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# Abrikosov lattice states in type-II superconductors within the Ginzburg-Landau theory (a mathematical approach)

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# Introduction

The Ginzburg-Landau theory is a phenomenological and macroscopic model that employs the thermodynamical formalism to describe continuous phase transitions. An early, famous application was the conductive-superconductive transition, which had been observed at the beginning of the century but had not yet been studied with a proper microscopic theory.
A remarkable result of the application of the Ginzburg-Landau theory to the study of the magnetic properties of superconductors was the prediction of an intermediate state (usually called mixed or vortex phase) in some materials that matched the experimental observations: the magnetic field nucleates in localized, isolated regions that serve as cores for vortices of superconductive current whose flow annhilates the field outside. This was carried out at first by A.A. Abrikosov in the work [2] (1957) and earned him the 2003 Nobel prize, along with Ginzburg and Leggett, "*for pioneering contributions to the theory of superconductors and superfluids*".

In the first chapter of this thesis we are going to present the Ginzburg-Landau theory for superconductivity for an axial-symmetric sample and we are going to see how a natural classification of superconducting materials follows from it.
In the second chapter we are going to describe the intermediate Abrikosov state giving it a more precise mathematical setting, mainly following the works [23, 29, 28] by I.M. Sigal and T. Tzaneteas. At first we will introduce a description of lattices and will exploit the dimensional reduction to define lattice shapes in a very natural way, then we will introduce the concept of equivariance of a superconducting state wrt actions of the lattice translations and gauge group and we will see a peculiar physical property that follows. Then, we will study the linearized Ginzburg-Landau equations close to the normal-mixed phase transition with a perturbative approach similar to the one introduced by Abrikosov. This will allow to compute the critical field and the most stable configuration close to it. At last, we are going to find an approximate expression for the critical field that marks the mixed-superconductive transition.

Throughout this thesis we are going to use the CGS system in the equations while the experimental measures will be given in SI units.
Vectors will be denoted with bold letters (e.g. $\boldsymbol{x}$), scalars and complex num-

bers with ordinary letters (both latin and greek, $i$ is the imaginary unit). The complex conjugate is expressed by starred characters. The gradient, divergence, curl and Laplace operators are marked respectively with $\nabla, \mathrm{div}, \mathrm{curl}, \triangle$.

# Contents

# Chapter 1

# Ginzburg - Landau theory

## 1.1 Sketch to the phenomenology of superconductors in magnetic field

The superconductive state of matter was first observed at the beginning of the XX century, its main feature being the vanishing of electrical resistivity when the transition occurs, i.e. when the termperature of the sample is brought below a certain critical value. The main difficulty in observing such state is that the critical temperature is usually quite low (between 0 and 5 K). Nevertheless, in the last centuries much progress have been made in cooling techniques and, consequently, in the experimential study of the superconducting state.
In particular, starting from the Thirties, important observations have been carried out regarding the interaction of a superconductor with an external magnetic field. We will focus our phenomenological description on these results.

| Material | $T_C$ [K] | $\Theta_D$ [K] | Material | $T_C$ [K] | $\Theta_D$ [K] |
|---|---|---|---|---|---|
| Al | 1.14 | 305 | Sn | 3.69 | 180 |
| Cd | 0.54 | 158 | Ta | 4.38 | 246 |
| Ga | 1.07 | 125 | Th | 1.32 | 200 |
| Hg | 4.12 | 69 | Tl | 2.38 | 100 |
| In | 3.37 | 150 | U | 1.25 | 141 |
| Pb | 7.26 | 86 | V | 4.3 | 69 |
| Re | 0.95 | 283 | Zn | 0.79 | 230 |

Table 1.1: Critical temperature of some elements compared to the Debye temperature.

### 1.1.1 Meissner effect

Chronologically, the first peculiar magnetic property of the superconductive phase is the so called Meissner effect, observed by Meissner and Ochsenfeld in

1933.
First of all they noticed that, for a given material, the temperature alone does not determine its state uniquely as the normal phase can occur below $T_C$ when an external magnetic field is switched on with a magnitude $H_0 > H_C(T)$. On the contrary, above $T_C$ no external field can induce the transition (i.e. $H_C = 0$ for all $T > T_C$).
Furthermore, they realized that when the material "enters" the superconductive phase, the external field starts being repelled, i.e. except for a thin surface layer, the field in the bulk of the sample is zero. This phenomenon is completely memory-free: if one lets the magnitude of the magnetic field oscillate around $H_C$, regardless of what has happened before the superconductor will repel all fields below the critical one and let all the other soak through. Hence a superconductor is not only a perfect conductor but also a perfect diamagnet.
The physical reason for a shielding of the field inside the superconductor is, of course, a generation of another field of opposite orientation and equal magnitude by some current that is somehow generated inside the superconductor.

Actually, a more precise analysis brought to the conclusion that the dimension of the region in the superconductor in which the field has still perceivable magnitude strongly depends on the geometrical properties of the material. As a consequence, we call hard superconductors (or of type I) those for which the penetration depth of the magnetic field is particularly mild.

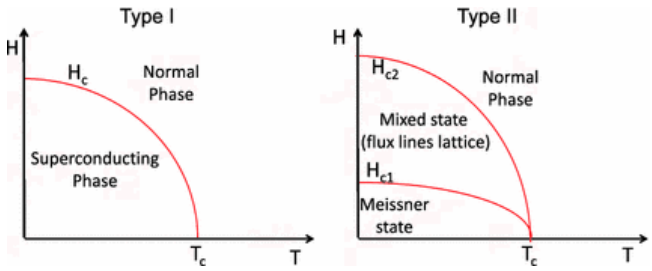


**Figure 1.1:** *Schematic phase diagrams for the two kinds of superconducting materials.*

### 1.1.2 Vortex state

However, an extremely different response to an applied field has been observed and has led to the conclusion that there exists a different class of superconductors, called of type II.
In such materials, at fixed temperature, two transitions occur at two different values for the magntiude of the external field, namely $H_{C1} < H_{C2}$ (both functions of $T$ as in Figure 1.1).
For fields $H < H_{C1}$, the superconductor behaves roughly as we have so far described, even though typically the penetration depth of the field is much broader, while for $H > H_{C2}$ the material behaves like an usual conductor. The

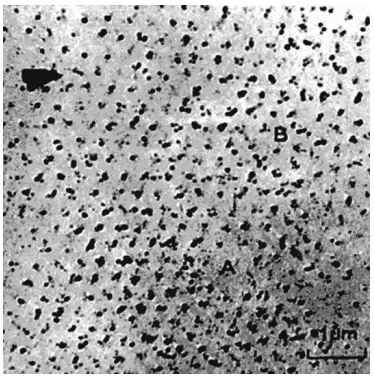


**Figure 1.2:** *One of the first vortex lattice images in a sample of Pb with 4% In at* $1.1\,\mathrm{K}$ *with a field of* $5 \times 10^{-2}\,\mathrm{T}$.

