \in \mathbb{R}$,*

$$\nabla_c(\alpha A^{i_1\dots i_k}{}_{j_1\dots j_\ell} + \beta B^{i_1\dots i_k}{}_{j_1\dots j_\ell}) = \alpha\nabla_c(A^{i_1\dots i_k}{}_{j_1\dots j_\ell}) + \beta\nabla_c B^{i_1\dots i_k}{}_{j_1\dots j_\ell}),$$

*ii. Leibniz rule: $\forall A$ of type $(k,\ell)$ and $B$ of type $(k',\ell')$,*

$$\begin{aligned}\nabla_c(A^{i_1\dots i_k}{}_{j_1\dots j_\ell} B^{r_1\dots r'_k}{}_{s_1\dots s'_\ell}) =&(\nabla_c A^{i_1\dots i_k}{}_{j_1\dots j_\ell})B^{r_1\dots r'_k}{}_{s_1\dots s'_\ell} \\ &+ A^{i_1\dots i_k}{}_{j_1\dots j_\ell}(\nabla_c B^{r_1\dots r'_k}{}_{s_1\dots s'_\ell}),\end{aligned}$$

*iii. commutativity with contraction:*

$$\nabla_c(A^{i_1\dots m\dots i_k}{}_{j_1\dots m\dots j_\ell}) = \nabla_c A^{i_1\dots m\dots i_k}{}_{j_1\dots m\dots j_\ell},$$

*iv. consistency with the notion of tangent vectors as directional derivatives on scalar fields: $\forall f : M \to \mathbb{R}$ and all $v \in T_pM$,*

$$v(f) = v^i \nabla_i f,$$

*v. torsion free:* $\forall f : M \to \mathbb{R}$,

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f.$$

The derivative operator is not unique, in fact, as shown in [35], any two derivative operators $\nabla$ and $\tilde{\nabla}$ are related by

$$\nabla_i w_j = \tilde{\nabla}_i w_j - C^k{}_{ij} w_k, \tag{A.5.1}$$

where $C^k{}_{ij}$ is a tensor of type (1,2) and $w_k$ a dual vector field. A symmetry property of $C^k{}_{ij}$ follows immediately from condition (v): if we put $w_k = \nabla_k f = \nabla_k \tilde{f}$ it follows that $C^k{}_{ij} = C^k{}_{ji}$. The general formula for $\nabla_i$ on an arbitrary tensor field $T$ of type $(k, \ell)$ in terms of $\tilde{\nabla}_i$ and $C^k{}_{ij}$ is

$$\begin{aligned}\nabla_i T^{j_1 \dots j_k}{}_{m_1 \dots m_\ell} =& \tilde{\nabla}_i T^{j_1 \dots j_k}{}_{m_1 \dots m_\ell} + \sum_{r=1}^{k} C^{j_r}{}_{ic} T^{j_1 \dots c \dots j_k}{}_{m_1 \dots m_\ell} \\ & - \sum_{s=1}^{\ell} C^c{}_{i m_s} T^{j_1 \dots j_k}{}_{m_1 \dots c \dots m_\ell}\end{aligned} \tag{A.5.2}$$

Thus, the difference between the two derivative operators $\nabla_i$ and $\tilde{\nabla}_i$ is completely characterized by the tensor field $C^k{}_{ij}$. The most important case of two different derivative operator is for $\tilde{\nabla}_i = \partial_i$. In this way, the tensor $C^k{}_{ij}$ is denoted as $\Gamma^k{}_{ij}$, named *Christoffel symbol*, that together with the ordinary derivative tells us how compute the derivative $\nabla_i$.

On a manifold $M$ with derivative operator $\nabla_i$, we can define the parallel transport of a vector $v^i$ along a curve $\gamma$ with a tangent $t^i$. The vector $v^i$, given at each point on the curve, is said to be *parallelly transported* as one moves along the curve if the equation

$$t^j \nabla_j v^i = 0$$

is satisfied along the curve.

Given only the manifold structure, many distinct derivative operators can be defined, but there is a natural choice requiring the constancy of inner product between two vector when they are parallely transported along any curve, i.e., $\nabla_k g_{ij} = 0$. It is easily provable that such request implies

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}).$$

and (A.5.1) takes the form

$$\nabla_i w_j = \partial_i w_j - \Gamma^k{}_{ij} w_k.$$

## A.6 Curvature: the Riemann tensor

The curvature of a manifold is completely described by the *Riemann curvature tensor* that is directly related to the failure of a vector to return to its initial value when parallel transported around a small closed curve. The Riemann tensor $R_{ijk}{}^{\ell}$ is defined as

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) w_k = R_{ijk}{}^{\ell} w_{\ell}, \tag{A.6.1}$$

where $w_k$ is a dual vector field. For a general tensor of type $(k, \ell)$

$$\begin{aligned}(\nabla_i \nabla_j - \nabla_j \nabla_i) T^{a_1 \ldots a_k}{}_{b_1 \ldots b_\ell} = &- \sum_{r=1}^{k} R_{ijm}{}^{a_r} T^{a_1 \ldots m \ldots a_k}{}_{b_1 \ldots b_\ell} \\ &+ \sum_{s=1}^{\ell} R_{ijb_s}{}^{m} T^{a_1 \ldots a_k}{}_{b_1 \ldots m \ldots b_\ell}.\end{aligned}$$

