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The Ising Model on Random Planar Graphs: a Review

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Chapter 1

Introduction

1.1 History of the Ising Model

The scope of this thesis is to give a description of the solution to the Ising model in a random 2-dimensional planar lattice given by Kazakov and Boulatov in [1]. Through the road to the solution, it will be necessary to develop different mathematical techniques, which will be described properly when needed.
The Ising model has firstly been theorized by Lenz in 1920 as an attempt to model ferromagnetism. It consists in a system of spin degrees of freedom locked on a square lattice, and obeying a Hamiltonian of the form

$$E = -J \sum_{(i,j)} S_i S_j - H \sum_{i}^{N} S_i$$

where S_i can have only integer values ± 1 and the sum over (i, j) is intended to be restricted to the nearest neighbours.
This model has been solved exactly by Lenz' student Ising in his thesis [2] in 1924 in the $1d$ case, using a transfer matrix. However, the $1d$ exact solution does not predict a ferromagnetic phase transition as the one is experimentally observed in $3d$ space, and the solution to the analogous higher dimensional model revealed itself to be a formidable task. Ising himself thought erroneously that the absence of phase transitions would have been preserved in higher dimensions. He brought qualitative arguments to support his idea, not recognising some peculiar aspects of the $1d$ model. Landau, instead, gives a qualitative argument that justifies the absence of a transition in one dimension at any finite temperature, but otherwise leads to the possibility of phase transition in two or more dimension.
The argument is the following: consider N spin sites on a d dimensional lattice, and let's start with a zero temperature state with all the spins aligned. Now, we calculate the variation of free energy at any temperature T for the creation of a single domain wall. In $1d$ this corresponds to a flip of a single spin, in $2d$ to the creation of an area of connected flipped spins. We have $\Delta F = \Delta E - T\Delta S$. In $1d$, $\Delta E = 2J$, and $\Delta S = k \log\left(\binom{1}{N-1}\right) = k \log(N - 1)$. Hence, $\Delta F \approx 2J - kT \log(N - 1)$. For every $T > 0$, in the thermodynamic limit $N \to \infty$ ΔF is less than zero, and hence it is energetically favourable to flip spin sites: no spontaneous magnetization domain holds, and hence no phase transitions are expected to be found.
On the other hand, consider a domain wall in $2d$ composed by L segments (sides dividing two spins with different orientation). The calculation of the exact number of states is difficult, since

we should choose the sides of an arbitrary domain imposing its closureness. However, we can easily observe that, starting from an arbitrary point, for each side between two spins in the domain wall there are at least two arbitrary possibilities of choice. If we now approximate $\Delta S \approx k \log 2^L$, we get $\Delta F = \Delta E - T\Delta S \approx L(2J - kT \log 2)$. In this case, $2J - kT \log 2$ can be either positive or negative depending on the temperature, there is a change of sign at $T = \frac{2J}{k \log 2}$. Hence, the creation of domains with different spin orientations could possibly be energetically disfavourable at certain temperatures, and thus the existence of two different phases is not excluded.
As a consequence of the 1d absence of phase transitions, the Ising model had been considered unuseful and too much idealized by physicists' community in the following years; quoting Heisenberg: "Ising succeeded in showing that also the assumption of directed sufficiently great forces between two neighbouring atoms of a chain is not suffcient to explain ferromagnetism" [3].
Physicists hence began to investigate other more complex models in the hope to find the missing phase transition, such as the Heisenberg model or the XY model. Heisenberg model, for example, treats spins as quantum mechanical degrees of freedom, and uses the Hamiltonian $H = -\frac{1}{2}\sum_{j=1}^{N}\left[\sum_{i=1}^{3}\left(J_i \sigma_j^{(i)}\sigma_{j+1}^{(i)}\right) + B\sigma_j^z\right]$, where the $\sigma^{(i)}$ are the x, y and z Pauli matrices, and J_i the coupling constants for x, y and z directions. XY model instead considers the spins as planar vectors of fixed length, which can freely rotate by an arbitrary angle.
Around 1935, the physical community found out that the Ising model was related to other interesting problems and a new debate started on the possibility of having a phase transition in higher dimensions. Peierls [4] in 1936 suggested that this indeed could have been the case, and new attempts to find an exact partition function were made.
In 1942, Lars Onsager ended the debate announcing at a meeting of the New York Academy of Science that he had found an exact solution of the $2d$ model in absence of magnetic field, which was later published in [5] (For another easier solution, see [6]). The solution indeed predicted a phase transition and the significance of the discovery was immediately recognized. Pauli, in a letter to Casimir, wrote: "nothing much of interest has happened except for Onsager's exact solution of the Two-Dimensional Ising Model" [7].
In the following years, the importance of the Ising model continued to grow, and it has been used to analyze different situations in statistical physics, such as the liquid-gas transition through a lattice gas model [8]. Since Onsager solution, many attempts had been made to find an exact $3d$ solution, but without any success.

1.2 Kazakov Ising model

In 1986, Kazakov gave a solution to the $2d$ Ising model on a random planar graph with zero magnetic field [9], followed by the full solution in another article in collaboration with Boulatov [1]. In contrast to the traditional treatment, which is based on a regular square lattice, he had the idea to consider the graph itself as a new degree of freedom. Fixing only the number of edges for vertex, he succeeds to calculate the partition function in the limit of the number of vertices of the graph going toward infinity.
To successfully achieve this result, Kazakov used a connection between random graphs and matrix models known as topological expansion, firstly pointed out by Gerard T'Hooft in his notorious article [10]. Hence, solving the matrix model, he was able to find the partition function analytically and to calculate the properties of the phase transition. In the following chapters, we will go through the various steps of the solution, developing the mathematical formalism needed to understand the process properly and reaching finally the desired result.

Chapter 2

Mapping Ising model to a matrix model

The Hamiltonian of the Ising model with N vertices in the variant of Kazakov can be written in the following fashion:

$$E = -J \sum_{i,j=1}^{N} G_{ij} S_i S_j - H \sum_i S_i$$

where G_{ij} is 1 if i and j are nearest neighbours and 0 elsewhere. We observe that the matrix G_{ij} is encoding the shape of the particular graph we are considering, and thus it has to be considered a variable by itself. We will hereafter consider$J = 1$. To extract the statistical properties of the system when interacting with a standard heat bath, the partition function has to be calculated:

$$Z_{\text{Ising}}(\beta, B) = \sum_{G,\{S_i\}} e^{-\beta E(G,\{S_i\})} \tag{2.1}$$

where $\{S_i\}$ is denoting a state (so an n-uple of spin values, one for each i) of the system and we are summing over all possible states and matrices G_{ij}. We shall see that considering the graph itself as a new degree of freedom is crucial to map the system to a matrix model and hence to calculate the partition function explicitly.
To enhance the aforementioned connection, we can rewrite the various terms of the sum in a different way. Let $V_\uparrow$ and $V_\downarrow$ be respectively the numer of vertices of a given graph in a given configuration which hosts a spin up particle, $E_{\uparrow\uparrow}$the number of edges connecting two spin-up vertices, $E_{\uparrow\downarrow}$ the number of edges connecting a spin-up and a spin down vertex and $E_{\downarrow\downarrow}$ the number of edges connecting two spin-down vertices. Let $E_p = E_{\uparrow\uparrow} + E_{\downarrow\downarrow}$ and $E_a = E_{\uparrow\downarrow}$. Then, we can rewrite the partition function as

$$Z_{\text{Ising}}(\beta, H) = \sum_{config} e^{\beta[(E_p - E_a) + H(V_\uparrow - V_\downarrow)]} \tag{2.2}$$

In this form, we see that we need to perform a sum over graphs, when each graph has a weight of $e^{\pm\beta H}$ for each $V_{\uparrow/\downarrow}$ vertex and of $e^{\pm\beta}$ for each parallel/antiparallel edge. We shall see in the following chapter that a sum of this kind can be performed through an integral over a matrix space.

Chapter 3

How to sum over graphs: Matrix models

We first consider, as an introductory case, a one dimensional Gaussian integral. Then, we will move to vector and matrix integrals.

3.1 Gaussian integrals and Feynman diagrams expansion

Consider the standard Gaussian integral:

$$I_0 = \int_{-\infty}^{\infty} dx e^{-a\frac{x^2}{2}} = \sqrt{\frac{2\pi}{a}}$$

The aim is to find expressions for integrals of the type

$$\frac{1}{I_0}\int_{-\infty}^{\infty} dx x^k e^{-\frac{a}{2}x^2}$$

If we consider $\frac{1}{I_0}e^{-a\frac{x^2}{2}}$ as a probability measure over $\mathbb{R}$, this is equivalent to calculate mean values of x^k. To solve this problem, we can introduce a so-called generating function for the moments of the Gaussian probability measure, which is similar to a Fourier transform without the imaginary unit, of the probability distribution. We hence consider

$$\frac{1}{I_0}\int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2+Jx} = e^{\frac{J^2}{2a}} = \frac{1}{I_0}\sum_{k=0}^{\infty}\frac{(J)^k}{k!}\int_{-\infty}^{\infty} dx x^k e^{-\frac{a}{2}x^2} = \sum_{k=0}^{\infty}\frac{(J)^k}{k!}\langle x^k\rangle$$

and we observe that

$$\sum_{k=0}^{\infty}\frac{(J)^k}{k!}\langle x^k\rangle = \sum_{l=0}^{\infty}\frac{J^{2l}}{(2a)^l l!}$$

Comparing these expressions term by term, we get $\langle x^{2k+1}\rangle = 0$ as expected, and:

$$\frac{\langle x^{2n}\rangle}{(2n)!} = \frac{1}{(2a)^n n!} \Rightarrow \langle x^{2n}\rangle = \frac{(2n)!}{(2a)^n n!} = \frac{1}{a^n}(2n-1)(2n-3)...3\cdot 1 = \frac{1}{a^n}(2n-1)!!$$

This number corresponds to all the possible connections in pairs of k dots. For example, for $2n = 4$, there will be the following pairings:

$$\langle xxxx \rangle \langle xxxx \rangle \langle xxxx \rangle$$

We in particular have that $\langle xx \rangle = \frac{1}{a}$.

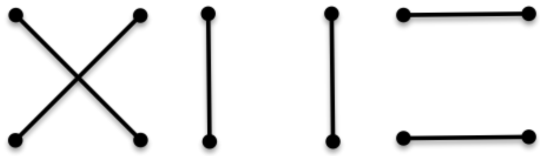

Figure 3.1: The three ways of coupling four points two by two

Since for each choice of pairing of the set of points there are $n = \frac{k}{2}$ connections between the points, we can look at the integral above as "composed" by a sum over all the graphs connecting $k = 2n$ points, where each edge (and then each pair) of every graph carries a multiplicative weight of $\frac{1}{a}$.

In the example $k = 2n = 4$, we get (we use $'$ symbol to mark the x's):

$$\left\langle x^4 \right\rangle = \langle x'x'' \rangle \langle x'''x'''' \rangle + \langle x'x''' \rangle \langle x''x'''' \rangle + \langle x'x'''' \rangle \langle x''x''' \rangle = \frac{1}{a} \cdot \frac{1}{a} + \frac{1}{a} \cdot \frac{1}{a} + \frac{1}{a} \cdot \frac{1}{a} = 3 \left(\frac{1}{a^2} \right)$$

We can show this result through a different, more direct way. We have that

$$\frac{1}{I_0} \int_{-\infty}^{\infty} dx x^k e^{-\frac{a}{2}x^2} = \frac{1}{2I_0} \int_{-\infty}^{\infty} dt e^{-\frac{1}{2}at} t^{n-\frac{1}{2}} = \frac{1}{I_0} \left(\frac{2}{a} \right)^{n-\frac{1}{2}} \int_{-\infty}^{\infty} ds e^{-s} s^{n-\frac{1}{2}} =$$
$$\left(\frac{2}{a} \right)^n \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \left(\frac{2}{a} \right)^n \left(n - \frac{1}{2} \right) \left(n - \frac{3}{2} \right) \cdot ...3 \cdot 1 = \frac{1}{a^n} (2n-1)!!$$

This is the 1D statement of the so-called Wick theorem [11], which we will prove in the general case after. Although the first method of calculation may seem involved and unnecessarily complicated, it is crucial in order to understand the final, more general result.
Starting from what we have just obtained, we are ready to find a series expansion for an integral of the type:

$$\frac{1}{I_0} \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2 + gx^k} \tag{3.1}$$

Expanding the exponential as a Taylor series in g, and exchanging the sum with the integral leads to

$$\frac{1}{I_0} \int_{-\infty}^{\infty} dx e^{-\frac{a}{2}x^2 + gx^l} = \frac{1}{I_0} \sum_{k=0}^{\infty} \frac{(g)^k}{k!} \int_{-\infty}^{\infty} dx x^{lk} e^{-\frac{a}{2}x^2}$$

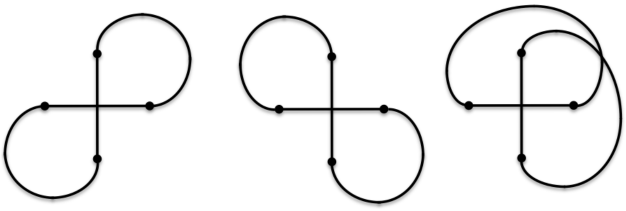

Figure 3.2: First order diagrams for quartic potential (observe the correspondance with Fig 3.1)

But now, each of the integrals in the sum can be understood using Wick theorem. We can construct a vertex with l edges, and consider k vertices of this kind. Connecting the different vertices in all possible ways gives us all the possible combinations for matching together $k \cdot l$ dots. The division by $k!$ accounts for the exchange symmetry between the different vertices, and leads us to consider one time only the equivalent diagrams. So, the original integral (3.1) can be considered as composed by a sum over all $k \in \mathbb{N}$ of all diagrams with k vertices, having l edges per vertex, in which each edge of each graph has a weight of $\frac{1}{a}$.
We observe that in this sum we are considering also disconnected diagrams, namely diagrams composed by different smaller connected parts, as in Fig 3.1. We show now that in order to obtain connected diagrams only we need to take the logarithm of the integral.