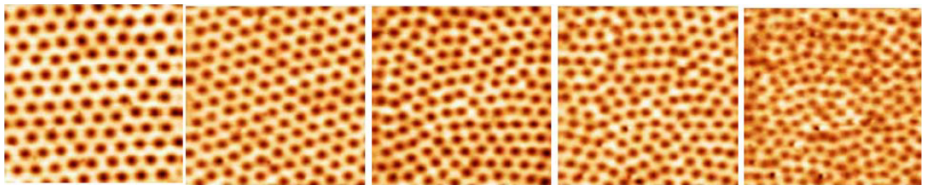

**Figure 1.3:** *Lattice images for increasing magnetic field (from left to right:* $1.8\,\mathrm{T}, 2.3\,\mathrm{T}, 2.5\,\mathrm{T}, 2.7\,\mathrm{T}$ *and* $3.3\,\mathrm{T}$*) in a sample of doped Co (0.4% Co).*


huge difference lies in the intermediate region $H \in (H_{C1}, H_{C2})$ where the material "lets" some of the magnetic field soak through, even though the penetration is not thorough because the superconductive properties have not been yet destroyed. Hence the sample exhibits a mixed state in which the field gathers in some confined and discretized regions and the superconductive currents "tries" not to let it leak by flowing around such areas thus generating an opposing field. In other words, the nonzero-field regions (which are, by any mean, areas in which the material is in the normal conductive phase) behave like cores of vortices of superconductive current. For this reason, the mixed state is sometimes called vortex phase.
The aim of this thesis is the study of such peculiar state within the macroscopic Ginzburg-Landau theory, following the work by Abrikosov [2] who first predicted its existence in the Fifties. We will see that the cores of the vortices tend not to arrange themselves in random fashion but form periodic lattices. We will study the energetic stability of such lattices and see which configuration is energetically favourable. Firts of all we are going to introduce the Ginzburg-Landau theory for the description of superconductors.

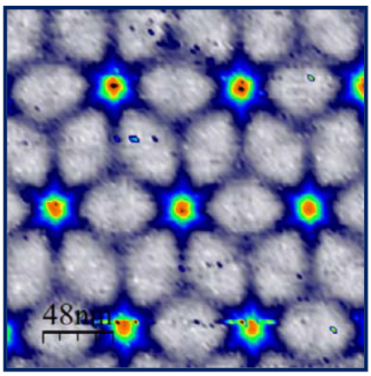


**Figure 1.4:** *Vortex core structures of 2 $H-NbSe_2$ for a field of* 0.15 T.

## 1.2 Ginzburg-Landau free energy functional

Ginzburg-Landau theory (G-L theory in the following) gives a macroscopical description of the superconductive state based on Thermodynamics. The key idea is to treat a superconductor as a thermodynamical system that can undergo some phase transitions along certain critical lines in the plane $T - H_0$.
In order to reach our aim, according to Landau theory for phase transitions, we need to postulate the existence of a *order parameter* $\psi$ which is zero in the normal phase and non-zero in the conductive phase. The physical significance of such parameter cannot be probed in such a macroscopic theory but there is a need for a more fundamental description. This has been done: the microscopical theory for superconductivity, named after Bardee, Cooper and Schrieffer (BCS theory), which we won't be going through here, interprets the squared modulus of the order parameter as the density of charge carriers. In the superconductive state such carriers are electrons in a coupled state which is found to be energetically more convenient at low temperature (Cooper pairs).
Going back to G-L theory, as a consequence of it being a thermodynamical theory, it is clear that it can describe only steady-states and that we need to introduce a proper potential. The problem is thus transposed to its minimization under certain experimental conditions (i.e. external constraints). A typical choice is that of fixed temperature and it is well-known that in such condition the equilibrum is found via minimization of the Helmholtz free energy. Our first task is, therefore, to find an analytic expression for $F$.

### 1.2.1 Landau expansion & Ginzburg kinetic term

An exact expression for $F$ is doomed to strongly depend on the microscopical phenomenon that give rise to what we macroscopically detect as "superconductivity". In other words, to make Termodynamics work we need to rely on some other theory that tells us where to start from. Since at this point we are completely ignorant about what goes on at small scales, the idea of Landau is to consider instead a series expansion of the free energy density $f$ in terms of the order parameter in a neighborhood of the critical point.

If we are to interpret $\psi$ as describing a density of superconductive corrent carriers, we expect the free energy to depend only on the observable quantity $|\psi|$. Actually, since we postulated $\psi = 0$ in the normal state and every neighborhood of the critical point is bound to contain a infinite number of conductive-state representatives, if we require analiticity of $f$, we cannot let it depend on odd powers of $|\psi|$ (which are not differentiable when $\psi \to 0$).
The problem then turns to picking the order at which we should truncate the expansion in order to have reasonable results. We immediately see that the first order alone is not enough: if $f = \alpha|\psi|^2$ (where $\alpha$ is a function of the other thermodynamical coordinates, namely the temperature), then the only solution for $|\psi|$ we find by minimization is $\psi = 0$, i.e. there exists no superconductive state. Let us, then, add one more term:

$$f = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4$$

Such density has infinitely-many stationary points in the complex plane: the origin $\psi = 0$ and the circle $\left\{z \in \mathbb{C} : \quad |z|^2 = -\frac{\alpha}{\beta}\right\}$. An evaluation of the second derivative of $f$ wrt to $|\psi|$ on the two sets leads to the conclusions:

- $\psi = 0$ is an energy minimizer iff $\alpha > 0$.
- $|\psi|^2 = -\frac{\alpha}{\beta}$ minimizes $f$ iff $\alpha < 0$.

So it is clear that in the expansion the sign of the coefficient $\alpha$ of the first order term determines whether the material is in the normal or in the superconductive state. Since we expect the expansion for $f$ to hold in a neighborhood of the critical point, which is, in the given thermodynamical setting, only determined by the critical temperature $T_C$, we may write:

$$\alpha(T) = \alpha_1\left[\frac{T}{T_C} - 1\right] + \mathcal{O}\left(\frac{T}{T_C} - 1\right)^3$$

where $\alpha_1$ is a positive constant ($\alpha(T_C) = 0$ because we require continuity).
Note that, since the superconductive state is never observed in ordinary conductors, we can infer that it is not allowed even as a maximum of $f$ (i.e. an unstable equilibrium). This is true iff, for $T > T_C$, $\psi = 0$ is the only stationary point for $f$, i.e. $\beta > 0$ (since we already established that $\alpha > 0$ in such region). Since $\beta$ must be positive even for $T < T_C$, its leading order in the expansion is the zeroth, i.e. $\beta(T) \approx \beta(T_C) > 0$ neglecting terms of order $(T - T_C)^2$.