Using equation (A.5.2) in (A.6.1), the components of the Riemann tensor are given by

$$R_{ijk}{}^{\ell} = -\partial_i \Gamma^{\ell}{}_{jk} + \partial_j \Gamma^{\ell}{}_{ik} + \Gamma^{m}{}_{ik} \Gamma^{\ell}{}_{mj} - \Gamma^{m}{}_{jk} \Gamma^{\ell}{}_{im}$$

The proof of the important properties listed below can be found in [35]:

i. $R_{ijk}{}^{\ell} = -R_{jik}{}^{\ell}$,

ii. $R_{[ijk]}{}^{\ell} = 0$,

iii. since $\nabla_i g_{jk} = 0$ we have $R_{ijk\ell} = -R_{ij\ell k}$,

iv. the Bianchi identities hold: $\nabla_{[i} R_{jk]\ell}{}^{m} = 0$.

The Riemann tensor can be decomposed into a trace part and a trace-free part. By the antisymmetry properties (i) and (iii), the only non-zero trace of the Riemann tensor is over the second and fourth (or equivalently, the first and third) indices, that defines the Ricci tensor

$$R_{ik} = R_{ijk}{}^{j}.$$

Since $R_{ijk\ell} = R_{k\ell ij}$ (it can be easily proved using the first three properties) the Ricci tensor is symmetric in its indices

$$R_{ik} = R_{ki}.$$

Finally, the *scalar curvature* $R$ is defined as the trace of the Ricci tensor

$$R = R^{i}{}_{i}.$$

The contractions of $i$ with $m$ and $j$ with $\ell$ of the Bianchi identities $\nabla_{[i}R_{jk]\ell}{}^{m} = 0$ lead to a fundamental equation for the consistency of Einstein's equation

$$\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R, \tag{A.6.2}$$

that can be written as

$$\nabla^i G_{ij} = 0,$$

where

$$G_{ij} = R_{ij} - \frac{1}{2}R g_{ij}$$

is named the *Einstein tensor*.

The trace-free part of the Riemann tensor is named Weyl tensor, $C_{ijk}{}^{\ell}$, and is defined for manifolds of dimension $n \geq 3$ by the equation:

$$R_{ijk\ell} = C_{ijk\ell} + \frac{2}{n-2}(g_{i[k}R_{\ell]j} - g_{j[k}R_{\ell]i}) - \frac{2}{(n-1)(n-2)}R g_{i[k}g_{\ell]j}.$$

By the definition, the Weyl tensor satisfies the first three properties of the Riemann tensor. Moreover, it can be verified that it is invariant under conformal transformations of the metric, then it is also called the *conformal tensor*.

As for the standard decomposition of the Maxwell tensor $F_{ij}$ into its electric and magnetic parts $\vec{E}$ and $\vec{B}$ with respect to an observer, i.e., a unit timelike vector $u^i$, we can define the *electric* and *magnetic* parts of the Weyl tensor as in [20]:

$$(C_+)^{ij}{}_{k\ell} = h^{im}h^{jn}h_k{}^r h_\ell{}^s C_{mnrs} + 4u^{[i}u_{[k}C^{j]m}{}_{\ell]n}u_m u^n$$
$$(C_-)^{ij}{}_{k\ell} = 2h^{im}h^{jn}C_{mnr[k}u_{\ell]}u^r + 2u_r u^{[i}C^{j]rmn}h_{km}h_{\ell n}$$

At a spacetime point (or region) the Weyl tensor is called *purely electric (magnetic)* with respect to $u^i$ if $C_- = 0$ $(C_+ = 0)$.

# Appendix B

# Riemann tensor on twisted and GRW spacetimes

## B.1 Twisted spacetimes

Christoffel symbol $\Gamma^k{}_{ij} = \frac{1}{2} g^{km}(g_{jm,i} + g_{im,j} - g_{ij,m})$:

$$\Gamma^0{}_{i0} = 0, \quad \Gamma^k{}_{00} = 0, \quad \Gamma^\rho{}_{\mu 0} = (\dot f/f)\delta^\rho{}_\mu, \quad \Gamma^0{}_{\mu\nu} = f\dot f g^*_{\mu\nu},$$
$$\Gamma^\rho{}_{\mu\nu} = \Gamma^{*\rho}{}_{\mu\nu} + (f_\nu/f)\delta^\rho{}_\mu + (f_\mu/f)\delta^\rho{}_\nu - (f^\rho/f) g^*_{\mu\nu},$$