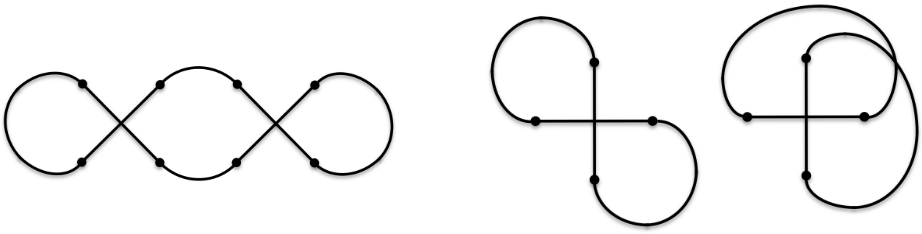

Figure 3.3: Examples of connected and disconnected diagrams with two vertices

Let

$$Z = \langle e^{gx^k} \rangle = e^W = \sum_{n=0}^{\infty} \frac{W^n}{n!}$$

we now show that W is the sum of all connected diagrams. Let D be a single possibly disconnected diagram, and C_n the list of all possible connected diagrams. Then,

$$D = \frac{1}{\prod_I n_I!} \prod_I (C_I)^{n_I}$$

Where n_I is the number of C_I diagrams that are present in D. The factor $\prod_I n_I!$ counts for the symmetry for vertices exchange of the single connected diagrams, so that we do not consider the same diagram more than once when we sum over all n_I. Then we get:

$$Z \propto \sum_{\{n_I\}} D = \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} = \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} = \prod_I e^{C_I} = e^{\sum_I C_I} = e^W$$

3.1.1 Vector Gaussian integrals

We now move on to the generalization of the technique exposed before to vector integrals. We shall see that an analogous result holds and we will prove Wick theorem for a vector integral in a way similar to the one shown above in the 1D case. Then, we will apply the result to matrix integrals, which will give us the right diagrammatic expansion to handle the original problem. Consider the Gaussian integral over $\mathbb{R}^N$:

$$I_0 = \int dx_1 ... dx_N e^{-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x}}$$

where A is a symmetric, positive definite matrix. We can calculate the integral with a change of variable diagonalizing the matrix A. Let Λ be the diagonalized matrix, and O be the orthogonal matrix such that $A = O^\dagger \Lambda O$. Then, if $\boldsymbol{y} = O\boldsymbol{x}$, since $\det(O) = 1$, the integral become:

$$\int dy_1 ... dy_N e^{-\frac{1}{2}\boldsymbol{y}^{\boldsymbol{T}} \Lambda \boldsymbol{y}} = \prod_{i=1}^{N} \int dy_i e^{-\frac{1}{2} y_i^2 \lambda_i} = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{|\det(A)|}}$$

We can now calculate mean values of expressions like $x_{i_1}...x_{i_k}$ using a method similar to the 1D method. We observe that

$$\frac{1}{I_0} \int dx_1 ... dx_N x_{i_1} ... x_{i_k} e^{-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x}} = \frac{1}{I_0} \frac{\partial}{\partial J_{i_1}} ... \frac{\partial}{\partial J_{i_k}} \int dx_1 ... dx_N e^{-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x} + J^T \boldsymbol{x}} \Big|_{J=0}$$

But, again with the change of variable $\boldsymbol{y} = O\boldsymbol{x}$ and remembering that $O^{-1} = O^T$, we have:

$$\int dx_1 ... dx_N e^{-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x} + J \boldsymbol{x}} = \int_{-\infty}^{+\infty} dy_i e^{-\frac{1}{2}\lambda_i y_i^2 + (JO)_i y_i} = \prod_{i=1}^{N} \sqrt{\frac{2\pi}{\lambda_i}} e^{\frac{1}{2}(JO)_i \frac{1}{\lambda_i} (JO)_i} = \sqrt{\frac{(2\pi)^N}{|\det(A)|}} e^{\frac{1}{2} J^T \cdot A^{-1} \cdot J}$$

Firstly, we can calculate mean values of $\langle x_i x_j \rangle$. Deriving the generating function, we obtain:

$$\langle x_i x_j \rangle = \frac{1}{I_0} \frac{\partial}{\partial J_i} \frac{\partial}{\partial J_j} \int dx_1 ... dx_N e^{-\frac{1}{2}\boldsymbol{x}^{\boldsymbol{T}} A \boldsymbol{x} + J \boldsymbol{x}} \Big|_{J=0} = A_{ij}^{-1} \tag{3.2}$$

We now expand both the exponentials and compare term by term:

$$\int dx_1 ... dx_N e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + J^T \boldsymbol{x}} = \sum_{k=0}^{\infty} \frac{\langle (J^T \boldsymbol{x})^k \rangle}{k!} = \sum_{l=0}^{\infty} \frac{\left(J^i A_{ij}^{-1} J^j \right)^l}{2^l l!}$$

The mean value of $x_{i_1}...x_{i_k}$ in the first series is multiplied by a factor of $\frac{1}{k!}\frac{k!}{n_1!\cdot...n_N!}J_{i_1}...J_{i_k}$ where n_i is the number of times that the i-th component occurs. The combinatorial factor $\frac{k!}{n_1!\cdot...n_N!}$ is the different ways to choose k different elements from k places, devided by the permutations given by the presence equal terms. On the other hand, the term containing $J_{i_1}...J_{i_k}$ on the right hand side of the equation correspond to $l = \frac{k}{2}$ and is made by $\frac{1}{n_1!...n_N!}$ times the sum of all possible pairings $\frac{1}{2^l l!}\sum_\sigma \left(A^{-1}\right)_{\sigma(i_1)\sigma(i_2)}\cdots\left(A^{-1}\right)_{\sigma(i_{k-1})\sigma(i_k)}$. The combinatorial factors cancels out, and we obtain the multi dimensional Wick Theorem:

Theorem 1 (Wick's theorem)**.** $\langle x_{i_1}...x_{i_k}\rangle = \frac{1}{2^l l!}\sum_\sigma \left(A^{-1}\right)_{\sigma(i_1)\sigma(i_2)}\cdots\left(A^{-1}\right)_{\sigma(i_{k-1})\sigma(i_k)}$

The factor $\frac{1}{2^l l!}$ is needed to count each Wick contraction once. We remark that the importance of this result is that we have expressed the mean value of a product on k coordinates as a sum of products of mean values of products of two coordinates. These fundamental quantities, which we have seen are in correspondance with edges of the graphs, are called propagators, borrowing the terminology from QFT.

3.2 Matrix integrals and topological expansion

We now move to an integral over the vector space of Hermitian matrices over $\mathbb{R}^N$. We set a Gaussian measure on this space of the type:

$$I_0 = \int dM e^{-\frac{N}{2}\operatorname{Tr} M^2}$$

The final scope is to expand diagrammatically an integral of the type:

$$\frac{1}{I_0}\int dM e^{-\frac{N}{2}\operatorname{Tr} M^2 + gN\operatorname{Tr}\left(M^4\right)} \tag{3.3}$$

Where dM has to be understood as $\prod_{i=1}^n dM_{ii}\prod_{i<j}\operatorname{Re}\{dM_{ij}\}\operatorname{Im}\{dM_{ij}\}$. The N multiplying the exponent, which is the size of the matricies, has only a rescaling effect which can be easily accounted for but is crucial in the topological expansion we are going to perform.
We can expand the integral applying the vectorial Wick theorem, so we first calculate the "propagators": $\langle M_{ij}M_{kl}\rangle = \delta_{il}\delta_{jk}$, as it's easily obtained deriving the generating function properly ($\frac{\partial}{\partial J_{ij}}$ means derivation respect the ij matrix component):

$$\langle M_{ij}M_{kl}\rangle = \frac{1}{I_0}\frac{\partial}{\partial J_{ij}}\frac{\partial}{\partial J_{lk}}\int dM e^{-\frac{N}{2}\operatorname{Tr} M^2 + \operatorname{Tr}(JM)}\Bigg|_{J=0} = \frac{\partial}{\partial J_{lk}}\frac{1}{N}J_{ij}\, e^{\frac{Tr\left(J^2\right)}{2N}}\Bigg|_{J=0} = \frac{1}{N}\delta_{il}\delta_{jk}$$

We are now ready to compute, firstly, the value of $\langle\operatorname{Tr}(M^n)\rangle$. This is probably better illustrated with an example, let's say, with $n = 4$.
Using Wick theorem, we get (summation over equal indices is implied):

$$\begin{gathered}\left\langle\operatorname{Tr}\left(M^4\right)\right\rangle = \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle = \\ \langle M_{ij}M_{jk}\rangle\langle M_{kl}M_{li}\rangle + \langle M_{ij}M_{kl}\rangle\langle M_{jk}M_{li}\rangle + \langle M_{ij}M_{li}\rangle\langle M_{jk}M_{kl}\rangle = \\ \frac{1}{N^2}\left(\delta_{ik}\delta_{jj}\delta_{ik}\delta_{ll} + \delta_{jl}\delta_{ii}\delta_{jl}\delta_{kk} + \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li}\right) = \\ \frac{1}{N^2}\left(N^3 + N^3 + N\right)\end{gathered}$$

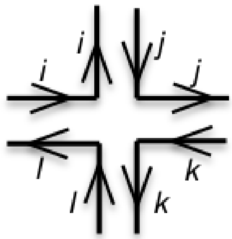

Figure 3.4: A quartic vertex, corresponding to (M^4)

We can better understand and generalize this example by slightly modifying the basic idea of graph expansion. For each term of the sum we get from Wick theorem, we create a vertex as in Figure (3.4). Then, we connect the ribbon edges with each other, using double (ribbon) lines, respecting the orientation of the arrows. Each double line connection corresponds to a mean value of a possible Wick contraction, each "ribbon" graph possibly obtained by this means to a term of the sum.

Considering the closed linear (not ribbon) loops which are hence created starting from an indexed line, we note that each loop is in correspondance with a closed sequence of indices (in the example, $ikki$, jj, ll for the first graph, then $jllj$, ii and kk for the second and $illkkjji$ in the third). Each pairing of two indices corresponds to a Kronecker δ with these two indices as subscripts. Hence each closed line, and then each closed sequence of indices, represents a summation over the "chain" of delta's which result finally in contributing by a factor of N.
The correspondence is now clear:

- the double connections in a given graph give us a certain choice of coupling the various terms in a way analog to Wick theorem.
- Each pairing gives us two Kronecker's deltas, which are represented by the two lines of the ribbon pairing connecting two indicies, and a $\frac{1}{N}$ factor.
- The various deltas obtained can be summed in cycles, each cycle gives an N factor when it's closed: this corresponds to a (multiplicative) contribution of N for each closed linear loop obtained.

Once we established this observation, we are finally ready to move to the full integral (3.3). We consider its development in a power series of g as usual:

$$\int dM e^{-\frac{N}{2}\operatorname{Tr} M^2 - gN\operatorname{Tr}\left(M^l\right)} = \sum_{k=1}^{\infty} \frac{(-g)^k N^k}{k!} \int dM \operatorname{Tr}\left(M^l\right)^k e^{-\frac{N}{2}\operatorname{Tr} M^2}$$

Now, every $\operatorname{Tr}\left(M^l\right)$ gives a single vertex, and for every order k we make the same procedure exposed before, creating all different possible ribbon graphs with k vertices. We observe that the N in the second part of the exponential gives an extra N factor for every vertex in the diagram we are considering. Now, something amazing happens. We recall all the various N factors for each diagram:

- One N factor for each vertex.

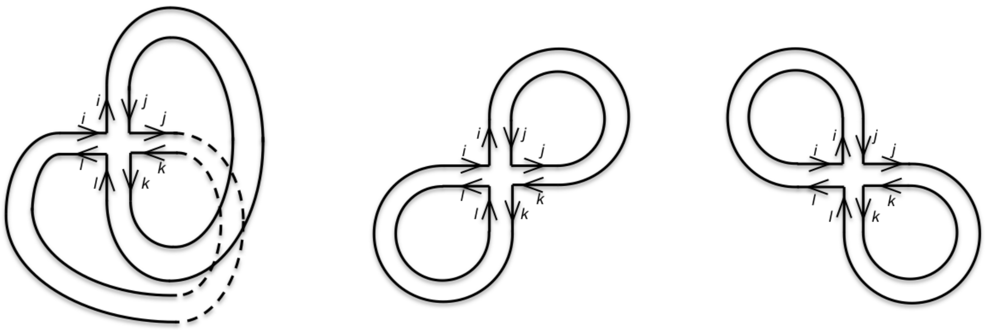

Figure 3.5: The three diagrams corresponding to $\langle M^4 \rangle$

- One $\frac{1}{N}$ factor for each ribbon edge.
- One N factor for each linear loop.

We can now define the Euler characteristics for fatgraphs as $\chi = V - E + L$, where V is the number of vertices, L of loops and E of edges [12]. This is the same of imagining each diagram as lying on a 2d surface, in such a way that different edges do not intersect. A diagram whose edges directly do not self intersect can lay on a sphere. For the other diagrams, each intersection needs a handle to be added to the sphere, so that one edge can pass over the other (in the example, the first diagram of 3.2 lies on a torus).
In a similar fashion, one can embed each of these graphs in a topologically different 2d surface. Moreover, every connected compact orientable 2d surface can be classified as a sphere with a number h of handles, and the Euler characteristic can be defined for it as $\chi = 2 - 2h$ (see Appendix 1). We now observe that each graph divides the surface on which is embedded in different regions (faces), and each loop encloses and hence corresponds to one face. Thus, the number of loops equals the number of faces. We get then that the previously defined Euler characteristic equals to the standard one: $\chi = V - E + F$.
Moreover, since each loop can be turned to a triangle adding two vertex and two edges, hence without affecting the Euler characteristic defined before, we see that the graph can be turned in a triangulation of the surface. We can then relate the definition of Euler characteristic for a graph and the one for a surface in term of the genus, and show that are equal. This is further discussed in Appendix 1.
Then, we established that each graph can be associated with a 2d surface with a certain genus, and that his weight in the integral is proportional to $N^{V-E+F} = N^{\chi}$. This non trivial and extremely important result is known as topological expansion, and will have application in the original problem: once we have mapped the partition function to a matrix integral, we will know that the leading term in N will corrispond to planar graphs only.

Chapter 4

Finding the right matrix model

Now that we have developed the formalism of matrix integrals, we can turn to the original problem: the calculation of the partition function (2.1). As we showed in chapter (2), we need to find an integral that can correspond to a sum over graphs with two different types of vertices and then edges with different weights. After that, we will need to find a method to solve it exactly in order to extract the information desired. To obtain this result, we should generalize the single matrix integral and use a double matrix integral:

$$Z(N,c,g,H) = \int dAdBe^{\left[-N\,\mathrm{tr}\left(A^2+B^2-2cAB+4ge^H A^4+4ge^{-H}B^4\right)\right]}$$

We observe that the 4 multipling ge^H and ge^{-H} is only dued to the fact that we removed the $\frac{1}{2}$ in front of N, hence rescaling the integral calling $A = \sqrt{2}A'$ and $B = \sqrt{2}B'$.
The mechanism is analogous to the one explained before, but guarantees the possibility to have vertices with different weights. $\mathrm{Tr}\left(A^2 + B^2 - 2cAB\right)$ is a quadratic form over the two matrices A and B, given by

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}$$

with determinant $\frac{1}{1-c^2}$. Hence,

$$\begin{pmatrix} 1 & c \\ c & 1 \end{pmatrix}^{-1} = \frac{1}{1-c^2} \begin{pmatrix} 1 & -c \\ -c & 1 \end{pmatrix}$$

and, following (3.2), we obtain the two weights for edges connecting A and B vertices.

$$\langle \mathrm{tr}\,AA \rangle = \langle \mathrm{tr}\,BB \rangle = \frac{1}{1-c^2}$$

and

$$\langle \mathrm{tr}\,AB \rangle = \frac{c}{1-c^2}$$

We will still have a weight proportional to the Euler characteristic of a given configuration, but with the additional contributions of the propagators connecting the edges, which are no more the unity. Moreover, the vertex of the two types will have additional weight of ge^H and ge^{-H} respectively. Thus, the weight of a diagram is:

$$Weight = N^{\chi}\left(ge^H\right)^{V_A}\left(ge^{-H}\right)^{V_B}\langle AA\rangle^{E_p}\langle AB\rangle^{E_a} = N^{\chi}\left[-\frac{gc}{\left(1-c^2\right)^2}\right]^V c^{-\frac{1}{2}(E_p-E_a)}e^{H(V_A-V_B)}$$

Where $V = V_A + V_B$ is the total number of vertices, and we used $E_p + E_a = E = 2V$. If we identify V_A with spin up vertices and V_B with spin down vertices, we can find the value of the constant c which makes this weight corresponding to the desired Ising weight of a configuration. Comparing with (2.2) gives $c = e^{-2\beta}$. We will keep on writing c in all the following sections. As before, we are now considering disconnected diagrams too. Taking the logarithm, we restrict ourselves only to connected diagrams. Then, we have:

$$F(c,g,H) = \log \frac{Z(c,g,H)}{Z(c,0,0)} = \sum_{configurations} N^{\chi} \left[-\frac{gc}{(1-c^2)^2} \right]^V e^{\beta[(E_p - E_a) + H(V_\uparrow - V_\downarrow)]} \tag{4.1}$$

We recall that we want to consider only the planar configurations, and that the thermodynamic calculation requires the number of vertices going towards infinity.
To account for the first problem, we use the topological expansion. Each topology brings a factor of N^{χ}, and any non planar topology has Euler characteristic ≤ 0 (Appendix 2). We can thus divide by N^2 and take the limit for the dimension of the matrix space going to infinity ($N \to \infty$):

$$F_{\text{pl}}(c,g,H) = -\lim_{N\to\infty} \frac{1}{N^2} F(c,g,H) \tag{4.2}$$

For planar diagrams, the $N^{\chi} = N^2$ simplifies, and for all other topologies the limit is 0. We are then left with a sum over all connected planar graphs with any numer of vertices. Of course (see (4.1) and chapter (3)) $F_{\text{pl}}(c,g,H)$ is a power series in g, where the exponent classifies the numer of vertices and each term $F_V(c,H)$ has all the contributions from that number:

$$F_{\text{pl}}(c,g,H) = \sum_{V=1}^{\infty} \left[-\frac{gc}{(1-c^2)^2} \right]^V F_V(c,H)$$

We need to estimate the coefficient for $V \to \infty$. In order to do this, we observe that F admits a series expansion, and is thus analytic in g. If the series has a finite radius of convergence (we shall see later that this is indeed the case), Hadamard's formula gives us:

$$\frac{1}{R} = \lim_{V\to+\infty} \sqrt[V]{F_V}$$

Hence, if we know $R = \frac{c|g_{cr}(c,H)|}{(1-c^2)^2}$ we can estimate $\lim_{V\to+\infty} F_V$:

$$Z_{\text{Ising}} = \lim_{V\to+\infty} F_V = \lim_{V\to+\infty} \left[\frac{c\,|g_{cr}(c,H)|}{(1-c^2)^2} \right]^{-V}$$

And finally, we get the free energy per site:

$$F_{\text{Ising}} = -\frac{1}{V\beta} \log Z_{\text{Ising}}(V,\beta,H) = \frac{1}{\beta} \log \left[\frac{c\,|g_{cr}(c,H)|}{(1-c^2)^2} \right] \tag{4.3}$$

Then, we now have to calculate the integral, perform the limits described here and obtain g_{cr}. To this calculation will be dedicated the following sections.