Now, since none of the coefficients of the expansion is a function of the position in the superconductor, there is a one-to-one correspondence between the temperature and a value of $\psi$ constant in $\Omega$. We want a model that can take into account the case in which the order parameter may vary from point to point. In order to do that it is necessary to add to the free energy density a term that depends on the gradient of $\psi$. If we want the above conditions to still hold, the lowest order term is proportional to $|\nabla\psi|^2$, i.e.:

$$f = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \gamma|\nabla\psi|^2$$

The interpretation of $\psi$ as a wavefunction suggests that we may identify the term proportional to the gradient as a sort of kinetic energy term, i.e. $\gamma = \frac{\hbar^2}{2m^*}$ for some effective mass $m^*$.
As a consequence, the free energy for a superconductor in a magnetic field with vector potential $\boldsymbol{A}$ is obtained via the usual sostitution $\boldsymbol{p} \mapsto \boldsymbol{p} + \frac{e^*}{c}\boldsymbol{A}$ for some effective charge $e^*$. The energy density of the magnetic field should also be added.
All in all, let $\Omega \subset \mathbb{R}^3$ be the volume of the superconductor, than the Helmholtz free energy is given as a functional over the order parameter $\psi$ and the vector potential $\boldsymbol{A}$:

$$\begin{aligned} F[\psi, \boldsymbol{A}] :=& F_{\text{sc}}[\psi, \boldsymbol{A}] + F_{\text{em}}[\boldsymbol{A}] \\ \equiv& \int_\Omega \mathrm{d}^3x \left[ F_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|\mathcal{D}_A\psi|^2 \right] + \int_{\mathbb{R}^3} \frac{\mathrm{d}^3x}{8\pi} |\text{curl}\, \boldsymbol{A}|^2 \end{aligned} \tag{1.1}$$

where $\mathcal{D}_A\psi := i\hbar\nabla\psi + \frac{e^*}{c}\psi\boldsymbol{A}$. Notice that $f$ is over the sample volume $\Omega$ while the field-density is distributed in the whole space (as it should because in the usual experimental setting one generates the field outside the superconductor).

*Remark* 1. The addition of the magnetic field energy density sets the physical dimensions of the order parameter: $[|\psi|^2] = [\ell]^{-3}$ enforcing the interpretation as a wavefunction (its squared-modulus has the dimensions of a volume density).

### 1.2.2 Gibbs free energy

Perhaps a more natural approach to Ginzburg-Landau theory comes from considering the Gibbs free-energy $G$ instead of the Helmholtz potential. Being the Gibbs free energy (in this context) the Legendre transform of the internal energy wrto the entropy and the magnetization, the natural extension of the extremuum principle for $U$ to $G$ is: the equilibrum state for a system held at fixed temperature and magnetic field minimizes the Gibbs free energy. This is of course the most natural experimental setting one sets up in order to probe into the properties of a superconductor.
Carrying out the computation yields:

$$\begin{aligned} G[\psi, \boldsymbol{A}] = \int_\Omega \mathrm{d}^3x \left[ g_n + \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{1}{2m^*}|\mathcal{D}_A\,\psi|^2 \right] + \\ + \int_{\mathbb{R}^3} \frac{\mathrm{d}^3x}{8\pi} |\text{curl}\boldsymbol{A} - \boldsymbol{H}_0|^2 \end{aligned} \tag{1.2}$$

where $\boldsymbol{H}_0$ is the applied field.

### 1.2.3 A more natural system of units

First of all we wish to group the constants in the expression for the Helmholtz free-energy in order to identify the typical dimensions of the system. It has

the clear aesthatical advantage of polishing out the integral (which immediately corresponds to the mathematical advantage of not having to carry lots of costants through), but it is also physically relevant because it outlines the scales we should expect to be able to peer into in order to detect such phenomenon.
As for the typical dimension of the order parameter, it is natural to use the equilibrum value without the field, i.e.:

$$|\psi_0|^2 := \frac{|\alpha|}{\beta} \tag{1.3}$$

The only other quantity we need to properly rescale is the vector potential, so we need to find the typical magnetic field and the characteristic length of the system.
As for the length, we can again make use of the mandatory homogenity of the last two terms in the expression for the free energy, which immediately yields:

$$\lambda := \frac{c}{2e^*}\sqrt{\frac{m^*}{\pi|\psi_0|^2}} \equiv \frac{c}{2e^*}\sqrt{\frac{m^*\beta}{\pi|\alpha|}} \tag{1.4}$$

As for the magnetic field, since $\alpha|\psi|^2$ and $|\operatorname{curl}\boldsymbol{A}|^2 \equiv |\boldsymbol{H}|^2$ must have the same dimensions, we get[1]:

$$H_C := 2\sqrt{\pi|\alpha||\psi_0|^2} \equiv 2|\alpha|\sqrt{\frac{\pi}{\beta}} \tag{1.5}$$

We, now, have all it takes to properly rescale the functional. If we let:

$$\begin{cases} |\psi'|(\boldsymbol{x}') := |\psi(\lambda\boldsymbol{x}')/\psi_0| \\ \boldsymbol{A}'(\boldsymbol{x}') := \boldsymbol{A}(\lambda\boldsymbol{x}')/(\sqrt{2}H_C\lambda) \end{cases} \tag{1.6}$$

then a substitution into the functional yields:

$$\begin{aligned} F =& \lambda^3\int_\Omega \mathrm{d}^3x'\left[F_n + \frac{|\alpha|^2}{\beta}\left(\frac{|\psi'|^4}{2} + \operatorname{sgn}(\alpha)|\psi'|^2\right)\right] + \\ &+ \frac{\lambda^3}{2m^*}\frac{|\alpha|}{\beta}\int_\Omega \mathrm{d}^3x'\left|\frac{i\hbar}{\lambda}\nabla'\psi' + \sqrt{2m^*|\alpha|}\boldsymbol{A}'\psi'\right|^2\Bigg] + \\ &+ \lambda^3\int_{\mathbb{R}^3}\mathrm{d}^3x'\frac{H_C^2}{4\pi}\left|\operatorname{curl}'\boldsymbol{A}'\right|^2 \\ =& \lambda^3\frac{|\alpha|^2}{\beta}\int_\Omega \mathrm{d}^3x'\left[F_n' + \operatorname{sgn}(\alpha)|\psi'|^2 + \frac{1}{2}|\psi'|^4 + \left|\frac{i}{\lambda}\frac{\hbar}{\sqrt{2m^*|\alpha|}}\nabla'\psi' + \boldsymbol{A}'\psi'\right|^2\right] + \\ &+ \lambda^3\frac{|\alpha|^2}{\beta}\int_{\mathbb{R}^3}\left|\operatorname{curl}'\boldsymbol{A}'\right|^2\mathrm{d}^3x' \end{aligned}$$

[1] The dimensionless factors are, of course, completely arbitrary. Here we introduce the commonely-used convention. In $H_C$ there's actually a $\sqrt{2}$ missing but it is immediately restored underneath.