where $\dot f = \partial_t f$, $f_\mu = \partial_\mu f$ and $f^\mu = g^{*\mu\nu} f_\nu$.
Riemann tensor $R_{jk\ell}{}^m = -\partial_j \Gamma^m{}_{k\ell} + \partial_k \Gamma^m{}_{j\ell} + \Gamma^i{}_{j\ell}\Gamma^m{}_{ki} - \Gamma^i{}_{k\ell}\Gamma^m{}_{ji}$:

$$\begin{aligned}
R_{\mu 0\rho}{}^0 &= f\ddot f g^*_{\mu\rho}, \\
R_{\mu\nu\rho}{}^0 &= g^*_{\mu\rho}(f\partial_\nu \dot f - \dot f f_\nu) - g^*_{\nu\rho}(f\partial_\mu \dot f - \dot f f_\mu), \\
R_{\mu\nu\rho}{}^\sigma &= R^*_{\mu\nu\rho}{}^\sigma + \left(\dot f^2 - \frac{f^\lambda f_\lambda}{f^2}\right)(g^*_{\mu\rho}\delta^\sigma{}_\nu - g^*_{\nu\rho}\delta^\sigma{}_\mu) \\
&\quad + \frac{2}{f^2}(f^\sigma f_\nu g^*_{\mu\rho} - f^\sigma f_\mu g^*_{\nu\rho} + f_\mu f_\rho \delta^\sigma{}_\nu - f_\nu f_\rho \delta^\sigma{}_\mu) \\
&\quad + \frac{1}{f}[\nabla^*_\mu(f^\sigma g^*_{\nu\rho} - f_\rho\delta^\sigma{}_\nu) - \nabla^*_\nu(f^\sigma g^*_{\mu\rho} - f_\rho \delta^\sigma{}_\mu)].
\end{aligned}$$

Ricci tensor $R_{j\ell} = R_{jk\ell}{}^k$:

$$\begin{aligned}
R_{00} &= -(n-1)(\ddot f/f), \\
R_{\mu 0} &= -(n-2)\partial_\mu(\dot f/f), \\
R_{\mu\nu} &= R^*_{\mu\nu} + g^*_{\mu\nu}[(n-2)\dot f^2 + f\ddot f] + 2(n-3)\frac{f_\mu f_\nu}{f^2} \\
&\quad - (n-4)\frac{f^\sigma f_\sigma}{f^2} g^*_{\mu\nu} - (n-3)\frac{1}{f}\nabla^*_\mu f_\nu - \frac{1}{f} g^*_{\mu\nu}\nabla^*_\sigma f^\sigma.
\end{aligned}$$

Scalar curvature $R = R^k{}_k$:

$$\begin{aligned} R =& \frac{R^*}{f^2} + \frac{1}{f^2}(n-1)[(n-2)\dot{f}^2 + 2f\ddot{f}] \\ & -(n-2)(n-5)\frac{f^\sigma f_\sigma}{f^4} - 2(n-2)\frac{\nabla^*_\sigma f^\sigma}{f^3} \end{aligned}$$

## B.2 GRW spacetimes

Christoffel symbol $\Gamma^k{}_{ij} = \frac{1}{2}g^{km}(g_{jm,i} + g_{im,j} - g_{ij,m})$:

$$\begin{aligned} &\Gamma^0{}_{i0} = 0, \quad \Gamma^k{}_{00} = 0, \quad \Gamma^\rho{}_{\mu 0} = (\dot{f}/f)\delta^\rho{}_\mu, \quad \Gamma^0{}_{\mu\nu} = f\dot{f}g^*_{\mu\nu}, \\ &\Gamma^\rho{}_{\mu\nu} = \Gamma^{*\rho}{}_{\mu\nu}. \end{aligned}$$

Riemann tensor $R_{jk\ell}{}^m = -\partial_j \Gamma^m{}_{k\ell} + \partial_k \Gamma^m{}_{j\ell} + \Gamma^i{}_{j\ell}\Gamma^m{}_{ki} - \Gamma^i{}_{k\ell}\Gamma^m{}_{ji}$:

$$\begin{aligned} R_{\mu 0\rho}{}^0 &= f\ddot{f}g^*_{\mu\rho}, \\ R_{\mu\nu\rho}{}^0 &= 0, \\ R_{\mu\nu\rho}{}^\sigma &= R^*_{\mu\nu\rho}{}^\sigma + \dot{f}^2(g^*_{\mu\rho}\delta^\sigma{}_\nu - g^*_{\nu\rho}\delta^\sigma{}_\mu). \end{aligned}$$

Ricci tensor $R_{j\ell} = R_{jk\ell}{}^k$:

$$\begin{aligned} R_{00} &= -(n-1)(\ddot{f}/f), \\ R_{\mu 0} &= 0, \\ R_{\mu\nu} &= R^*_{\mu\nu} + g^*_{\mu\nu}[(n-2)\dot{f}^2 + f\ddot{f}]. \end{aligned}$$

Scalar curvature $R = R^k{}_k$:

$$R = \frac{R^*}{f^2} + \frac{1}{f^2}(n-1)[(n-2)\dot{f}^2 + 2f\ddot{f}$$

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