Chapter 5

Performing the calculation

To actually solve the integral in the matrix model, we need to resort to an integral formula found by Harish-Chandra in [13]. In the following chapter, we develop the mathematical formalism to understand it, and we expose a proof valid in our particular case following Metha [14]

5.1 Haar measure on Lie groups

Consider a Lie Group $(G, \cdot)$ and denote, for every $g \in G$ by L_g the diffeomorphism induced by left multiplication by g. Then we can, as usual, build the left invariant vector fields defined by the condition $dL'_g\left(X|_g\right) = X|_{g' \circ g}$ for every left invariant field X and for every $g, g' \in G$. The group operation offers a possibility to naturally induce a metric on the entire manifold starting from an arbitrary metric on the tangent space at the identity $\mathfrak{g}$. We can use left multiplication by group elements to "move" the vector from the tangent space they belong to to the Lie algebra, so defining the metric at a point g by $\langle X, Y\rangle_g = \langle dL_{g^{-1}}\left(X|_g\right), dL_{g^{-1}}\left(Y|_g\right)\rangle$. This is the same to define the metric at g as the pullback of the metric on $\mathfrak{g}$ to T_g by $L_{g^{-1}}$. With this definition, we are actually transporting an orthonormal basis on $\mathfrak{g}$ to every tangent space in the manifold. Moreover, this metric has the property of being left invariant, that is, every left translation is an isometry. $\langle dL_{g'}\left(X|_g\right), dL_{g'}\left(Y|_g\right)\rangle = \langle X, Y\rangle_g$.
Having defined a metric on G, we can now measure lengths and, as a consequence, volumes. We want to define the volume form in such a way that the infinitesimal volume of a parallelepiped made up by an orthonormal base is 1. We should first define the manifolds which can have a volume form consistently defined everywhere: these manifolds are called orientable. The precise definition is that a manifold is orientable if there exists an everywhere non zero n form. Since the vector space of n forms is one dimensional, and a volume form is an n-form, the two statements are actually the same. Moreover, every Lie group is orientable: you can just define an n form at the identity and pull-back it on every point in G similarly to how we did with the metric.

So, for an orientable Riemannian Manifold of dimension n, we define the volume form as the unique n form which in an orthonormal frame $e_1 ... e_n$ has the form $\omega = e^1 \wedge ... e^n$. Before showing that this is indeed a good definition, we need to find the expression for ω in a generic coordinate sistem. Let A be the change of basis matrix, so that $e'_i = A^j{}_i e_j$. we have that $\omega' dx'^1 \wedge ... dx'^n = \frac{1}{n!}\epsilon_{j_1 ... j_n} A^{j_1}{}_{i_1} ... A^{j_n}{}_{i_n} dx'^{i_1} \wedge ... dx'^{i_n} = det(A) dx'^1 \wedge ... dx'^n$. But, for every change of coordinaates between two orthonormal bases, $det(A) = 1$, so ω has the same action on the two bases and the definition is well posed. Moreover, for an orientation preserving change of coordinates, we find that $\det(A) = \sqrt{A^t A} = \sqrt{\det(g)}$. Then, we get:

$$\omega = \sqrt{\det(g)} dx^1 \wedge \cdots \wedge dx^n \tag{5.1}$$

Using the same procedure, we can define on a Lie group a right invariant metric and measure as well. However, in general the two are not equivalent. On the other hand, in the special case of a (connected) compact Lie group we can show that exists a both left and right invariant measure:

Theorem 2. *Every connected compact Lie Group admits a bi-invariant measure [15].*

Proof. Let $dimG = n$, and $d\omega$ a left invariant measure on G.
Consider now, for every $g \in G$, the n-form $R_g^* (d\omega)$. We want to show that $R_g^* (d\omega) = d\omega$.
This form is still left invariant since left and right translations commute. Moreover, since $dim\Lambda^n (V) = 1$, $R_g^* (d\omega) = h(g) d\omega$. We can thus define the function $\Delta : G \ni g \to h(g) \in \mathbb{R}$. Since $R_{g \circ g'}^* = R_{g'}^* \circ R_g^*$, $\Delta(g' \circ g) = \Delta(g')\Delta(g)$. But $\Delta(e) = 1$ and G is connected, so $\Delta(g) > 0 \; \forall g \in G$, otherwise would exist a g such that $R_g^* (d\omega) = 0$. Moreover, G is compact and Δ is countinuous, so $\Delta(G)$ is compact. If there exists $g \in G$ such that $\Delta(g) \neq 1$, then the sequence $\Delta(g^n)$ for $n \in \mathbb{Z}$ would be unbounded and this contradicts the compactness of $\Delta(G)$. Hence, $\Delta(g) = 1 \; \forall g \in G$, and as a consequence, $d\omega$ is right-invariant. ☐

In Appendix 2 we will show how a bi-invariant measure is found in $U(N)$ and explicitely calculate the metric tensor in suitable coordinates.

5.2 Integrating over a Lie algebra: eigenvalues and unitary sector

To actually perform the integral we want to calculate, we need to integrate over the vector space of Hermitian matrices over $\mathbb{C}^N$, which is the Lie algebra of $U(N)$. We now show how we can make a change of coordinates which parametrizes the matrices with their eigenvalues and a unitary matrix.
The concept is similar to the one of the Weyl integration formula over Lie groups. In that case, one observes that we can choose a maximal abelian subgroup of a Lie group, called the maximal torus, which consists in the case of a matrix group of a maximal set of commuting (and hence simultaneously diagonalizable in $U(N)$ case) matrices. Then, since every element of a Lie group is conjugate to an element of the maximal torus (spectral theorem guarantees that every unitary matrix is conjugate to a diagonal one), one can integrate separately over the maximal torus and over the subset of the group which is needed to perform the conjugation.
This leads to the following formula for integration over Lie groups:

$$\int_G f(g) dg = |W|^{-1} \int_T \Delta(t)^2 \int_{G/T} f\left(gtg^{-1}\right) d[g] dt$$

where $|W|$ is the order of the Weyl group, $\Delta(t)$ the Vandermonde determinant (cfr: App 3) and $d[g]$ the Haar measure on G/T. We are now going to obtain a similar expression for integration over a Lie algebra. Of course, corresponding to the maximal torus there will be a maximal abelian subalgebra (the Cartan subalgebra), and we can write an arbitrary Hermitian matrix H as $H = U^\dagger \Lambda U$, where Λ is diagonal. Starting from a flat metric tensor $g_{ij} = \delta_{ij}$, we can calculate the metric tensor in the new coordinates. We begin with:

$$dH = d\left(U \Lambda U^\dagger\right) = dU \Lambda U^\dagger + U d\Lambda U^\dagger + U \Lambda dU^\dagger$$

Since $d\left(UU^\dagger\right) = dI = 0$, we have:

$$dU^\dagger = -U^\dagger dU U^\dagger$$

Then, we obtain:

$$\begin{aligned} dH = dU\Lambda U^\dagger + Ud\Lambda U^\dagger - U\Lambda U^\dagger dU U^\dagger = \\ = U\left(U^\dagger dU\Lambda + d\Lambda - \Lambda U^\dagger dU\right)U^\dagger = \\ U\left(\left[U^\dagger dU, \Lambda\right] + d\Lambda\right)U^\dagger \end{aligned}$$

We can now obtain the form of the metric tensor:

$$ds = \mathrm{Tr}\left(dHdH\right) = \mathrm{Tr}\left(\left[U^\dagger dU, \Lambda\right] + d\Lambda\right)^2$$

Since $\left(U^{-1}dU\right)^\dagger = dU^\dagger U = -U^{-1}dU$, the matrix $U^{-1}dU = idT$ is anti Hermitian, so dT is Hermitian.
We get:

$$\begin{aligned} ds^2 &= \mathrm{Tr}\left(idT\Lambda - \Lambda idT + d\Lambda\right)^2 = \mathrm{Tr}\left(dT\Lambda dT\Lambda - dT\Lambda^2 dT + \Lambda dT\Lambda dT + d\Lambda^2\right) \\ &= -\mathrm{Tr}\left(\left[dT, \Lambda\right]^2\right) + \mathrm{Tr}\, d\Lambda^2 \\ &= \mathrm{Tr}\, d\Lambda^2 + \mathrm{Tr}\left(\left(dT\Lambda\right)^2 + \left(\Lambda dT\right)^2 - dT\Lambda dT^2 - \Lambda dT^2\Lambda\right) \\ &= \sum_k \left(d\lambda_k^2\right) - 2\,\mathrm{tr}\left(dT\Lambda dT\Lambda - (dT)^2(d\Lambda)^2\right) \\ &= \sum_k \left(d\lambda_k^2\right) - 2\sum_{ij}(dT)_{ij}(dT)_{ji}\left(\lambda_i\lambda_j - \lambda_i^2\right) \\ &= \sum_k \left(d\lambda_k^2\right) - 2\sum_{ij}\left|dT_{ij}\right|^2\lambda_j\left(\lambda_i\lambda_j - \lambda_i^2\right) \\ &= \sum_k \left(d\lambda_k^2\right) + 2\sum_{ij}\left[d\left(\mathrm{Re}\,T_{ij}\right)^2 + d\left(\mathrm{Im}\,T_{ij}\right)^2\right]\left(\lambda_i - \lambda_j\right)^2 \end{aligned}$$

From the metric tensor in the new coordinates, it's easy to obtain the measure. From (5.1), and since $\det g = 2^{n^2-n}\prod_{i<j}\left(\lambda_i - \lambda_j\right)^4$ we get:

$$d\mu = 2^{\frac{1}{2}n(n-1)}\Delta\left(\lambda_1 \ldots \lambda_n\right)^2 \prod_{i=1}^{n} d\lambda_i \prod_{i<j} dT_{ij} \tag{5.2}$$

where $\Delta\left(\lambda_1 \ldots \lambda_n\right)$ is the Vandermonde determinant(cfr: App 3).
We thus devided the integration in a part over the eigenvaules and in a part over unitary matrices. We observe that the integration over unitary matrices is not over the whole $U(N)$: to avoid multiple matrices counting the same Hermitian matrix, we need to fix n parameters, and we are in fact integrating over $U(N)/\exp(\Lambda)$ [16]. However, we will keep on writing the integration as it would be over the whole group.

5.3 Divergence and Laplacian On a Manifold

To prove the Harish-Chandra formula, we need to introduce a last topic, the divergence on a manifold. We consider a manifold endowed with a volume form ω, and a vector field X on that manifold. The vector field induces a local diffemorphism on the manifold, which in turn modifes the volume form at every point. The divergence of the vector field X at a point is the first order change of the volume form at this point induced by the infinitesimal diffeomorphism related to X. We are then led to the definition:

Definition 1 (Divergence of a vector field)**.** *Let X be a vector field on a manifold endowed with a volume form ω. Then, $\boldsymbol{\nabla} \cdot X$ is defined by the equation $(\boldsymbol{\nabla} \cdot X)\,\omega = \mathfrak{L}_X \omega$*

We observe that, since $\Lambda^n\left(T_p^*\right)$ has dimension one, every n form is proportional to the volume form. Then the divergence is well definded. Moreover, we can use Cartan's homoopy formula: $\mathfrak{L}_X \omega = i_X d\omega + d i_X \omega$, and knowing that $d\omega = 0$, we find that $(\boldsymbol{\nabla} \cdot X)\omega = d i_X \omega$. If the manifold is endowed with a metric, there is a corrispondence beetween k forms and n-k forms. Let $a \in \Lambda^k(T_p^*)$ be a k form. Then, the metric maps naturally a in a multivector $a^* \in \Lambda^k(T_p)$. Contracting the volume form with this multivector gives a n-k form, whihc is called the (Hodge) dual form $\star a$. In this corrispondence, every n form is mapped to a zero form: a function. Then we obtain $\boldsymbol{\nabla} \cdot X = \star \mathfrak{L}_X \omega = \star d \star X^\flat$. We can understand this duality relation thinking that the Hodge star send a k form α to a "perpendicular" n-k form $\star\alpha$: a form which kernel is the orthogonal subspace to the kernel of α and such that $\alpha \wedge \star\alpha = \|\alpha\|^2 \omega$ (or equivalently $\|\alpha\| = \|\star\alpha\|$).
We can now compute an explicit formula for the divergence in arbitrary coordinates. Using the definition, we calculate $\mathfrak{L}_X \omega$:

$$\mathfrak{L}_X \omega = \mathfrak{L}_X(\sqrt{|\det(g)|})dx^1 \wedge ...dx^n = d i_X \left(\sqrt{|\det(g)|})dx^1 \wedge ...dx^n\right)$$
$$= d\left(\sqrt{|\det(g)|}(X_1 dx^2 \wedge ...dx^n - X_2 dx^1 \wedge dx^3 ...dx^n + ...(-1)^{n-1} X_n dx^1 \wedge ...dx^{n-1})\right)$$
$$= \sum_{i=1}^{n} \partial_i(\sqrt{|\det(g)|} X_i) dx^1 \wedge dx^2 \wedge ...dx^n$$

Now, since $(\boldsymbol{\nabla} \cdot X)\,\omega = (\boldsymbol{\nabla} \cdot X)\sqrt{|\det(g)|}dx^1 \wedge ...dx^n = \mathfrak{L}_X \omega$ by definition, we compare the two expressions and we find:

$$\boldsymbol{\nabla} \cdot X = \frac{1}{\sqrt{|\det(g)|}} \partial_a \sqrt{|\det(g)|} X^a$$

Having found the divergence of a vector field, we are directly brought to the general form of the Laplacian operator:

Definition 2 (Laplacian of a scalar function)**.** *Let $\phi : M \to \mathbb{R}$ be a scalar function on a Riemannian manifold. Then,*

$$\nabla^2 \phi = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \phi = \frac{1}{\sqrt{|\det(g)|}} \partial_a \sqrt{|\det(g)|} g^{aj} \frac{\partial \phi}{\partial x_j} \tag{5.3}$$

5.4 Harish-Chandra formula for U(N)

We turn back to the original integral (4.2). Firstly, we diagonalize the matrices: $A = U^\dagger X U$ and $B = V^\dagger Y V$ where X and Y are diagonal matrices. We observe that:

$$\mathrm{Tr}(AB) = \mathrm{Tr}\big(U^\dagger X U V^\dagger Y V\big) = \mathrm{Tr}\big(V U^\dagger X U V^\dagger Y\big) = \mathrm{Tr}\big(W X W^\dagger Y\big)$$

where we defined $(VU^\dagger) = W$. Using the above calculation, we can handle the $2c\,\mathrm{Tr}(AB)$ in the exponential, and we obtain:

$$\int dX dY \Delta^2(X)\Delta^2(Y) \exp\left[-N \sum_i \left(x_i^2 + y_i^2 + 4ge^H x_i^4 + 4ge^{-H} y_i^4\right)\right] \int dW \exp\left[2Nc\,\mathrm{tr}\left(W X W^\dagger Y\right)\right]$$

We used formula (5.2) to handle the differential of the change of variable. The difficult part to integrate is now the one in dW. We will follow the path used by Metha, in [14].