It is, then, natural to introduce a further length:

$$\xi := \frac{\hbar}{\sqrt{2m^*|\alpha|}} \tag{1.7}$$

The free energy only depends on the dimensionless ratio $\kappa := \lambda/\xi$.
Furthermore, since it is obvious that the transformation $S \mapsto \mu S$ leaves the extremum of the action $S$ invariant for all $\mu > 0$, we can equivalently consider (dropping the primes):

$$\begin{aligned} F[\psi, \boldsymbol{A}] = \int_\Omega \mathrm{d}^3x \left[ F_n + \frac{1}{2}(1 + \mathrm{sgn}(\alpha)|\psi|^2)^2 + \left| \frac{i}{\kappa}\nabla\psi + \boldsymbol{A}\psi \right|^2 \right] + \\ + \int_{\mathbb{R}^3} |\mathrm{curl}\, \boldsymbol{A}|^2 \mathrm{d}^3x \end{aligned} \tag{1.8}$$

where $F_n := F_n' - \frac{1}{2}$ (still a dimensionless constant). As for the sign of $\alpha$, we have already noticed that $\alpha < 0$ in the superconductive phase and $\alpha > 0$ in the normal phase. Since we are particularly interested in the former, we take $\mathrm{sgn}(\alpha) = -1$.
To abide by the standard notation we introduce a further rescaling of the vector potential $\boldsymbol{A} \mapsto \kappa\boldsymbol{A}$, which finally leads to (suppressing a scaling factor $\kappa^{-2}$):

$$F[\psi, \boldsymbol{A}] = \int_\Omega \mathrm{d}^3x \left[ f_n + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |\mathcal{D}_A\psi|^2 \right] + \int_{\mathbb{R}^3} |\mathrm{curl}\, \boldsymbol{A}|^2 \mathrm{d}^3x \tag{1.9}$$

where the rescaled $\mathcal{D}_A$ is $\mathcal{D}_A\psi = i\nabla\psi + \psi\boldsymbol{A}$.
The physical significance of $|\psi_0|$ is clear while we will return on the role of $\xi, \lambda, H_C$ and $\kappa$ in Ginzburg-Landau theory.

The rescaled version of the Gibbs free energy is obtained by analogous substitutions in Eq. (1.2):

$$\begin{aligned} G[\psi, \boldsymbol{A}] = \int_\Omega \mathrm{d}^3x \left[ g_n + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |\mathcal{D}_A\psi|^2 \right] + \\ + \int_{\mathbb{R}^3} |\mathrm{curl}\boldsymbol{A} - \boldsymbol{h}_0|^2 \mathrm{d}^3x \end{aligned} \tag{1.10}$$

where $\boldsymbol{h}_0 := \frac{\kappa}{\sqrt{2}} \frac{\boldsymbol{H}_0}{H_C}$.

## 1.3 Reduction to $\mathbb{R}^2$

A thorough study of the three-dimensional model is far too complicated and no analytic solution to the problem has been obtained thus far, even though some estimations have been made by means of asymptotical analysis (see e.g. [11]). Hence, we wish to reduce the dimension of the problem to two, and that is done by assuming a particular symmetric shape for the superconducting sample that

allows us neglect one dimension. Specifically we assume that the volume $\Omega^{(3)}$ occupied by the sample in 3-space may be written as the product $\Omega^{(3)} = \Omega^{(2)} \times \mathbb{R}$ where $\Omega^{(2)}$ is a bounded, smooth and simply-connected domain of $\mathbb{R}^2$.
We further assume that such symmetry is reflected in our solution $(\psi, \boldsymbol{A})$ to the minimization problem for the free energy (1.10), i.e. translational invariance along the "axis" of the "generalised cylinder" $\Omega^{(3)}$ and reflection invariance through a plane perpendicular to such axis. As a consequence, let $\boldsymbol{e}_z$ be the unit vector in the direction of the axis, then neither $\psi$ nor $\boldsymbol{A}$ can depend on the $z$ coordinate and $\boldsymbol{A} \cdot \boldsymbol{e}_z = 0$. As a further simplification we take $\boldsymbol{h}_0 = h_0 \boldsymbol{e}_z$.
Under these assumptions, the Gibbs free energy per unit length (i.e. on each plane perpendicular to $\boldsymbol{e}_z$) in natural units (dropping the subscripts $\Omega := \Omega^{(2)}$) is given by:

$$G[\psi, \boldsymbol{A}] = \int_\Omega \mathrm{d}^2x \left[ g_n + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |\mathcal{D}_A \psi|^2 \right] + \int_{\mathbb{R}^2} |\mathrm{curl}\boldsymbol{A} - h_0|^2 \mathrm{d}^2x \quad (1.11)$$

Let us now focus for a moment on the question of well-definiteness of the functional $G$: the integral exists iff $\psi$, $\boldsymbol{A}$ and their first order derivatives are square-integrable functions. The set of such functions has a precise name: *Sobolev spaces*. Given $U \subseteq \mathbb{R}^d$ we define[2]

$$H^1(U, V) := \{u : U \to V, \qquad u, \partial_\alpha u \in L^2(U) \quad \forall \alpha = 1, \ldots, d\}$$

The significance of such set is that it can very easily be given the structure of a Hilbert space because there it is canonically isomorphic to $\bigoplus_{i=1}^{d+1} L^2(U)$ via the map $u \mapsto (u, \partial_1 u, \ldots, \partial_d u)$. The inner product in $H^1(U, V)$ is then simply the one induced to the direct-sum space by the products on each of the "original" spaces:

$$(u, v)_{H^1(U,V)} := (u, v)_{L^2(U)} + \sum_{\alpha=1}^{d} (\partial_\alpha u, \partial_\alpha v)_{L^2(U)}$$

Just like for the spaces $L^p$, $H^1$ is just a special case of the sets $H^k$ (for $k \in \mathbb{N}$) of functions whose derivatives up to the $k$-th order are square-integrable in the given set. Each of these spaces is of course canonically isomorphic to $\bigoplus_{i=1}^{d^k+1} L^2(U)$ (where $U \subseteq \mathbb{R}^d$) and can thus be given the structure of a Hilbert space.
With such definition, the most natural setting for the variational problem of $G$ is the functional space:

$$\mathcal{X} := \{(\psi, \boldsymbol{A}) \in H^1(\Omega, \mathbb{C}) \times H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2), \quad (\mathrm{curl}\, \boldsymbol{A} - h_0) \in L^2(\mathbb{R}^2)\}$$

where $H^1_{\mathrm{loc}}(U, V)$ is the space of functions $u : U \to V$ such that $u|_A \in H^1(A, V)$ for every $A$ compact subset of $U$.