Consider the heat equation for a scalar function over the vector space of Hermitian matrices (A and B will be Hermitian matrices hereafter) with initial condition $\xi(A,0) = \eta(A)$ which we will suppose that depends only on the eigenvalues:

$$\frac{\partial \xi(A_{ij},t)}{\partial t} = \frac{1}{2}\nabla^2 \xi(A_{ij},t) = \frac{1}{2}\left[\sum_i \frac{\partial^2}{\partial A_{ii}^2} + \frac{1}{2}\sum_{i<j} \frac{\partial^2}{\partial\left(\operatorname{Re} A_{ij}\right)^2} + \frac{\partial^2}{\partial\left(\operatorname{Im} A_{ij}\right)^2}\right]\xi(A_{ij},t)$$

The solution is the spatial convolution by the heat kernel $\frac{1}{(2\pi t)^{n^2/2}} \exp\left[-\frac{1}{2t}\operatorname{tr}(A-B)^2\right]$:

$$\xi(A,t) = \int dB \frac{1}{(2\pi t)^{n^2/2}} \exp\left[-\frac{1}{2t}\operatorname{tr}(A-B)^2\right]\eta(B)$$

We can now make a change of variable diagonalizing B and A:

$$A = U^\dagger X U \text{ and } B = V^\dagger Y V$$

Using (5.2) for B we get:

$$\xi(X,V,t) = \frac{1}{(2\pi t)^{n^2/2}} \int dY \Delta(Y)^2 \eta(Y) \int dU \exp\left[-\frac{1}{2t}\operatorname{tr}\left(V^\dagger X V - U^\dagger Y U\right)^2\right]$$

But

$$\operatorname{Tr}\left(V^\dagger X V - U^\dagger Y U\right)^2 = \operatorname{Tr}\left[U(U^\dagger X U - V^\dagger Y V)U^\dagger U(U^\dagger X U - V^\dagger Y V)U^\dagger\right] = \operatorname{Tr}\left(X - (UV^\dagger)Y\left(VU^\dagger\right)\right)^2$$

We call $VU^\dagger = W$ and we get:

$$\xi(X,t) = \frac{1}{(2\pi t)^{n^2/2}} \int dY \Delta(Y)^2 \eta(Y) \int dW \exp\left[-\frac{1}{2t}\operatorname{tr}\left(X - W^\dagger Y W\right)^2\right]$$

This change of variable (which is possible because η does not depend on U) shows that the solution too depends only on the eigenvalues. We can further expand the term in the exponential:

$$\operatorname{Tr}\left(X - W^\dagger Y W\right)^2 = \operatorname{Tr} X^2 - 2\operatorname{Tr}\left(XW^\dagger Y W\right) + \operatorname{Tr}\left(W^\dagger Y W\right)^2 = \operatorname{Tr} X^2 - 2\operatorname{Tr}\left(XW^\dagger Y W\right) + \operatorname{Tr} Y^2$$

This gives us:

$$\xi(X,t) = \frac{1}{(2\pi t)^{n^2/2}} \int dY \Delta(Y)^2 \eta(Y) \exp\left(-\frac{1}{2t}\operatorname{Tr} X^2\right)\exp\left(-\frac{1}{2t}\operatorname{Tr} Y^2\right)\int dW \exp\left[\frac{1}{t}\operatorname{Tr}\left(XW^\dagger Y W\right)\right] \tag{5.4}$$

Consider ξ as a function of X is the same as making the change of coordinates in A before solving the differential equation and then solving the differential equation in the new coordinates. We can hence use the laplacian in his general-coordinate form (5.3) to obtain the new equation solved by $\xi(A(X))$:

$$\frac{\partial \xi(X,t)}{\partial t} = \frac{1}{2}\frac{1}{\Delta^2(X)}\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Delta^2(X)\frac{\partial}{\partial x_i}\xi(X,t)$$

Since the initial condition does not depend on U, the derivatives of ξ with respect to U and hence the corrispondent part of the laplacian vanishes, and only eigenvalues part is left. Taking the derivatives, we get:

$$\frac{\partial\xi(X,t)}{\partial t} = \frac{1}{2}\frac{1}{\Delta^2(X)}\left[2\Delta(X)\frac{\partial\Delta(X)}{\partial x^i}\frac{\partial\xi(X,t)}{\partial x^i} + \Delta^2(X)\frac{\partial^2\xi(X,t)}{\partial^2 x^i}\right]$$

and

$$\frac{\partial}{\partial t}(\Delta(X)\xi(X,t)) = \frac{1}{2}\left[2\frac{\partial\Delta(X)}{\partial x^i}\frac{\partial\xi(X,t)}{\partial x^i} + \Delta(X)\frac{\partial^2\xi(X,t)}{\partial^2 x^i}\right]$$

We now use the fact that Vandermonde determinant is an harmonic function of the eigenvalues (see Appendix 3) and then $\frac{\partial^2\Delta(X)}{\partial^2 x^i} = 0$

$$\frac{\partial}{\partial t}(\Delta(X)\xi(X,t)) = \frac{1}{2}\left[\frac{\partial^2\Delta(X)}{\partial^2 x^i} + 2\frac{\partial\Delta(X)}{\partial x^i}\frac{\partial\xi(X,t)}{\partial x^i} + \Delta(X)\frac{\partial^2\xi(X,t)}{\partial^2 x^i}\right] = \frac{1}{2}\frac{\partial^2}{\partial^2 x^i}\Delta(X)\xi(X,t)$$

Hence, $F(X;t) = \Delta(X)\xi(X,t)$ satisfies the diffusion equation again:

$$\frac{\partial}{\partial t}(F(X,t)) = \nabla^2 F(X,t)$$

With initial condition $F(X,0) = \Delta(X)\eta(X)$.
Again, the solution is provided by heat kernel:

$$F(X,t) = \int dY\,\Delta(Y)\eta(Y)\frac{1}{(2\pi t)^{n/2}}\exp\left[-\frac{1}{2t}\operatorname{tr}(X-Y)^2\right] \tag{5.5}$$

We can now compare (5.4) and (5.5), to get:

$$\Delta(X)\int dY\,\Delta(Y)^2\eta(Y)\exp\Big(-\frac{1}{2t}\operatorname{Tr}X^2\Big)\exp\Big(-\frac{1}{2t}\operatorname{Tr}Y^2\Big)\int dW\exp\left[\frac{1}{t}\operatorname{Tr}\big(XW^\dagger YW\big)\right] =$$
$$(2\pi t)^{\frac{n(n-1)}{2}}\int dY\,\Delta(Y)\eta(Y)\exp\left[-\frac{1}{2t}\operatorname{tr}(X-Y)^2\right]$$
$$= (2\pi t)^{\frac{n(n-1)}{2}}\int dY\,\Delta(Y)\eta(Y)\exp\Big(-\frac{1}{2t}\operatorname{Tr}X^2\Big)\exp\Big(-\frac{1}{2t}\operatorname{Tr}Y^2\Big)\exp\left[\frac{1}{t}\operatorname{tr}(XY)\right]$$

We can now simplify $\exp\big(-\frac{1}{2t}\operatorname{Tr}X^2\big)$. Now, we choose $\eta(Y)$ in order to get our desired integral. Hence, we set $\eta(Y) = \exp\big(\frac{1}{2t}\operatorname{Tr}Y^2\big)\exp\big(-N(\sum_i y_i^2 + 4ge^{-H}y_i^4)\big)$ to obtain the desired expression. Setting now $t = \frac{1}{2Nc}$, multiplying by $\Delta(X)\exp\big(-N(\sum_i x_i^2 + 4ge^{H}x_i^4)\big)$ and integrating in dX gives us:

$$\int dXdY\,\Delta(X)^2\Delta(Y)^2\exp\left[-N\sum_i\big(x_i^2+y_i^2+4ge^Hx_i^4+4ge^{-H}y_i^4\big)\right]\int dW\exp\big[2Nc\operatorname{Tr}\big(XW^\dagger YW\big)\big] =$$
$$\Big(\frac{\pi}{Nc}\Big)^{\frac{n(n-1)}{2}}\int dXdY\,\Delta(X)\Delta(Y)\exp\left[-N\sum_i\big(x_i^2+y_i^2+4ge^Hx_i^4+4ge^{-H}y_i^4+2cx_iy_i\big)\right]$$

This result is equivalent to the one obtained by using the Harish-Chandra-Itzkyson-Zuber formula, which exactely calculates the integral $\int dWe^{2Nc\operatorname{tr}\left(WXW^\dagger Y\right)}$.

$$\int_{SU(n)} dW\exp\left[\frac{1}{t}\operatorname{tr}\left(XWYW^\dagger\right)\right] = t^{\frac{1}{2}n(n-1)}\prod_{j=0}^{n-1}j!\frac{\det\left[\exp\frac{1}{t}\left(x_iy_j\right)\right]}{\Delta(X)\Delta(Y)}$$

5.5 Bi-orthogonal polynomials

After the tour de force proving Harish-Chandra formula, we ended with the result (neglecting numerical factors which will finally simplifies when normalizing):

$$Z = \int dXdY \Delta(X)\Delta(Y) \exp\left[-N\sum_i \left(x_i^2 + y_i^2 + 2cx_iy_i + 4ge^H x_i^4 + 4ge^{-H}y_i^4\right)\right] \tag{5.6}$$

To solve the integral, we exploit the property of the determinant to be invariant under linear combinations of columns, and then we will develop the technique of bi-orthogonal polynomials. First, let's call $v(x,y) = x^2 + y^2 - 2cxy + 4ge^H x^4 + 4ge^{-H}y^4$. We now observe that thanks to the antisimmetry of det under colums exchange, summing to each column of the Vandermonde matrix a linear combination of the preceding ones does not affect the determinant. Then, we can rearrange the matrix in such a way that on each m column we have an arbitrary monic polynomial $[P_m(x_k)]_{k=1...N}^{m=0...N-1}$ of degree m in indeterminate x_k. This allows us to choose the polynomials in a convenient way, and hence to caclulate the integral.
Equation (5.6) becomes:

$$Z = \int dXdY \det\left[P_r(x_k)\right]\det\left[Q_r(y_k)\right]e^{-N\sum_i v(x_i,y_i)} \tag{5.7}$$

and we impose that polynomials satisfy the following relationship of bi-orthogonality:

$$\int dxdye^{-Nv(x,y)}P_k(x)Q_j(y) = h_k\delta_{kj} \tag{5.8}$$

We will prove that the condition can indeed be satisfied if h_k are chosen properly, and that this is sufficient to completely define the polynomials. Before summarizing the principal properties of bi-orthogonal polynomials, we show why this choice will permit us to calculate the integral. We write (5.7) using Levi-Civita symbol:

$$\begin{aligned} Z &= \epsilon^{i_1...i_N}\epsilon^{j_1...j_N}\int dx_1dy_1...dx_Ndy_Ne^{-N\sum_i v(x_i,y_i)}P_{i_1}(x_1)...P_{i_N}(x_N)Q_{j_1}(y_1)...Q_{j_N}(y_N) \\ &= \epsilon^{i_1...i_N}\epsilon^{j_1...j_N}\int dx_1dy_1...dx_Ndy_Ne^{-N\sum_i v(x_i,y_i)}\prod_{l=1}^N P_{i_l}(x_l)\prod_{k=1}^N Q_{j_k}(y_k) = \\ &\epsilon^{i_1...i_N}\epsilon^{j_1...j_N}\prod_{l=1}^N\int dx_ldy_le^{-Nv(x_l,y_l)}P_{i_l}(x_l)Q_{j_l}(y_l) \end{aligned}$$

This is the sum of $(N!)^2$ integrals. Now, using bi-orthogonality condition, we have that the integrals are non zero only when $i_l = j_l \;\forall\; l$. This happens once for every permutation on the N indices (we can change the first N indeces arbitrarily, the second N are determinate by the first since they must be equal), and gives a factor of $\prod_{k=1}^N h_k$. Thus, the final integral is:

$$Z = N!\prod_{k=1}^N h_k$$

5.5.1 Properties of bi-orthogonal polynomials

In this section we expose a series of properties of bi-orthogonal polynomials that will allow us to find the h_k. We firstly recall two properties of orthogonal polynomials which will have an analogous in the bi-orthogonal case:

Proposition 1. *Let $\{P_i(x)\}$ a family of orthogonal polynomials. Then, $P_k(x)$ is orthogonal to every polynomial of degree $\leq k$*

Proof. Let $Q(x)$ be a polynomial of deg $m \leq k$. The orthogonal polynomials of degree $\leq k$ are linarly independent, since they are all of different degree. Moreover, since the space of polynomials of degree $\leq k$ has dimension $k+1$, they are a basis of this space. Then, $Q = \sum_{i=0}^{k} c_i P_i$, and using orthogonality, Q is orthogonal to P_k. □

Proposition 2. *Orthogonal polynomials satisfy a three-term recurrence relation with constants A_k, B_k and C_k [17]:*

$$xP_k(x) = A_k P_{k+1}(x) + B_k P_k(x) + C_k P_{k-1}(x)$$

Proof. Suppose that the recurrence contains also the term $D_k P_{k-2}(x)$, then:

$$\int_\sigma dx x P_k(x) P_{k-2}(x) = D_k$$

Since xP_{k-2} is a linear combination of polynomial of degree $\leq k-1$, the left integral vanishes by orthogonality and gives $0 = D_k$. In the same way one proves the absence of all lower order terms in the recurrence. □

Similar results hold for bi-hortogonal polynomials:

Proposition 3. *Let $Q_k(y)$ a bi-orthogonal polynomial of degree k. Then,*

$$\int dxdy e^{-Nv(x,y)} P(x) Q_k(y) = 0$$

for all polynomials $P(x)$ of degree $m \leq k$.