[2] Actually $\partial_\alpha u$ is required to exist only in the *weak sense*, which is an integral notion of derivative resembling that of regular distributions. For the details see [9].

### 1.3.1 Symmetries

It is important to ask whether there exists one or more groups whose action on $(\psi, \boldsymbol{A})$ leave the free energy (Eq. 1.11) invariant. It is easy to see that:

1. Due to the translational invariance of the Lebesgue measure, $G$ is unchanged under the action of the abelian translation group, i.e. if for all $\boldsymbol{r} \in \mathbb{R}^2$ we let $(\mathcal{T}_r u)(\boldsymbol{x}) := u(\boldsymbol{x} + \boldsymbol{r})$, then $G[\psi, \boldsymbol{A}, \Omega] = G[\mathcal{T}_r\psi, \mathcal{T}_r\boldsymbol{A}, \mathcal{T}_r\Omega]$ (the set should of course also be translated: $\mathcal{T}_r\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} - \boldsymbol{r} \in \Omega\}$).

2. Due to the invariance of the modulus under rotation, $G$ is left invariant by the action of $\mathcal{O}(2)$, i.e. for all $R \in \mathcal{O}(2)$, $G[\psi, \boldsymbol{A}, \Omega] = G[U_R\psi, U_R\boldsymbol{A}, U_R\Omega]$ where the actions are $(U_R u)(\boldsymbol{x}) := u(R^{-1}\boldsymbol{x})$, $(U_R\boldsymbol{v})(\boldsymbol{x}) := R\boldsymbol{v}(R^{-1}\boldsymbol{x})$ and $U_R\Omega := \{\boldsymbol{x} \in \mathbb{R}^2 : R^{-1}\boldsymbol{x} \in \Omega\}$.

There is a further group which is not as immediate and are the so-called gauge transformations.
First of all it is clear that the magnetic field energy density term is invariant under the transformation $\boldsymbol{A} \mapsto \boldsymbol{A} + \nabla\lambda$. As for $\psi$, since the first two terms in the superconductive energy density depend only on its modulus, it follows that they are invariant under the transformation $\psi \mapsto \psi e^{i\chi}$. The relation between $\lambda$ and $\chi$ can be found by computing the coupling term:

$$\mathcal{D}_{A'}\psi' = e^{i\chi}\left[i\nabla\psi - \psi\nabla\chi + \psi\boldsymbol{A} + \psi\nabla\lambda\right]$$

So, in order to have $|\mathcal{D}_A\psi| = |\mathcal{D}_{A'}\psi'|$, $\lambda$ and $\chi$ must satisfy $\nabla\lambda = \nabla\chi$. Wlog we can choose $\lambda = \chi$.

**Theorem 1** (Gauge invariance of the G-L free energy)**.** *The Ginzburg-Landau free energy (1.9) is invariant under the transformation:*

$$(\psi, \boldsymbol{A}) \mapsto (\mathcal{G}_\chi\psi, \mathcal{G}_\chi\boldsymbol{A}) := (\psi e^{i\chi},\ \boldsymbol{A} + \nabla\chi) \tag{1.12}$$

*for any* $\chi \in H^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R})$.

*Remark* 2. If $\Omega$ is connected, any solution of $\nabla\chi = \nabla\lambda$ can be found as $\chi = \lambda + C$ for $C \in \mathbb{R}$. Suppose we choose $C \neq 0$, then our gauge transformation would be $(\psi, \boldsymbol{A}) \mapsto (\psi e^{i(\lambda+C)}, \boldsymbol{A} + \nabla\lambda)$. This transformation is obviously the composition of two gauge maps with $\chi_1 = \lambda$ and $\chi_2 = C$. This is what we meant when we said we could always pick $C = 0$ (i.e. $\lambda = \chi$): any other choice is gauge-equivalent.

As usual, fixing $\chi$ is called making a *gauge choice*. A popular choice is the following:

**Lemma 1** (Coulomb gauge)**.** *Let* $(\psi, \boldsymbol{A})$ *satisfy the G-L equations. There always exists* $\chi$ *such that:*

$$\begin{cases} \operatorname{div}(\mathcal{G}_\chi\boldsymbol{A}) = 0 & \boldsymbol{x} \in \Omega \\ (\mathcal{G}_\chi\boldsymbol{A}) \cdot \boldsymbol{n} = 0 & \boldsymbol{x} \in \partial\Omega \end{cases} \tag{1.13}$$

*Proof.* A substitution of $\mathcal{G}_\chi \boldsymbol{A} = \boldsymbol{A} + \nabla\chi$ gives the Neumann problem for the Poisson equation:

$$\begin{cases} \triangle\chi = -(\operatorname{div} \boldsymbol{A}) \\ \partial_n\chi = -\boldsymbol{A}\cdot\boldsymbol{n} \qquad x \in \partial\Omega \end{cases}$$

in the set $\Omega$. The source and the boundary data clearly satisfy the compatibility condition $\int_\Omega (\operatorname{div}\boldsymbol{A})\mathrm{d}^2x = \int_{\partial\Omega}(\boldsymbol{A}\cdot\boldsymbol{n})\,\mathrm{d}\ell$, so a solution $\chi$ for the problem exists. □

*Remark* 3. Note that the Coulomb-gauge condition does not determine $(\psi, \boldsymbol{A})$ uniquely: it sets no constraints on $\psi$ and even $\boldsymbol{A}$ is specified up to the gradient of an harmonic function. The remaning freedom will allow us to prescribe some additional conditions when required.

### 1.3.2 Reduction to $\Omega$

The functional space $\mathcal{X}$ is not the best choice to pick our couple $(\psi, \boldsymbol{A})$ from: it is both physically redundant (we are particularly interested in the magnetic field outside the superconductor) and it involves mathematical technicalities we wish to avoid. We would rather choose $(\psi, \boldsymbol{A}) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2) =: \mathcal{X}_\Omega$ but we first need to check that the two settings for the problem are physically equivalent, i.e. the minimizers for the restricted functional:

$$G_\Omega[\psi, \boldsymbol{A}] := \int_\Omega \left[ g_n + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |\mathcal{D}_A\psi|^2 + |\mathrm{curl}\boldsymbol{A} - h_0|^2 \right] \mathrm{d}^2x \qquad (1.14)$$

in $\mathcal{X}_\Omega$ are in a one-to-one correspondence with those of $G$ in $\mathcal{X}$.
First of all let us notice that for all $(\psi, \boldsymbol{A}) \in \mathcal{X}$ we have $(\psi, \boldsymbol{A}|_\Omega) \in \mathcal{X}_\Omega$ and, since the magnetic field free energy density is positive, $G_\Omega[\psi, \boldsymbol{A}|_\Omega] \leq G[\psi, \boldsymbol{A}]$. Moreover:

**Lemma 2.** *Let $(\psi, \boldsymbol{A}) \in \mathcal{X}_\Omega$. Then $\boldsymbol{A}$ can be extended to $\boldsymbol{A}' \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ in such a way that $G_\Omega[\psi, \boldsymbol{A}] = G[\psi, \boldsymbol{A}']$.*

*Proof.* As a first step let us construct a vector field $\tilde{\boldsymbol{A}} \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying the minumum requirements:

1. The (gauge-invariant) curl of the two vector fields coincides in the superconductor: $(\operatorname{curl}\tilde{\boldsymbol{A}})|_\Omega = \operatorname{curl}\boldsymbol{A}$.
2. A necessary condition for the difference between the two functionals evaluated respectively in $(\psi, \boldsymbol{A})$ and $(\psi, \tilde{\boldsymbol{A}})$ to be zero is $G[\psi, \tilde{\boldsymbol{A}}] - G_\Omega[\psi, \boldsymbol{A}] = \int_{\mathbb{R}^2 \setminus \Omega} \left|\mathrm{curl}\,\tilde{\boldsymbol{A}} - h_0\right|^2 \mathrm{d}^2x = 0$, hence we choose $(\operatorname{curl}\tilde{\boldsymbol{A}})|_{\mathbb{R}^2\setminus\Omega} = h_0$.

This can be easily done by means of a function $\phi$ satisfying:

$$\begin{cases} \triangle\phi = \operatorname{curl}\boldsymbol{A} & \boldsymbol{x} \in \Omega \\ \triangle\phi = h_0 & \boldsymbol{x} \in \mathbb{R}^2 \setminus \Omega \end{cases}$$

Then, if we let $\tilde{\boldsymbol{A}} := (-\phi_y, \phi_x)$, it follows $\operatorname{curl} \tilde{\boldsymbol{A}} = \phi_{xx} + \phi_{yy} = \triangle\phi$.
However, the first requirement does not suffice to have $\tilde{\boldsymbol{A}}|_\Omega = \boldsymbol{A}$ because, since $\Omega$ is simply-connected, by standard vector-potential theory the two vector fields may differ for the gradient of a function $\chi \in H^2(\Omega, \mathbb{R}^2)$, i.e. $\tilde{\boldsymbol{A}}|_\Omega = \boldsymbol{A} + \nabla\chi$. In order to get rid of $\chi$, we need to construct a suitable gauge transformation. First of all let us notice that, as a consequence, thanks the requirement for the curl of $\tilde{\boldsymbol{A}}$ outside $\Omega$, $G_\Omega[\psi, \boldsymbol{A}] = G_\Omega[\psi e^{i\chi}, \tilde{\boldsymbol{A}}|_\Omega] = G[\psi e^{i\chi}, \tilde{\boldsymbol{A}}]$. Then we only need to arbitrarily extend $\chi$ to a function $\tilde{\chi} \in H^2_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and gauge transform $(\psi e^{i\chi}, \tilde{\boldsymbol{A}})$ by $-\tilde{\chi}$ to have $\boldsymbol{A}'|_\Omega := (\tilde{\boldsymbol{A}} - \nabla\tilde{\chi})|_\Omega = \boldsymbol{A} + \nabla\chi - (\nabla\tilde{\chi})|_\Omega = \boldsymbol{A}$ and to keep the identity $G_\Omega[\psi, \boldsymbol{A}] = G[\psi, \boldsymbol{A}']$. □

As a consequence we get:

**Proposition 1.** *Each of the following holds:*

1. $\min_{\mathcal{X}} G = \min_{\mathcal{X}_\Omega} G_\Omega$.
2. *Each minimizer $(\psi, \boldsymbol{A}) \in \mathcal{X}_\Omega$ of $G_\Omega$ can be extended to a minimizer of $G$.*
3. *Each minimizer $(\psi, \boldsymbol{A}) \in \mathcal{X}$ of $G$ can be restricted to a minimizer of $G_\Omega$.*

*Proof.* The crucial remark here is that a necessary condition for $(\psi, \boldsymbol{A}) \in \mathcal{X}$ to be a minimizer of $G$ is $(\operatorname{curl} \boldsymbol{A})|_{\mathbb{R}^2 \setminus \Omega} = h_0$ (see next section), thus it can be obtained by a minimizer of $G_\Omega$ in $\mathcal{X}_\Omega$ through the construction described in the Lemma. The converse follows from the inequality $G_\Omega[\psi, \boldsymbol{A}|_\Omega] \leq G[\psi, \boldsymbol{A}]$. □

## 1.4 Ginzburg-Landau equations

According to standard Thermodynamics, the equilibrum condition is found via minimization of the free energy. The more common experimental condition is, as we already noticed, that of fixed temperature and external field, so the right free energy to minimize is Gibbs'.
Before doing that, let us prove the following useful identities:

**Lemma 3.** *Let $\Omega \subseteq \mathbb{R}^2$ be a regular open set. Then*[3]*:*

$$(f, \operatorname{curl} \boldsymbol{X})_{L^2(\Omega,\mathbb{R})} = \int_{\partial\Omega} f(\boldsymbol{n} \wedge \boldsymbol{X})\, \mathrm{d}s + (\operatorname{curl}^\dagger f, \boldsymbol{X})_{L^2(\Omega,\mathbb{R}^2)} \tag{1.15}$$

$$(\boldsymbol{\xi}, \nabla u)_{L^2(\Omega,\mathbb{C}^2)} = \int_{\partial\Omega} u \langle \boldsymbol{\xi}, \boldsymbol{n} \rangle\, \mathrm{d}s - (\operatorname{div} \boldsymbol{\xi}, u)_{L^2(\Omega,\mathbb{C})} \tag{1.16}$$

*where* $\operatorname{curl}^\dagger f := (f_y, -f_x)$.

[3] $L^2(\Omega, \mathbb{C}^2)$ is to be interpreted as the space of square-integrable vector fields $\boldsymbol{X} : \Omega \to \mathbb{C}^2$, i.e. those whose squared standard hermitian norm has finite integral over $\Omega$. The inner product is clearly $(\boldsymbol{X}, \boldsymbol{Y}) := \int_\Omega \langle \boldsymbol{X}, \boldsymbol{Y} \rangle \mathrm{d}^2x$ where $\langle \cdot, \cdot \rangle$ is the canonical scalar product. The same goes for $L^2(\Omega, \mathbb{R}^2)$ by substitution of the hermitian product with the Euclidian one.