Proof. Bi-orthogonal polynomials in x variable $P_j(x)$ with $j \leq k$ are a basis for polynomials in x of degree $j \leq k$. Then, $P = \sum_{j=0}^{k} P_j$, and using orthogonality condition (5.8), we get the thesis. □

Proposition 4. *Bi-orthogonal polynomials satisfy the following recursion relations:*

$$xP_k(x) = P_{k+1}(x) + R_k P_{k-1}(x) + S_k P_{k-3}(x) \tag{5.9}$$
$$yQ_k(x) = Q_{k+1}(x) + R'_k Q_{k-1}(x) + S'_k Q_{k-3}(x) \tag{5.10}$$

Proof. $xP_k(x)$ is a polynomial of degree $k+1$, then is certainly a linear combination of the $k+2$ $P_j(x)$ $j \leq k+1$. We suppose that in (5.9) there is a term $T_k P_{k-5}(x)$. We can multiply both sides of (5.9) by $Q_{k-5}(y)$ and integrate with the usual measure. We get:

$$\int dxdy \exp(-Nv) x P_k(x) Q_{k-5}(y) = T_k h_{k-5}$$

Now, to solve the LHS, we need to transform multiplication by x in operations involving y. We observe that

$$\frac{\partial}{\partial y} e^{-Nv(x,y)} = 2N \left(cxe^{-Nv(x,y)} - e^{-Nv(x,y)} \left(y + 8ge^{-H} y^3 \right) \right)$$

Then,

$$cxe^{-Nv(x,y)} = \frac{1}{2N} \frac{\partial}{\partial y} e^{-Nv(x,y)} + e^{-Nv(x,y)} \left(y + 8ge^{-H} y^3 \right) \tag{5.11}$$

Inserting this result in the previous integral, we get:

$$\int dxdy \left[\frac{1}{2N}\frac{\partial}{\partial y}e^{-Nv(x,y)} + e^{-Nv(x,y)}\left(y + 8ge^{-H}y^3\right)\right] P_k(x)Q_{k-5}(y) = cT_k h_{k-5}$$

The second part of the integral gives a polynomial in y of degree $k-2$. Integrating by parts the first part gives a polynomial of degree $k-6$. Then, by proposition (3) the integral is 0, and since $h_k \neq 0$ must be $T_k = 0$. An identical procedure gives (5.10). □

We now search for additional relationships which allow us to determine h_k. Let $f_k = \frac{h_k}{h_{k-1}}$. Then:

Proposition 5. *The following relationships hold:*

$$\begin{aligned}
cS_k &= 8ge^{-H} f_k f_{k-1} f_{k-2} \\
cS'_k &= 8ge^{H} f_k f_{k-1} f_{k-2} \\
cR_k &= \left[1 + 8ge^{-H}\left(R'_{k+1} + R'_k + R'_{k-1}\right)\right] f_k \\
cR'_k &= \left[1 + 8ge^{H}\left(R_{k+1} + R_k + R_{k-1}\right)\right] f_k \\
\frac{k}{2N} &= -cf_k + 8ge^{-H}\left[R'_k\left(R'_{k+1} + R'_k + R'_{k-1}\right) + S'_{k+2} + S'_{k+1} + S'_k\right] + R'_k \\
\frac{k}{2N} &= -cf_k + 8ge^{H}\left[R_k\left(R_{k+1} + R_k + R_{k-1}\right) + S_{k+2} + S_{k+1} + S_k\right] + R_k
\end{aligned}$$

Proof. We will prove relations 1,3 and 5. The other three are completely analogous.

- To prove the first relation, we multiply (5.9) by cQ_{k-3} and integrate with the measure. The result is:

  $$cS_k h_{k-3} = \int dxdy e^{-Nv(x,y)} cxP_k(x)Q_{k-3}(y)$$

  Using (5.11) we get

  $$cS_k h_{k-3} = \int dxdy \left[\frac{1}{2N}\frac{\partial}{\partial y}e^{-Nv(x,y)} + e^{-Nv(x,y)}\left(y + 8ge^{-H}y^3\right)\right] P_k(x)Q_{k-3}(y)$$

  The first two terms in the integral give polynomials in y of degree $\leq k$ and than null contributes. To handle $y^3 Q_{k-3}(y)$, we use (5.10) three times: observe that the only term of degree $\geq k$ which comes out from the recursion relation is exactly $Q_k(y)$. Then

  $$cS_k h_{k-3} = 8ge^{-H} h_k$$

- To prove the second relation, we multiply (5.9) by cQ_{k-1} and integrate with the measure. The result is:

  $$cR_k h_{k-1} = \int dxdy e^{-Nv(x,y)} cxP_k(x)Q_{k-1}(y)$$

  Using (5.11) we get:

  $$cR_k h_{k-1} = \int dxdy \left[\frac{1}{2N}\frac{\partial}{\partial y}e^{-Nv(x,y)} + e^{-Nv(x,y)}\left(y + 8ge^{-H}y^3\right)\right] P_k(x)Q_{k-1}(y)$$

  The first term in the integral is 0. To handle

  $$\int dxdy e^{-Nv(x,y)}\left(y + 8ge^{-H}y^3\right) Q_{k-1}(y)P_k(x)$$

we use again (5.10). The yQ_{k-1} term gives Q_k as the only term of degree $\geq k$. The term y^3Q_{k-1} is more complicated. We need to apply (5.10) three times, and find out all the possible terms of degree k. There are three possible ways to obtain terms of degree k, in total we get $(R_{k+1}+R_k+R_{k-1})P_k$. Using orthogonality relation, this leads to

$$cR_k h_{k-1} = h_k \left[1 + 8ge^{-H}\left(R'_{k+1}+R'_k+R'_{k-1}\right)\right]$$

- To prove the third relation, we multiply (5.9) by cQ_{k+1} and integrate with the measure. The result is:

$$ch_{k+1} = \int dxdy e^{-Nv(x,y)} cxP_k(x)Q_{k+1}(y)$$

Using (5.11) we get:

$$ch_{k+1} = \int dxdy \left[\frac{1}{2N}\frac{\partial}{\partial y}e^{-Nv(x,y)} + e^{-Nv(x,y)}\left(y+8ge^{-H}y^3\right)\right] P_k(x)Q_{k+1}(y)$$

The $yQ_{k+1}(y)$ term gives contribution $R'_{k+1}Q_k(y)$. To handle the term containing the derivative, we integrate by parts: since $Q_{k+1}(y)$ is a monic polynomial, taking the derivative gives a degree k polynomial with coefficent $(k+1)$ for the term of degree k. This polynomial can be written as $(k+1)Q_k + \sum_{i=0}^{k-1} c_i Q_i(y)$(we recall that $\{Q_i(y)\}_{i=0\ldots k}$ are a basis for polynomials o degree $\leq k$). Then, the only contribution is $-\frac{k+1}{2N}h_k$. For the other therms we use a procedure analogous to the one exposed before. The result is:

$$ch_{k+1} = 8ge^{-H}h_k\left[R'_{k+1}\left(R'_{k+2}+R'_k+R'_{k-2}\right)+S'_{k+3}+S'_{k+2}+S'_{k+1}\right]+R'_{k+1}h_k - \frac{k+1}{2N}h_k$$

$\square$

5.6 The limit $N \to \infty$: going to continuum

In principle, we could use relationships of proposition (5) to find the h_k and the appropriate coefficents. However, we recall that we are interested in the $N\to\infty$ limit. We will see that it's more convinient to firstly perform the limit, and only after to solve recursion relations. In fact, when $N\to\infty$ we can make a passage to continuum, and approximate the discrete relations in k of (5) by algebraic equations in a continuum variable.
We observe that $k = 1...N$. Then, $x_k = \frac{k}{N}$ goes from 0 to 1, and the difference between two consecutive values $x_{k+1}-x_k = dx = \frac{1}{N}$ goes to zero in the limit consedered. The relation between k and x_k is one to one, so we can equally use x_k as an index. Hence, we can replace k by Nx. If we accept that $f_k - f_{k-1} \to 0$ if $N\to\infty$, f_k become a continuous function $f(x)$, with $0 \leq x \leq 1$. Then, the relationships (5) become:

$$cS(x) = 8ge^{-H}f^3(x) \tag{5.12}$$
$$cS'(x) = 8ge^{H}f^3(x)$$
$$cR(x) = \left[1+24ge^{-H}R'(x)\right]f(x) \tag{5.13}$$
$$cR'(x) = \left[1+24ge^{H}R(x)\right]f(x) \tag{5.14}$$
$$cx + 2c^2f(x) - 24(4g)^2f^3(x) = 2cR'(x)\left[1+24ge^{-H}R'(x)\right] \tag{5.15}$$
$$cx + 2c^2f(x) - 24(4g)^2f^3(x) = 2cR(x)\left[1+24ge^{H}R(x)\right]$$

We can now use (5.12) (5.13) and (5.14) to obtain $S(x)$ $R(x)$ and $R'(x)$ as a function of $f(x)$:

$$cS(x) = 8ge^{-H} f^3(x)$$
$$R(x) = f(x)\frac{c + 24ge^{-H} f(x)}{c^2 - (24g)^2 f^2(x)}$$
$$R'(x) = f(x)\frac{c + 24ge^{H} f(x)}{c^2 - (24g)^2 f^2(x)}$$

and then, inserting in (5.15), we can find an equation which relates $f(x)$ to x, g and H. In principle, solving the equation gives $f(x)$ and then all the terms of the orthogonality condition. The equation is:

$$\frac{x}{2} = -cf(x) + \frac{12(4g)^2}{c} f^3(x) + c\frac{f(x)}{[c - 24gf(x)]^2} + 48gc^2 f^2(x)\frac{(\cosh H - 1)}{[c^2 - (24g)^2 f^2(x)]^2}$$

which can be written more compactly defining $z(x) = (24g/c)f(x)$:

$$4gx = -\frac{1}{3}c^2 z + \frac{1}{9}c^2 z^3 + \frac{1}{3}\frac{z}{(1-z)^2} + \frac{2}{3}\frac{z^2}{(1-z^2)^2}(\cosh H - 1) \equiv w(z) \tag{5.16}$$

Solving the equation $4gx = w(z)$ would give us f. However, it's clear that the equation is too complex to be solved analytically. Moreover, we are interested not in $f(x)$, but in the continuum limit of $Z = N! \prod_{k=1}^{N} h_k$ or, to be precise, to the logarithm, which will takes connected diagrams. Keeping $f(x)$ implicitely defined by (5.16), we now search for the right expression for the continuum limit desired.
We want to find:

$$F = \log\left(\frac{Z(c, g, H)}{Z_0(c, 0, 0)}\right)$$

We have $Z = N! \prod_{k=1}^{N} h_k$, and $Z_0 = N! \prod_{k=1}^{N} h_k^{(0)}$, where $h_k^{(0)}$ are calculated using the previous relationships and setting $H = g = 0$. For each h_k, holds: $h_k = h_0 \prod_{i=1}^{k} f_i$. Then,

$$Z = N! \prod_{k=1}^{N} h_k = N! h_0^N f_1^{N-1} \cdot ... f_{N-1}^{1} = N! h_0^N \prod_{k=1}^{N-1} f_k^{N-k}$$

Then,

$$F = \log\left(\frac{h_0^N \prod_{k=1}^{N-1} f_k^{N-k}}{{h_0^{(0)}}^N \prod_{k=1}^{N-1} {f_k^{(0)}}^{N-k}}\right) = N \log\left(\frac{h_0}{h_0^{(0)}}\right) + \sum_{k=1}^{N-1} (N-k) \log\left(\frac{f_k}{f_k^{(0)}}\right)$$

We can now select the planar topology by taking the limit:

$$F_{\text{pl}} = \lim_{N\to\infty} \frac{1}{N^2} F = \lim_{N\to\infty} \frac{1}{N} \log\left(\frac{h_0}{h_0^{(0)}}\right) + \frac{1}{N}\sum_{k=1}^{N-1}\left(1 - \frac{k}{N}\right)\log\left(\frac{f_k}{f_k^{(0)}}\right)$$

$\frac{1}{N}\log\left(\frac{h_0}{h_0^{(0)}}\right) \longrightarrow_{n\to\infty} 0$ since the argument of the logarithm is finite. We recall that $x_k = \frac{k}{N}$ and $x_{k+1} - x_k = dx = \frac{1}{N}$. The sum approximates a Riemannian sum, which then become an integral:

$$F_{\text{pl}}(c, g, H) = \int_0^1 dx(1-x)\log\frac{f(x)}{f^{(0)}(x)}$$

Finally, $f^{(0)}(x)$ is easily found, since (5.16) is immediately solved when $g = H = 0$.

$$f_0(x) = \frac{1}{2}\frac{cx}{1-c^2}$$

The last thing to do is to evaluate the integral we have obtained. The integral is in the variable x, but we know $f(x)$ only implicitely (we are supposing that $x(f)$ is bijective near $x = f = 0$). However, we know $x(f)$. We can then make a change of variable, integrating in df, so that the only thing to be found implicitely are the extremes of integration. We will use $z(x)$ instead of f for more simplicity. We have: $x = w(z)$, $dx = \frac{dw}{dz}dz$. Hence:

$$\int_{z(0)}^{z(1)} dz w'(z)(1-w(z)) \log\left(\frac{c(1-c^2)z}{12gcw(z)}\right)$$

The integral can be done, however it is fairly complicated becouse of the logarithm. Integrating by parts before doing the change of variable makes the calculation easier:

$$\int_0^1 dx(1-x)\log\frac{2(1-c^2)f(x)}{cx} =$$
$$\left(x-\frac{x^2}{2}\right)\log\frac{2(1-c^2)f(x)}{cx}\Bigg|_0^1 - \int_0^1 dx\left(x-\frac{x^2}{2}\right)\frac{c}{2(1-c^2)}\frac{x}{f(x)}\frac{d}{dx}\left(\frac{2f(x)(1-c^2)}{cx}\right) =$$
$$\frac{1}{2}\log\left(\frac{z(1)(1-c^2)}{12g}\right) - \int_0^1 dx\left(x-\frac{x}{2}\right)\frac{f'(x)}{f(x)} + \int_0^1 dx\left(x-\frac{x^2}{2}\right)x =$$
$$\frac{1}{2}\log\left(\frac{z(1)(1-c^2)}{12g}\right) + \frac{3}{4} - \int_0^1 dx\left(x-\frac{x}{2}\right)\frac{f'(x)}{f(x)}$$

Now, we use $\frac{dz}{dx} = \frac{24g}{c}f'(x)$, $dx = dz\frac{c}{24gf'(x)}$. We get:

$$F_{\text{pl}}(c,g,H) = \frac{1}{2}\log\left(\frac{z(1)\left(1-c^2\right)}{12g}\right) + \frac{3}{4} - \int_{z(0)}^{z(1)} dz\frac{c}{24gf'(x)}\left(x(z)-\frac{x(z)}{2}\right)\frac{f'(x)}{f(z)}$$
$$= \frac{1}{2}\log\left(\frac{z(1)\left(1-c^2\right)}{12g}\right) + \frac{3}{4} - \int_{z(0)}^{z(1)}\frac{dz}{z}\left(\frac{w(z)}{4g}-\frac{w^2(z)}{32g^2}\right)$$

We consider $z(0) = 0$: the result is hence a function depending on $z(1)$.

5.7 The partition function

Having found the planar free energy of the matrix model, we now have to take the last step towards the partition function of the Ising model.

We recall that F_{pl} is a valid free energy when is analytical in g. Consider now a plot of $y = w(z)$. We find $z(1)$ by intersecting the graph with an horizontal axis of height $4gx|_{x=1} = 4g$. Now, since analycity depends only on the module of g, we can let $4g$ varying on both positive and negative w-axis. Hence, if we find a value of g for which the corresponding $z(1,g)$ has a discontinuity, the integral which gives us F_{pl} can no longer be analytic. This situation happens if $w'(z) = 0$. Imposing this condition, we are thus able to find the critical points g_{cr}, which, using (4.3), will give us F_{Ising}.