*Proof.* Both are a rather straightforward application of the integration-by-parts formula:

$$\int_\Omega f\frac{\partial g}{\partial x_i}\mathrm{d}^d x = \int_{\partial\Omega} fgn_i\,\mathrm{d}\sigma - \int_\Omega g\frac{\partial f}{\partial x_i}\mathrm{d}^d x \tag{1.17}$$

for $d = 2$, where $\boldsymbol{n}$ is the outgoing normal unit vector to $\partial\Omega$. □

*Remark* 4. We highlighted the formulas in the previous Lemma because they have a clear geometrical interpretation:

1. The operator $\mathrm{curl}^\dagger$ is the formal adjoint of the curl over the set of vector fields satisfying $\boldsymbol{n}\wedge\boldsymbol{X} = \boldsymbol{0}$ on $\partial\Omega$, i.e. those orthogonal to the boundary of $\Omega$ (or, conversely, curl is the formal adjoint of $\mathrm{curl}^\dagger$ when restricted to functions such that $f = 0$ on $\partial\Omega$).

2. The divergence is the formal adjoint of the gradient over the domain of vector fields satisfying $\langle\boldsymbol{\xi},\boldsymbol{n}\rangle = 0$ on $\partial\Omega$, i.e. those parallel to the boundary of $\Omega$ (or the converse just like above).

**Theorem 2** (Ginzburg-Landau equations for the Gibbs free energy)**.** *A necessary condition for $(\psi,\boldsymbol{A})\in\mathcal{X}_\Omega$ to be a minimizer of the Gibbs free energy in natural units $G_\Omega$ given by Eq. (1.14) is that it satisfies the system of PDEs:*

$$\boxed{\begin{cases}\triangle_A\psi + \kappa^2(|\psi|^2-1)\psi = 0 & \boldsymbol{x}\in\Omega\\ \mathrm{curl}^\dagger(\mathrm{curl}\,\boldsymbol{A}-h_0) + \mathrm{Re}\{\psi^*\mathcal{D}_A\psi\} = \boldsymbol{0} & \boldsymbol{x}\in\Omega\\ \langle\mathcal{D}_A\psi,\,\boldsymbol{n}\rangle = 0 & \boldsymbol{x}\in\partial\Omega\\ \mathrm{curl}\,\boldsymbol{A} = h_0 & \boldsymbol{x}\in\partial\Omega\end{cases}} \tag{1.18}$$

*where $\triangle_A := \mathcal{D}_A^\dagger\circ\mathcal{D}_A$ and $\mathcal{D}_A^\dagger\,\boldsymbol{X} := i\,\mathrm{div}\,\boldsymbol{X} + \langle\boldsymbol{A},\boldsymbol{X}\rangle$ being the formal adjoint of $\mathcal{D}_A$ on the domain of functions satisfying the first b.c.*

*Proof.* For clarity's sake, we consider the variation of the free energy density $g_\Omega$ wrt to $\delta\psi$ and $\delta\boldsymbol{A}$ term by term (let us also drop the subscript in $g_\Omega$).

1. From the variation of $g_{\mathrm{sc},1}$, containing the even powers of $|\psi|$, we get:

$$\begin{aligned}\delta g_{\mathrm{sc},1} =& \big(|\psi+\delta\psi|^2-1\big)^2 - \big(|\psi|^2-1\big)^2 =\\ =& \big(|\psi|^2 - 1 + 2\,\mathrm{Re}\{(\delta\psi)^*\psi\}\big)^2 - \big(|\psi|^2-1\big)^2 + \mathcal{O}\Big(|\delta\psi|^2\Big)\\ =& 4\,\mathrm{Re}\Big\{(\delta\psi)^*(|\psi|^2-1)\psi\Big\} + \mathcal{O}\Big(|\delta\psi|^2\Big)\end{aligned}$$

2. As for the kinetic term $g_{\mathrm{sc},2} = |\mathcal{D}_A\psi|^2$:

$$\begin{aligned}\delta g_{\mathrm{sc},2} =& |i\nabla(\psi+\delta\psi) + (\boldsymbol{A}+\delta\boldsymbol{A})(\psi+\delta\psi)|^2 - |\mathcal{D}_A\psi|^2\\ =& |\mathcal{D}_A\psi + \mathcal{D}_A\,\delta\psi + \psi\delta\boldsymbol{A}|^2 - |\mathcal{D}_A\psi|^2 + \mathcal{O}\Big(|\delta\psi|^2, |\delta\psi\,\delta\boldsymbol{A}|, |\delta\boldsymbol{A}|^2\Big)\\ =& 2\,\mathrm{Re}\big\{\big\langle\mathcal{D}_A(\delta\psi) + \psi\delta\boldsymbol{A}, \mathcal{D}_A\psi\big\rangle\big\} + \mathcal{O}\Big(|\delta\psi|^2, |\delta\psi\,\delta\boldsymbol{A}|, |\delta\boldsymbol{A}|^2\Big)\end{aligned}$$

3. The variation of the magnetic field energy-density is a straightforward calculation: $\delta g_{\mathrm{em}} = 2(\operatorname{curl}\boldsymbol{A} - h_0)\operatorname{curl}(\delta\boldsymbol{A}) + \mathcal{O}\big(|\delta\boldsymbol{A}|^2\big)$.

As for the total free energy $G = \int_\Omega g \mathrm{d}^2x$, it is convenient to group the terms according to the variation that appears in each of them. Using the Lemma:

- For those containing $\delta\psi$:

$$\begin{aligned}\delta G_1 =& 2\,\mathrm{Re}\Big\{\kappa^2\big(\delta\psi,\,(|\psi|^2-1)\psi\big)_{L^2(\Omega,\mathbb{C})} + \big(\mathcal{D}_A(\delta\psi),\mathcal{D}_A\psi\big)_{L^2(\Omega,\mathbb{C}^2)}\Big\}\\ =& 2\,\mathrm{Re}\Big\{\big(\delta\psi,\,\kappa^2(|\psi|^2-1)\psi + i\operatorname{div}\mathcal{D}_A\psi + \langle\boldsymbol{A},\mathcal{D}_A\psi\rangle\big)_{L^2(\Omega,\mathbb{C})}\Big\} +\\ &+ 2\,\mathrm{Im}\int_{\partial\Omega}(\delta\psi)^*\langle\boldsymbol{n},\mathcal{D}_A\psi\rangle\,\mathrm{d}\sigma\\ =& 2\,\mathrm{Re}\Big\{\big(\delta\psi,\,\kappa^2(|\psi|^2-1)\psi + \triangle_A\psi\big)_{L^2(\Omega,\mathbb{C})} - i\int_{\partial\Omega}(\delta\psi)^*\langle\boldsymbol{n},\mathcal{D}_A\psi\rangle\,\mathrm{d}\sigma\Big\}\end{aligned}$$

  Since $\delta\psi$ is arbitrary, setting $\delta G_1 = 0$ gives the first G-L equation and boundary condition.