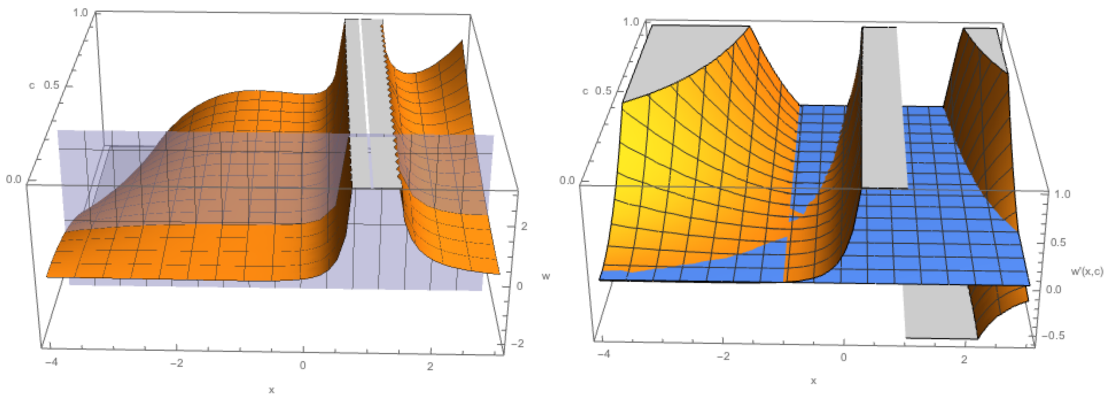

Figure 5.1: The function $w(z, c, 0)$ cut by the plane $c = 1/4$, and its derivative $\frac{\partial w}{\partial z}(z, c, 0)$, cut by $z = 0$. Note that at $c = 1/4$ the two zeros of the derivative superpose at $x = -1$

We get:

$$w'(z, c, H) = \frac{1}{3}\frac{1+z}{(1-z)^3}\left[1 - c^2(1-z)^4\right] + \frac{4}{3}\frac{z\left(1+z^2\right)}{\left(1-z^2\right)^3}(\cosh H - 1) = 0$$

from which we can obtain the condition:

$$\cosh H - 1 = -\frac{(1+z_{cr})^4\left[1 - c^2(1-z_{cr})^4\right]}{4z_{cr}\left(1+z_{cr}^2\right)} \tag{5.17}$$

Using (5.17), we can eliminate $\cosh(H) - 1$ in (5.16), obtaining $4g_{cr} = w_{cr}$ as a function of z_{cr}, and implicitely of H. We get:

$$4g_{cr} = w_{\mathrm{cr}}(z_{cr}(c, H), c) = \frac{5}{18}c^2 z_{cr}\left(z_{cr}^2 - 3\right) + \frac{1}{6}\left(4c^2 + 1\right)\frac{z_{cr}}{1+z_{cr}^2} \tag{5.18}$$

For $H = 0$, we can find the zeros of the derivative exactly. The five solutions are:

$$z_{cr} = -1, \; z_{cr} = 1 \pm \frac{1}{\sqrt{c}}, \; z_{cr} = 1 \pm \frac{i}{\sqrt{c}}$$

Of course we exclude the complex ones. According to the argument given above, only the critical value of z nearer in module to 0 will be the one determinant for the partition function. We recall that $0 < c < 1$. Then, $z_{cr} = 1 + \frac{1}{\sqrt{c}}$ is always greater than 1 and as a consequence unimportant. The two important roots are hence $z_{cr\_1} = -1$ and $z_{cr\_2} = 1 - \frac{1}{\sqrt{c}}$, and marque the two phases. In fact, for $c \le \frac{1}{4}$, z_2 is nearer to 0 than z_1, and for $c \ge \frac{1}{4}$ the roles exchange. At $c = \frac{1}{4}$, the two zeros of the derivative superpose, and there is a minimum of fourth order in the function. Inserting z_{cr} in (5.18), we can find the values of g_{cr}:

$$4g_{cr}(c, 0) = \begin{cases} \frac{2}{9}c^2 - \frac{1}{12} & 0 < c \le \frac{1}{4} \\ -\frac{2}{9}c^2 + \frac{2}{3}c - \frac{4}{9}\sqrt{c} & \frac{1}{4} < c < 1 \end{cases}$$

Finally, we've obtained the so long desired partition function and Ising free energy, for $H = 0$.

Using (4.3), we get:

$$F_{\text{Ising}} = \log\left[\frac{c\,|g_{cr}(c,H)|}{(1-c^2)^2}\right] = \begin{cases} \log\left[\frac{c\left(\frac{2}{9}c^2-\frac{1}{12}\right)}{(1-c^2)^2}\right] & 0 < c \leq \frac{1}{4} \\ \log\left[\frac{c\left(-\frac{2}{9}c^2+\frac{2}{3}c-\frac{4}{9}\sqrt{c}\right)}{(1-c^2)^2}\right] & \frac{1}{4} < c < 1 \end{cases}$$

Of course, F_{Ising} is analytical when $c \neq \frac{1}{4}$. For $c = \frac{1}{4}$, we have a phase transition.
If $H \neq 0$, we can find the perturbations to $z_{cr}(c,0)$ and then to g_{cr} and F_{Ising} to the leading order in H. On the leading order, for H small $\cosh(H) - 1 \approx \frac{1}{2}H^2$. Then, let $z_{cr}(c,H) = z_{cr}(c,0) + \epsilon$.
For $z_{cr}(c,0) = -1$, we get:

$$\frac{1}{2}H^2 = \frac{-\epsilon^4\left[1 - c^2(2-\epsilon)^4\right]}{4(\epsilon-1)(2+\epsilon^2-2\epsilon)} \approx \frac{\epsilon^4(1-16c^2)}{8}$$
$$\implies \epsilon = \frac{\sqrt{2H}}{(1-16c^2)^{1/4}}$$

For $z_{cr}(c,0) = 1 - \frac{1}{\sqrt{c}}$, we get:

$$\frac{1}{2}H^2 \approx \frac{-\left(2-\frac{1}{\sqrt{c}}+\epsilon\right)^4\left[1-c^2(\frac{1}{\sqrt{c}}+\epsilon)^4\right]}{4\left(1-\frac{1}{\sqrt{c}}+\epsilon\right)\left[1+\left(1-\frac{1}{\sqrt{c}}\right)^2\right]} \approx \frac{\epsilon\sqrt{c}\left(2-\frac{1}{\sqrt{c}}\right)^4}{\left(1-\frac{1}{\sqrt{c}}\right)\left(2-\frac{2}{\sqrt{c}}+\frac{1}{c}\right)}$$
$$\implies \epsilon = \frac{H^2\left(1-\frac{1}{\sqrt{c}}\right)\left(2-\frac{2}{\sqrt{c}}+\frac{1}{c}\right)}{2\sqrt{c}\left(2-\frac{1}{\sqrt{c}}\right)^4}$$

We can insert this development in (5.18), and we thus obtain the values of g_{cr}:

$$4g_{cr}(c,H) = \begin{cases} \frac{2}{9}c^2 - \frac{1}{12} + \frac{\sqrt{1-16c^2}}{12}H + o(H) & 0 < c \leq \frac{1}{4} \\ -\frac{2}{9}c^2 + \frac{2}{3}c - \frac{4}{9}\sqrt{c} + \kappa H^2 + o(H^2) & \frac{1}{4} \leq c < 1 \end{cases}$$

Inserting in (4.3) gives the free energy.

Chapter 6

Critical Exponents

Now that we found the free energy of the Ising model on a random quartic planar lattice, we are ready to calculate critical exponents. We recall in the following chapter what a critical exponent is, and why do they arise (see Appendix 4 for further details on the theory).
In statistical physics there are two types of phase transitions. The ones with discontinuous Gibbs free energy are called first-order phase transitions, the others, which have continuous Gibbs free energy but discontinuities in higher order derivatives are called continuous phase transitions (or n-th order phase transition, according to the order of the first discontinuous derivative in the free energy).
Examples of first order phase transitions are the liquid-gas transition in fluid or abrupt change of magnetization in a ferromagnetic medium at low temperature induced by a change of orientation of the magnetic field to which ferromagnet is exposed. Phase transitions of this kind happen when the probability distribution of the system (given by the free energy of the configuration) presents two unequal minima depending on a parameter; the system stays on the absolute minimum (thermodynamics confuses mean value with most probable value invoking central limit theorem). When changes in the parameter switch the relative height of the two minima, the system abruptly passes from one state to another, and a first order phase transition occurs.
When we consider a second (or n-th) order phase transition, a different mechanism occurs. Here, when the temperature changes, the free energy passes from a state with one minimum to a state with two separate, possibly equal, minima. The system chooses one of the minima, broking the symmetry of the original distribution. As a consequence, a quantity with non zero mean value appears: the so-called order parameter. When this happens a second order phase transition occurred; the temperature at which it happens is called critical temperature. The first general theory of second order phase transitions was given by Landau and Ginzburg. However, for subtle reasons it failed to predict right critical exponents. More advanced theories succeded, as scaling theory (see Appendix 4).
The quantities describing the most characterizing properties of a system near a continuous phase transition are critical exponents. These are defined in the following way: near the critical point, we can expand the various thermodynamic quantities in the parameter $(T - T_c)$. The critical exponents will be in general associated to the exponent which governs the behaviour of a certain thermodynamic parameter near the critical point.
We concentrate now on the case of a ferromagnetic system. The parameters on which the free energy depends are the temperature T, the magnetic field H, the magnetization $\langle s \rangle = \frac{\partial F}{\partial H}$, its conjugate parameter. Other interesting thermodynamic quantities are the spacific heat at constant temperature $\frac{1}{N} \frac{\partial U}{\partial T}\big|_H$ and the magnetic susceptivity $\chi = \frac{\partial M}{\partial H}\big|_{H=0}$. We define the

following critical exponents:

- α: specific heat at constant H.
  α is defined as the negative leading exponent of $C_H = C = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{H=0}$ near the critial point, as a function of the adimensional parameter $\epsilon = (T - T_c)/T_c$. Then we search $\alpha :\ C_H\left(\frac{T-T_c}{T_c}\right) = C_H(\epsilon) \approx \epsilon^{-\alpha}$. Since at $c = \frac{1}{4}$ there's a discontinuity of third derivative, which is the derivative of specific heat, we get $\alpha = -1$.

- β: spontaneous magnetization.
  β is defined as the leading exponent governing the behaviour of M near the critical point, as a function of ϵ, in zero magnetic field:$M(c, 0) \approx (-\epsilon)^\beta$. Since M is the conjugate parameter of H, we get:

  $$M(c,0) = \frac{\partial}{\partial H}\log g_{\mathrm{cr}}(c,H)\bigg|_{H=0} = \begin{cases} \frac{3\sqrt{1-16c^2}}{8c^2-3} & 0 < c \le \frac{1}{4} \\ 0 & \frac{1}{4} < c < 1 \end{cases}$$

  Considering that $c_{cr} = \frac{1}{4}$, we have, calling $c_{cr} - c = \epsilon \ge 0$:

  $$M(\epsilon,0) \approx \frac{3\sqrt{1-16(1/4+\epsilon)^2}}{8\cdot\frac{1}{16}-3} \approx \frac{3\sqrt{8\epsilon}}{-\frac{5}{2}} \approx -\frac{12\sqrt{2}}{5}\sqrt{\epsilon}$$

  Then, $M \approx (T_c - T)^{1/2}$ and $\beta = \frac{1}{2}$

- γ: magnetic susceptivity.
  γ is defined as the negative leading exponent which governs the magnetic susceptivity $\chi = \frac{\partial M}{\partial H}\big|_{H=0}$ near the critical point, as a function of ϵ. We get $\gamma = 2$.

- δ.
  δ represent the variation in the magnetization when a magnetic field is turned on near critical point. It is defined by: $M(c_{cr}, H) = |H|^{1/\delta}\,\mathrm{sign}\,H$. To find this exponent, we need to find how z_{cr}, and then g_{cr} and F_{Ising} varies for small H at the critical temperature. We call as before $z_{cr}(c, H) = z_{cr}(c, 0) + \epsilon$ and we get:

  $$\frac{1}{2}H^2 \approx \frac{-\epsilon^4\left[1-\frac{1}{16}(2-\epsilon)^4\right]}{4(-1)(2)} \approx \frac{2\epsilon^5}{8}$$
  $$\Longrightarrow \epsilon = \sqrt[5]{2H^2}$$

  We insert in (5.18) to find:

  $$4g_{cr}\left(z_{cr\_1}, c_{cr}, H\right) = -\frac{5}{72} + \frac{\sqrt[5]{2}}{24}H^{6/5} + o(H^{6/5})$$

  We find:

  $$M\left(c_{cr}, H\right) = \frac{\partial}{\partial H}\log w_{cr}\left(z_{cr\_1}, c_{cr}, H\right) \propto H^{1/5}$$

This exponents are different from the classical ones (to which can be related by KPZ formula), but however they satisfies equally the relationships obtained by scaling theory (Appendix 4). Moreover, Kazakov and Boulatov tested universality solving the problem with cubic vertices, and found the same exponents.

Appendix A

Euler characteristic and classification of 2d compact manifolds

Euler characteristic has been originally defined by Euler on regular polyhedra, when he observed that for a regular polyhedron held $V - E + F = 2$. More generally, is possible to define Euler characteristic for any triangulable surface extending the above definition: this reveals itself to be a topological invariant, and gives a topological classification of $2d$ compact orientable surfaces. We will first describe the process involved in an intuitive way. The general rigorous definition of Euler characteristic for arbitrary manifolds is given in terms of the Betti numbers of Homology group, which will expose briefly later.
We start now defining Euler characteristic for a sphere. The idea is to create a triangulation, namely a non intersecting triangle graph, which grows in points number until it matches the surface at the limit, and to find, starting from a single triangle, a quantity invariant for the addition of new vertices and edges.
Consider hence a tetrahedron. It's immediately verified that $V - E + F = 2$. Moreover, we note that adding a new vertex and connecting it to the tetrahedron requires adding 3 new edges, 2 new faces, since three faces are created and one removed, and of course one new vertex. Hence, $\chi = V - E + F$ remains unchanged. We can then start from a tetrahedron whose vertices lay on the sphere, and then iterate the aforementioned procedure adding more and more vertices on the surface. As the number of points grows, the triangulation matches the surface and we see that $\chi = V - E + F$ is hence a feature of the surface itself. Moreover, in the limit of the graph approaching the surface, any continuous deformation only alters the positions of the vertices but not their connections and their mutual relationships. We can imagine a continuous deformation which makes the edges intersect, but creating a triangulation more strictly matching the surface resolves the problem. We see then that χ is a topological invariant.
Having found Euler characteristic for a sphere, we want to extend it to every other $2d$ manifold. We can hence analyze how cutting holes and gluing together objects affect χ. The effect of an infinitesimal hole is evident: a triangle is removed and everything else is unchanged. Hence, Euler characteristic is diminished by 1. Now, stretching the hole arbitrary does not change χ as before: hence every hole has the same effect of reducing χ by one. It follows immediately that a cylindric surface without the two basic circles has $\chi = 0$. Now, we want to glue a cylinder to a sphere to create a torus: we cut two holes in the sphere, making χ decreasing by two, and then we

glue the two removed bases of the cylinder to the holes. Of course the two Euler characteristics sum together, but since the cylinder has $\chi = 0$, we find that the Euler characteristic of the torus is $\chi = 0$.
To create further surfaces, we introduce now the connected sum of two surfaces. The procedure is the following: we cut one hole in each surface, and then we glue together the edges of the two holes. For example, the connected sum of a manifold with a torus adds a handle to the initial manifold. Let's consider two surfaces M_1 and M_2, and denote the connected sum of the two as $M_1 \sharp M_2$. It's easy to see that

$$\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - 2$$

This holds because cutting the two holes diminishes Euler characteristic by one for each manifold, and gluing together the edges sums the two Euler characteristics. Then, we know that a sphere with a certain number N_{Ha} of handles and a certain number N_{Ho} of holes will have an Euler characteristic of:

$$\chi = 2 - 2N_{Ha} - N_{Ho}$$

We now try to justify heuristically a rigorous proven theorem, which states that every connected compact orientable surface is homeomorphic to the sphere or to the connected sum of a certain number of tori, with some holes if it has a boundary. This, together with the result staten before, implies that χ completely classifies $2d$ orientable surfaces (the result is extandable to non orientable surfaces too, these are obtained summing together a certain number of projective planes).
We first define an n-simplex as the n dimensional analogous of the triangle or tetrahedron. Formally, an n simplex in R^n is, given $n+1$ linerly independent points $v_0, \ldots, v_n$, the set of points Δ^n such that:

$$\Delta^n = \sum_{i=0}^{n} c_i v_i : \sum_{i=0}^{n} c_i = 1 \ \wedge \ c_i \geq 0 \ \forall i$$

$v_0, \ldots, v_n$ are the vertices of the simplex. The order of the vertices is also important: they define the orientation of the simplex. We will denote an n-simplex with an orientation as $[v_0, \cdots, v_n]$. The faces of an n-simplex are defined by taking convex linear combinations of n of the $n+1$ original vertices. The set of all the faces, each one taken with the right orientation, is defined to be the boundary of the simplex (the hat means that the marked vertex is excluded):

$$\partial \Delta^n = \sum_i (-1)^i \left[v_0, \cdots, \hat{v}_i, \cdots, v_n\right]$$

We define a simplicial complex as a set of simplices such that each face of a simplex is contained in the complex (for example, all faces, edges and vertices of a tetrahedron are in a simplicial complex which contains the tetrahedron) and that the intersection of two simplices is a face of both (no superposition). A simplicial complex is homogeneous of dimension n if the only simplicies of dimension different to n are faces of other simplices. In $2d$, homogeneous simplicial complexes are just triangles sharing edges and vertices.
Now, we can define a triangulation of a surface (manifold) as a homomorphism from a simplicial complex to the surface (manifold).
It's a nontrivial theorem in topology that every compact $2d$ surface admits a triangulation. Although the proof is non obvious, we can justify this assertion by adopting an extrinsic approach, which follows the reasoning used to find Euler characteristic for the sphere. We embed the manifold in R^n, and we use the result of the embedding to define a triangulation in R^n which vertices are on the surface. Then, we continuously deform the surfaces of the triangles in such a

way that they approach the surface: this is the desired triangulation.
From now, we follow intuitively Conway's ZIP proof [18]. Take the triangulation and separate all the triangles of the simplicial complex from each other. We know that each triangle is homeomorphic to a sphere with a hole. We want to reconstruct the surface gluing triangles, showing that this can only cause to close holes and adding handles, if the surface is orientable. The only thing that gluing sides of triangles can do is to glue parts of the boundary of a hole with each other or to glue parts of the boundaries of two different holes with each other.
If we glue the whole boundaries of two holes together we have the following possibilities:

- If the two surfaces are initially disconnected, we simply connect them. The number of handles and holes other than the two glued simply adds and nothing else happens.
- If the two holes are on the same connected surface, completely gluing them together preserving orientation adds a handle to the surface (it would have added a Klein bottle if glued not preserving orientation)

If we glue parts of a single hole together we have:

- If two complementary sides of the hole are glued, preserving orientation, the hole is closed and nothing else changes.
- If the two parts glued together are not the entire hole, they just create one more hole (without preserving orientation they could have added a Moebius strip).

If we do not glue all the boundaries of two holes but only a part, we can imagine shrinking all the parts of the boundaries that we are not gluing together to small regions. This means that the procedure is equal to gluing together two whole holes, with a little hole remaining unclosed. Gluing two holes together preserving orientation gives a handle, we then get a handle and some more holes.
We started with a triangle, which is a sphere with a hole. We showed heuristically that gluing triangles together can only create/closing holes, connecting disconnected components or creating handles. Hence, the theorem follows. Then, we found out that Euler characteristic measures the number of handles of a $2d$ surface. We now see why it can equivalently be defined as: $\chi = 2-2g$ where g, the genus of a surface, is the maximum number of cuts along non-intersecting closed simple curves which do not make the resultant manifold disconnected.

A.1 Simplicial homology

The modern approach to Euler characteristic goes through Homology groups. Here we will give only a short introduction which will allow us to give the modern definition. Starting from a simplicial complex, we will now define the simplicial homology of a topological space. The idea is similar to find a triangulation: we try to glue an n simplicial complex and its boundaries, which are of course triangulable, to a space and then we will "move" the triangulation from the simplex to the space. We can hence work with the complex instead of the space. We give the following definition [19]:
A Δ complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$, with n depending on the index α, such that:

- The restriction $\sigma_\alpha| \overset{\circ}{\Delta}{}^n$is injective, and each point of X is in the image of exactly one such restriction $\sigma_\alpha| \overset{\circ}{\Delta}{}^n$.

- Each restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$, where we identify faces of Δ^n and Δ^{n-1} in the obvious way.
- A set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α

We now define $\Delta_n(X)$ as the free abelian group with basis $\sigma_{\alpha_n}\left(\overset{\circ}{\Delta^n}\right)$, namely, all the possible combinations of element of the simplicial complex we choose: $\sum_\alpha n_\alpha \sigma_\alpha \; n_\alpha \in \mathbb{Z}$.
We can extend the definition of a boundary for each $\Delta_n(X)$ taking the boundary of its basis components σ_{α_n}:

$$\partial_n\left(\sigma_{\alpha_n}\right) = \sum_i (-1)^i \sigma_{\alpha_n} | \left[v_0, \cdots, \hat{v}_i, \cdots, v_n\right]$$

Hence, $\partial_n : \Delta_n(X) \to \Delta_{n-1}(X)$.
The core property of the boundary is that the boundary of a boundary is zero: we will always take two sides twice with opposite orientations. Although this is intuitive, we can formally prove it:

$$\begin{aligned}\partial_{n-1}\partial_n(\sigma) = &\sum_{j<i}(-1)^i(-1)^j \sigma | \left[v_0, \cdots, \hat{v}_j, \cdots, \hat{v}_i, \cdots, v_n\right] \\ &+ \sum_{j>i}(-1)^i(-1)^{j-1} \sigma | \left[v_0, \cdots, \hat{v}_i, \cdots, \hat{v}_j, \cdots, v_n\right]\end{aligned}$$

and the two terms cancels out. Since the boundary of a boundary is zero, the boundary of an element of $\Delta_k(X)$ will have a null boundary. However, the converse it's not true: an element with a null boundary, which is called a cycle, not always is a boundary of something else: this depends only on the topology of the space. We then define the k-th homology group $H_k = \operatorname{Ker}\partial_k / \operatorname{Im}\partial_{k+1}$. Thus, we are identifiying two k cycles if they differ by a boundary of an element of Δ_{k+1}. Finding the dimension (the number of independent generators) of H_k it's like counting the k dimensional holes of a manifold.
We report as an example in (A.1) three cases of Δ complexes (which corrisponds to triangu-

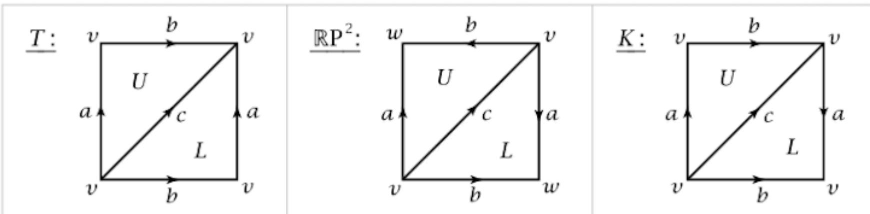

Figure A.1: Three triangulations (Δ complexes) on torus, projective plane and Klein bottle [19]

lations) defined of the torus, the projective plane and the Klein bottle. If we focus now on $2d$ surfaces, we can see the similarity between the two approaches: we searched a way to describe a surface with a triangulation, and we counted the holes inside. If we define the k-th Betti number as the dimension of H_k, then Euler characteristic for two dimensional surfaces can be defined as:

$$\chi = b_0 - b_1 + b_2$$

This is the same as old definition $\chi = V - E + F$: Betti numbers are counting independent vertices, loops and closed faces that does not come from higher dimensional boundaries and hence that give a non zero contribution to Euler characteristic.

Appendix B

An example of Haar measure: the U(N) case

We can give an example of a bi-invariant metric for the Lie Group of unitary matrices of dimension n. For a matrix group, the grorup operation is simply matrix multiplication, which is a linear operation on the vector space $Mat(n, \mathbb{C})$ which contain the group, so action of dL_V on a vector dU is just VdU. Defining on the Lie algebra $\mathfrak{g}$ a metric of the usual form $\langle dX, dY \rangle = tr\,(dXdY)$, we obtain that the metric at some point U is

$$\langle dX, dY \rangle_U = tr\left(U^{-1}dXU^{-1}dY\right) \tag{B.1}$$

It is easy to verify, that this is indeed a bi-invariant metric: we have that $(UV)^{-1}\,d(UV) = V^{-1}U^{-1}dUV$ (this expression has to be considered as a linear combination of the components ot the vector dU). Hence, we obtain $\mathrm{Tr}\left(U^{-1}V^{-1}VdXU^{-1}V^{-1}VdY\right) = \mathrm{Tr}\left(U^{-1}dXU^{-1}dY\right)$ for a left translation by V and, using the cyclic property of the trace, $\mathrm{Tr}\left(V^{-1}U^{-1}dXVV^{-1}U^{-1}dYV\right) = \mathrm{Tr}\left(U^{-1}dXU^{-1}dY\right)$ for a right translation by V. This construction gives us the possibility to calculate volume integrals on classical matrix Lie groups.

Having found an explicit form for the metric, the issue of calculating a volume integral reduces to find a good, global coordinate system which allows us to perform the calculation explicitly. This can be achieved in a quite general fashion for a compact Lie Group [Molinari, non saprei come citarla]. We first observe that the Lie algebra $\mathfrak{g}$ of a Lie group G, being a vector space, admits obviously a global coordinate system. On the other hand, the exponential map is surjective for a compact Lie group, and hence can be used to integrate over the whole group, with appropriate domain restrictions. We can move on the algebra, and use the exponential map to obtain a coordinate system on the group.

We consider on $\mathfrak{g}$ a basis $\{T_1..T_n\}$ orthonormal with respect to the scalar product chosen at the identity (we recall that the tangent space at the identity is canonically isomorphic with $\mathfrak{g}$). The coordinates for a point in the Lie algebra $H = \lambda^i T_i$ will be $\{\lambda_1...\lambda_n\}$. The exponential map send $\mathfrak{g} \ni H \to \exp\{H\} \in G$. Then, we can use $\{\lambda_1...\lambda_n\}$ as a coordinate system for the group. The metric on a point $\exp\{\left(\lambda^i T_i\right)\}$ can now be pulled pack on the point $\lambda^i T_i$ of the algebra using the differential of the exponential: if $\partial_i = \frac{\partial}{\partial T_i}$ are the basis of the tangent space at H in the algebra, we have $\langle \partial_i, \partial_j \rangle_H = \langle d\exp\{(\partial_i)\}, d\exp\{(\partial_j)\}\rangle_{\exp\{H\}}$. The differential of the exponential map, which is far from trivial being the group non commutative, is given by the following:

Theorem 3 (Lie-Trotter formula)**.**

$$de^H = \int_0^1 dte^{(1-t)H}(dH)e^{tH}$$

Proof. A formal proof make use of the parametric expression $\Gamma(s,t) = e^{-sX(t)}\frac{\partial}{\partial t}e^{sX(t)}$ and shows (by a simple derivation by s) that satisfies the differential equation: $\frac{\partial \Gamma}{\partial s} = e^{-sX(t)}\frac{\partial X}{\partial t}e^{sX(t)} = e^{ad_{-sX}}\frac{\partial X}{\partial t}$. Solving the equation and evaluating Γ in $s = 1$ gives the desired result.
However, we can give a more intuitive, less rigorous proof. We have that:

$$\frac{d}{dt}e^{X(t)} = \lim_{n\to+\infty}\left(1 + \frac{X(t)}{n}\right)^n$$

Recalling that $X(t)$ does not commute with $\frac{X(t)}{dt}$, we find, applying the product rule and keeping the right order:

$$\frac{d}{dt}e^{X(t)} = \lim_{n\to+\infty}\sum_{i=1}^{n}\left(1 + \frac{X}{n}\right)^{n-i}\frac{1}{n}\frac{dX}{dt}\left(1 + \frac{X}{n}\right)^i$$

Taking the continuum limit when $n \to +\infty$ we obtain that $\frac{1}{n} = dk$ and $\frac{i}{n} = k$. This is equivalent to write $k = \frac{i}{n}, i \in \mathbb{N}$ and summing over all possible values of k from 0 to 1.

$$\frac{d}{dt}e^{X(t)} = \lim_{n\to+\infty}\sum_{k=\frac{1}{n}}^{1}\left(1 + \frac{X}{n}\right)^{n(1-k)}\frac{1}{n}\frac{dX}{dt}\left(1 + \frac{X}{n}\right)^{nk}$$

When $n \to +\infty$, $\frac{1}{n} = k_i - k_{i-1} \to dk$, the $\sum_k \to \int_0^1 dk$ and $\left(1 + \frac{X}{n}\right)^n \to e^X$. We hence obtain:

$$\frac{d}{dt}e^{X(t)} = \int_0^1 dke^{(1-k)X}\frac{dX}{dt}e^{kX}$$

$\square$

Having in hand a general formula for the differential of the exponential, we now possess an explicit formula to calculate integrals on Lie groups. We can now apply, following Molinari, the construction above to the case of the unitary group of dimension N.
Let $T_1...T_n$ an orthonormal basis of $\mathfrak{g}$, and $H = x^aT_a$ a generic point in the algebra. Let $U = \exp\{(H)\}$ the point in the group G corresponding to coordinates $x^1, ...x^n$ with the previous construction. Then, $dU = de^H = e^H\int_0^1 dte^{-tH}dHe^{tH} = Udx^a\int_0^1 dte^{-x^aT_a}T_ae^{x^aT_a}$. Inserting in the metric (B.1), we obtain,

$$g_{ab}dx^adx^b = dx^adx^b\,\mathrm{Tr}\left[\int_0^1 \mathrm{Ad}_{e^{-tH}}(T_a)\int_0^1 \mathrm{Ad}_{e^{-tH}}(T_b)\right] =$$
$$dx^adx^b\int_0^1\int_0^1 dt_1dt_2\,\mathrm{Tr}\left[e^{t_1H}T_ae^{(t_1-t_2)H}T_be^{t_2H}\right]$$

We perform now the change of variable: $t = t_2 - t_1$ and $s = \frac{t_1+t_2}{2}$. The new coordinates are to be integrated in the intervals: $-1 < t < 1$ and$|\frac{t}{2}| < s < 1 - |\frac{t}{2}|$. Using the cyclic property of the trace, we get

$$dx^adx^b\int_{-1}^1\int_{|\frac{t}{2}|}^{1-|\frac{t}{2}|} dtds\,\mathrm{Tr}\left[e^{\left(-s+\frac{t}{2}\right)H}T_ae^{-tH}T_be^{\left(s+\frac{t}{2}\right)H}\right] =$$
$$dx^adx^b\int_{-1}^1 dt\,\mathrm{Tr}\left[\mathrm{Ad}_{e^{tH}}(T_a)\,T_b\right]\int_{|\frac{t}{2}|}^{1-|\frac{t}{2}|} ds$$

Now, let $[T_a, T_b] = f_{ab}{}^{k} T_k$, where $f_{ab}{}^{k}$ are the structure constants of the group in the chosen basis. Then, $\mathrm{ad}_{\lambda^i T_i}\left(\mu^j T_j\right) = \lambda^i f_{ij}{}^{k} \mu_j T_k$. Moreover, since the adjoint representation of the group is the exponential of the one of the algebra, we get $\mathrm{Ad}_{e^{\lambda^i T_i}} = \exp\{(\mathrm{ad}_{\lambda^i T_i})\}$ and thus $\mathrm{Ad}_{e^{tH}}\left(T_a\right) = e^{x^i f_{ia}{}^{l}} T_l$. Then, $\mathrm{Tr}\left[\mathrm{Ad}_{e^{tH}}\left(T_a\right) T_b\right] = e^{tx^i f_{ia}{}^{l}}\,\mathrm{Tr}\left[T_l T_b\right] = e^{tx^i f_{iab}}$. We then obtain

$$g_{ab} dx^a dx^b = \int_{-1}^{1} dt\,(1-|t|)\,e^{tx^i f_{iab}}$$

If we call $x^i f_{iab} = M_{ab}$ $(M : \mathfrak{g} \to \mathfrak{g})$, we have, from the antisymmetry of the structure constants, that $M = -M^\dagger$. Then M is diagonalizable and we get, if $i\lambda_k$ are the eigenvalues of M:

$$g_{ab} = \int_{-1}^{1} dt\,(1-|t|)\,e^{i\lambda_a}\delta_{ab} = \frac{\sin^2\left(\lambda_a/2\right)}{\left(\lambda_a/2\right)^2}\delta_{ab}$$

Appendix C

The Vandermonde determinant

Here we introduce briefly Vandermonde matrices and Vandermonde determinant. Consider a vector with components:$\{\lambda_i\}_{i=1}^n$. Then, a Vandermonde matrix is a matrix of the form $T_{ij} = [\lambda_i]^{j-1}$:

$$\begin{pmatrix} 1 & \lambda_1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \dots & \lambda_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_n & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{pmatrix}$$

Vamdermonde matrices has different applications through mathematics and physics: for example, the DFT matrix is a Vandermonde matrix containing roots of unity. Given an n dimensional vector, the matrix which impement DFT is:

$$\mathcal{F}_{jk} = \frac{1}{\sqrt{n}} \left[e^{i\frac{2\pi}{n}} \right]^{(j-1)(k-1)} \qquad 1 \leq i, j \leq n$$

In the present work, the determinant of a Vandermonde matrix arise in connection with the volume element in matrix Lie Groups, and λ_i are the eigenvalues of a matrix. We hence will refer to λ_i as eigenvalues hereafter, and call Λ the vector composed by λ_i. We now define the Vandermonde determinant:

$$\Delta(\Lambda) = \det\big([\lambda_i]^{j-1}\big)$$

The principal property of this determinant is that it can be written in a simple form in terms of the eigenvalues. The following Proposition holds:

Proposition 6. $\Delta(\Lambda) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i)$

Proof. $\Delta(\Lambda)$ is an homogeneus polynomial in the λ_i. Since every term of the determinant is a prodoct of n different factors, each one having degree from 0 to $n-1$ in its indeterminate λ_i, the total degree of every term is $\sum_{i=0}^{n-1} i = \frac{n(n-1)}{2}$.
We can now consider $\Delta(\Lambda)$ as a polynomial of degree $n-1$ on every λ_i on turn. Evaluating this new polynomial in every other $\lambda_j,\ j \neq i$ corresponds to substitute λ_j to λ_i in the determinant. This gives two equal columns in the matrix, hence the determinant is 0, hence λ_j is a root of this polynomial. Since this holds for every λ_i and λ_j, using unique factorization theorem for all the polynomials in the various λ_i we get that $(\lambda_i - \lambda_j)$ are factors of the original polynomial:

$$\Delta(\Lambda) = Q(\lambda_1, ..\lambda_n) \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \tag{C.1}$$

But the product is already of degree $\frac{n(n-1)}{2}$, hence Q is a constant. Since all the terms in the original sum have coefficent one, and the product in (C.1) gives factors with coefficent 1, $Q = 1$. Hence we get:

$$\Delta(\Lambda) = \prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)$$

This can also be proved constructively by using elementary rows and columns operations and properties of the determinant. □

We state and prove now a last useful fact.

Proposition 7. *Let $\{\lambda_i\}_{i=1}^n \in \mathbb{R}^n/\mathbb{C}^n$ be n continuous parameters. Then, $\Delta(\Lambda)$ is an harmonic function of $\{\lambda_i\}_{i=1}^n$:*

$$\nabla^2(\Delta(\Lambda)) = 0$$

Proof. Let $i = 0...(n-1)$ and $\sigma(i)$ a permutation of i's. Then, $\Delta(\Lambda) = \sum_\sigma (-1)^{\epsilon(\sigma)} x_1^{\sigma(1)} ... x_n^{\sigma(n)} = \sum_\sigma (-1)^{\epsilon(\sigma)} \prod_{i=0}^{n-1} x_i^{\sigma(i)}$. For each j, $\frac{\partial^2 \Delta(\Lambda)}{\partial \lambda_j^2}$ is an homogeneous polynomial of degree $\frac{n(n-1)}{2} - 2$, where the maximum degree of λ_j is $n-2$. Each term in $\nabla^2 \Delta(\Lambda)$ can come from two possible terms in the original $\Delta(\Lambda)$: from $\frac{\partial^2}{\partial \lambda_i^2} x_1^{a_1} ... x_i^{k+2} x_j^k ... x_n^{a_k}$ and from $\frac{\partial^2}{\partial \lambda_j^2} x_1^{a_1} ... x_i^k x_j^{k+2} ... x_n^{a_k}$ the same term $k(k-1) x_1^{a_1} ... x_i^k x_j^k ... x_n^{a_k}$ is obtained (for all other possible choices in the a_i, such that the exponents are overall a permutation $\sigma(i)$), and no other term will contribute to this monomial. But $x_1^{a_1} ... x_i^{k+2} x_j^k ... x_n^{a_k}$ and $x_1^{a_1} ... x_i^k x_j^{k+2} ... x_n^{a_k}$ coem in opposite sign in $\Delta(\Lambda)$ because of the antisymmetry of the determinant for exchange of columns. Hence, the two contributes cancels out, and the Vandermonde determinant is harmonic. □

Appendix D

Ginzburg-Landau theory and scaling

We give here an example of Ginzburg-Landau theory. Landau observed that, in a continuous phase transition, there is always an order parameter which is zero in the first phase, and becomes non zero because of the symmetry breaking is the second phase. Consider the free energy as a function of the order parameter ϕ: $G = G\,(T, P, \phi, N)$. Since near the critical point the order parameter is small, this suggests an expansion of the free energy in a power series on the order parameter:

$$G = G_0(T, P) + G_1(T, P)\phi + G_2(T, P)\phi^2 + G_3(T, P)\phi^3 + \cdots$$

Now we exploit the symmetries of the system to remove terms from the free energy. We consider the case of an interaction with a magnetic field: the order parameter in this case would be the magnetization m, and the conjugate intensive parameter the magnetic field $B = \frac{\partial G}{\partial m}$. In a one dimensional system m is a scalar. Since the system, at $B = 0$ is invariant for flipping the magnetization, only terms with even power can appear. If we had considered the model in more dimensions, the order parameter would have become a vector. Its rotational invariance would have implied the dependence of G only to the module of the magnetization. For a general system, this reasoning brings to different consequences case by case. We now focus on the $1d$ example, since already contains all the element of interest. The free energy is reduced to:

$$G\,(T, P, \phi, N) = G_0(T, P) + G_2(T, P)\phi^2 + G_4(T, P)\phi^4 + ...$$

For small ϕ, we can retain the first two terms only, and we will suppress the dependence from the pressure, which is unrelevant for now. We observe that the qualitative plot of G depends on the signs of G_2 and G_4. When both $G_2 > 0$ and $G_4 > 0$ the plot has a single minimum for $\phi = 0$, if G_2 changes sign, then two symmetrical minima appears on opposite ϕ values and $\phi = 0$ becomes an unstable maximum point. Hence, in this second region the real stable thermodynamic curve will have an horizontal plateau connecting the two minima (and hence a discontinuous first derivative if G is seen as a function of B), and the passage from one to another will denote a classical first order phase transition. As a consequence, we define the critical temperature as the temperature T_c at which G_2 change sign. This is the point at which second order phase transition happens. We can moreover expand G_2 in powers of $(T - T_c)$:

$$G_2[T, P(T)] = (T - T_{cr})\, G_2^0 + O(T - T_c)^2$$

Since G_4 remains positive, we will keep implicit its temperature dependence: it does not influence the qualitative behaviour. Now we show what is intuitively clear: under T_c, for a zero magnetic

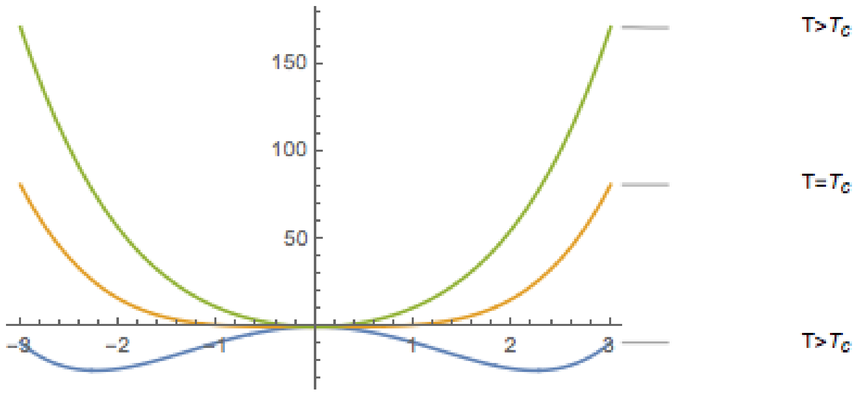

Figure D.1: The Ginzburg Landau free energy for $T < T_c$, $T = T_c$ and $T > T_c$

field there will be two opposite symmetric stabel states with nonzero magnetization. We just impose $B = \frac{\partial G}{\partial \phi} = 0$ and solve the equation:

$$\frac{\partial G}{\partial \phi} = 2\left(T - T_c\right) G_2^0 \phi + 4 G_4(T) \phi^3 = 0 \tag{D.1}$$

We see that the sign of the coefficent of ϕ $2\left(T - T_c\right) G_2^0$ determines the number of solutions: for $(T - T_c) > 0$ there is only one real solution: 0. For $(T - T - c) < 0$, two minima appears in

$$\phi = \pm \left[2 \frac{G_2^0}{G_4(T)} \left(T_{cr} - T\right) \right]^{1/2}$$

and $\phi = 0$ become a maximum: a second order phase transition has occured. This theory allows us to calculate the behaviour of the interesting thermodynamical quantities near T_c, as power laws in $(T - T_c)$. Critical exponents are the leading order exponents related to the behaviour of the various quantities.

We see immediately that spontaneous magnetization at zero magnetic field goes as $\left(T_{cr} - T\right)^{1/2}$, hence $\beta = \frac{1}{2}$. The specific heat at constant temperature $-T \left(\frac{\partial^2 G}{\partial T^2} \right)_B$,which gives the exponent α, does not diverge nor goes to 0 in general, it simply approach a constant. Hence, $\alpha = 0$. The susceptibility $\chi = \frac{\partial \phi}{\partial B} = \left[\left(\frac{\partial^2 G}{\partial \phi^2} \right)_{T, \phi \to 0} \right]^{-1}$ is find deriving G twice:

$$\frac{\partial^2 G}{\partial \phi^2} = 2\left(T - T_{cr}\right) G_2^0 + 12 G_4(T) \phi^2$$

Taking the limit $\phi \to 0$, we obtain:

$$\chi = 2\left[\left(T - T_{cr}\right) G_2^0 \right]^{-1}$$

The critical exponent γ is then $\gamma = 1$. The last exponent δ is realated to $\left.\frac{\partial \phi}{\partial B}\right|_{T_{cr}}$ and tells how ϕ react to changes is his conjugate parameter B at T_c. This can be read directly from (D.1), and gives $\delta = 3$.

Landau Ginzburg theory can also considered as a mean field theory: one consider G as a function of both ϕ and his conjugate parameter B:

$$G(T, B, \phi) = G_0(T) + G_2(T)\phi^2 + G_4(T)\phi^4 - B\phi$$

Then one minimizes G, and use the abslute minimum value as the actual value of G, finding hence the relationship betwenn ϕ and B. This theory seems to be really general, relying only on the postulates of thermodynamics, and on symmetry considerations. However, although it predicts correctly the qualitative behaviour, it fails in predicting the right critical exponents. This is because of an oversimplification in thermodynamics: it substitutes the mean value with the most probable value [20]. Since the probability distribution is generally really sharp, this does not lead to errors in the thermodynamic limit. However, near the critical point the distribution loses this characteristic, and long range correlations become important. Hence, to obtain the right exponents, one has to deal with the full statistical mechanic treatment of the system, starting from its partition function.

In spite of this, a general theory has been developed, which offers at least relationships between various critical exponents: the scaling theory. The idea is to use as an help the long range correlations emerging at critical point: we thus hypotize that, near T_{cr}, moving away from critical point does not change the form of the free energy, but only its scale. This means, if x is a parameter on which free energy depends, that holds:

$$G(\lambda x) = g(\lambda)G(x)$$

We now show that this is equivalent to say that $G(\lambda x) = \lambda^p G(x)$, or that $g(\lambda)$ is a polynomial. From here, the discussion is primarily taken from [21].

First, consider

$$G(\lambda\mu x) = g(\lambda\mu)G(x) = g(\lambda)g(\mu)G(x)$$

Hence, we have that

$$g(\lambda\mu) = g(\lambda)g(\mu) \tag{D.2}$$

Now, if we derive (D.2) with respect to μ, we get:

$$\frac{\partial}{\partial\mu}g(\lambda\mu) = \lambda g'(\lambda\mu) = g(\lambda)g'(\mu)$$

Now, we pose $\mu = 1$, $g'(1) = p$. We get:

$$\lambda g'(\lambda) = pg(\lambda) \implies \frac{\lambda}{p} = \frac{d}{d\lambda}(\log(g))$$

Hence, $g(\lambda) = \lambda^p$, and $G(\lambda x) = \lambda^p G(x)$. In two variables, this becomes

$$G(\lambda^p x, \lambda^q y) = \lambda G(x, y) \tag{D.3}$$

A relationship of this kind means that G does not depend freely on the two parameters: the two are correlated, and we can exploit this correlation to find relationships between p and q and the various critical exponents.

Let now be $x = (T - T_c)$ and $y = B$. We calculate the various critical exponents:

- For the heat capacity, we have $C = -T\left(\frac{\partial^2 G}{\partial T^2}\right)_{B=0}$. We differentiate both sides twice (D.3):

  $$\lambda^{2p}C(\lambda^p(T - T_{cr}), \lambda^q B|_{B=0}) = \lambda C((T - T_{cr}), 0)$$

  Since λ can be chosen freely, we can set is so that it makes the first argument of G equal to 1, balancing $(T - T_{cr})$. We choose hence $\lambda = (T - T_{cr})^{-1/p}$, which gives:

  $$C((T - T_{cr}), 0) = (T - T_{cr})^{(1-2p)/p}C(1, 0)$$

  So we find $\alpha = 2 - \frac{1}{p}$: we moved the dependency by $(T - T_{cr})$ outside the function thanks to the scaling hypotesis.

- For magnetization we follow an analogous procedure: We derive (D.3) with respect to B and set $B = 0$ and $\lambda = (T_{cr} - T)^{-1/p}$. The result is

$$\begin{aligned} \lambda^q M\left(\lambda^p (T - T_{cr}), \lambda^q B\right) &= \lambda M((T - T_{cr}), B) \\ M(T - T_{cr}, 0) &= (T_{cr} - T)^{(1-q)/p} M(-1, 0) \end{aligned} \tag{D.4}$$

  and $\beta = \frac{1-q}{p}$.

- For susceptibility, again we follow the same procedure. We derive (D.3) respect to B, set λ to the usual value and get:

$$\chi(T - T_{cr}, 0) = (T - T_{cr})^{(1-2q)/p} \chi(1, 0)$$

  Hence, $\gamma = \frac{2q-1}{p}$

- Finally we find δ. We start from (D.4) and we impose $T = T_c$ and $\lambda = B^{-1/q}$. We obtain:

$$M(0, B) = B^{(1-q)/q} M(0, 1)$$

  and $\delta = \frac{q}{1-q}$.

We found all critical exponents in terms of the unknown p and q. We can hence find p and q in terms of, for example, α and δ:

$$p = (2 - \alpha)^{-1} \text{ and } q = \frac{\delta}{1 + \delta}$$

Inserting in the expression for β and γ we get $\beta(1 + \delta) = 2 - \alpha$ and $\gamma = \frac{\delta-1}{\delta+1}(2 - \alpha)$. Combining the equations gives Widom's scaling law:

$$\frac{\gamma}{\beta} = \delta - 1$$

And Rushbrooke law:

$$\alpha + 2\beta + \gamma = 2$$

These very general relationships need to be satisfied by every set of critical exponents of a system which respects the general hypothesis on the basis of scaling theory. In reference to the present work, it's immediately checked that the exponents found for Ising model in a random planar lattice indeed satisfy scaling relationships.

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