- As for the terms containing the variation $\delta\boldsymbol{A}$:

$$\begin{aligned}\delta G_2 =& 2\,\mathrm{Re}\Big\{\big(\psi\delta\boldsymbol{A},\,\mathcal{D}_A\psi\big)_{L^2(\Omega,\mathbb{C}^2)}\Big\} + 2\big(\operatorname{curl}(\delta\boldsymbol{A}),\,\operatorname{curl}\boldsymbol{A} - h_0\big)_{L^2(\Omega,\mathbb{R})}\\ =& 2\int_{\partial\Omega}(\boldsymbol{n}\wedge\delta\boldsymbol{A})(\operatorname{curl}\boldsymbol{A} - h_0)\,\mathrm{d}\sigma +\\ &+ 2\big(\delta\boldsymbol{A},\,\operatorname{curl}^\dagger(\operatorname{curl}\boldsymbol{A} - h_0) + \mathrm{Re}\{\psi^*\mathcal{D}_A\psi\}\big)_{L^2(\Omega,\mathbb{R}^2)}\end{aligned}$$

  which yields the second equation and b.c. if set to zero.

□

*Remark* 5 (Minimization of the other functionals). Very little changes if one considers the other functionals:

1. The minimization of $G$ instead of $G_\Omega$ extends the condition $\operatorname{curl}\boldsymbol{A} = h_0$ on the entire $\mathbb{R}^2\setminus\Omega$.

2. Evaluating the variation of the Helmholtz free energy gives $\operatorname{curl}\boldsymbol{A} = \boldsymbol{0}$ on $\mathbb{R}^2\setminus\Omega$, which is not what one would "physically" expect (unless the applied field is zero, but then there would be no difference between the expressions for the Helmholtz and the Gibbs free energies).

Both Ginzburg-Landau equations very much resemble well-known PDE of "standard" Mathematical Physics:

1. If we neglect the non-linear term proportional to the squared modulus of the order parameter, the first equation is equivalent to the Schrödinger equation for a point-like particle of energy $\kappa^2$ and unit mass and charge moving in an electromagnetic field described by the potentials $(\varphi = 0, \boldsymbol{A})$.

2. If we let $\boldsymbol{B} := \operatorname{curl} \boldsymbol{A}$, then one immediately recognizes the fourth Maxwell equation in the second PDE of our system with source given by:

$$\boldsymbol{j}_{\text{sc}} := -\operatorname{Re}\{\psi^* \mathcal{D}_A \psi\} \equiv \operatorname{Im}\{\psi^* \nabla \psi\} - |\psi|^2 \boldsymbol{A} \tag{1.19}$$

The first term is not very surprising because it is the usual probability current from non-relativistic Quantum Mechanics times the charge $e^*$ (if one restores physical units, i.e. inverse tranforming via Eq. (1.6) plus dividing the vector potential by $\kappa$). The second term is non-linear in $\psi$, so it should be expected as well since the first equation is Schrödinger-like only up to non-linear terms.

## 1.5 Classification of superconductors

We have seen that one could distinguish two types of superconductive materials based on the existence of an intermediate state (and, consequently, of two df-ferent critical fields). Energetically, the difference must lie in the fact that it is more convenient for a superconductor of the second kind, under some condition of temperature and applied field, to let some of it get through its surface, while for a superconductor of the first kind the field is still repelled. It has, thus, something to do with the region in which the field is annhilated by the Meissner effect. Furthermore, since the only parameter on which the free energy depends on after our reduction to normal units is $\kappa$, it is fair to expect that such number can somehow distinguish between the two behaviours.
To give some better understanding of our conjecture, we need to introduce the so called *surface energy density* $\sigma$, i.e. the difference between the free energy in the bulk of the superconductor and in the outside region.

### 1.5.1 One dimensional problem

Unfortunately, a two-dimensional approach to this problem can only be discussed via numerical simulations (see e.g. [3]), so we are "forced" to further reduce the number of dimensions of our problem to one, i.e. we introduce a further translational invariance in a given direction (and we may call the unit vector that identifies such line $\boldsymbol{e}_y$) in each section of our superconductor. As a consequence, all physical quantities can only depend on the coordinate of the position $\boldsymbol{x}$ orthogonal to $y$, which we are going to call $x$.
We will work in the Coulomb gauge because the condition $\operatorname{div}\boldsymbol{A} = \frac{\partial A_1}{\partial x} = 0$ forces $A_1$ to be constant (and we can always set such value to zero). All the freedom that's left in the vector potential, then, lies in a single function $A(x) := A_2(x)$.
With such choices, the G-L system (1.18) becomes:

$$\begin{cases} -\psi'' + A^2\psi + \kappa^2(|\psi|^2 - 1)\psi = 0 \\ \operatorname{Im}\{\psi^*\psi'\} = 0 \\ -A'' + |\psi|^2 A = 0 \end{cases} \tag{1.20}$$

As for the b.c., the translational invariance in the $y$-direction prescribes the "shape" of $\Omega$ which must be something like $I \times \mathbb{R}$ for some connected interval $I \subseteq \mathbb{R}$. As a consequence $\boldsymbol{n}$ is parallel (or anti-parallel) to $\boldsymbol{e}_x$, so:

$$\begin{cases} \psi'(\partial I) = \{0\} \\ A'(\partial I) = \{h_0\} \end{cases} \tag{1.21}$$

The second equation in (1.20) tells us that $\psi^*\psi'$ is a real number, and that can hold iff both $\psi'$ and $\psi^*$ are either real or imaginary. Neither of the remaining equations can distinguish between the two cases (i.e. if $\psi$ solves both of them, so does $i\psi$), so we can take $\psi \in \mathbb{R}$ wlog. The system is thus reduced to:

$$\begin{cases} \psi'' = \kappa^2(\psi^2 - 1)\psi + A^2\psi \\ A'' = \psi^2 A \end{cases} \tag{1.22}$$

plus the b.c. The fact that neither $\psi'$ nor $A'$ show up in the ODEs gives us an obvious integral of motion that can be computed by multiplying both sides of the first equation by $2\psi'$ and both sides of the second by $2A'$:

$$\begin{cases} ((\psi')^2)' = \kappa^2\left[\left(\frac{1}{2}\psi^2 - 1\right)\psi^2\right]' + A^2(\psi^2)' \\ ((A')^2)' = \psi^2(A^2)' \end{cases}$$

Summing up the equations and using Leibniz rule:

$$0 = \left[(\psi')^2 + (A')^2 - \left(\frac{\kappa^2}{2}(\psi^2 - 2) + A^2\right)\psi^2\right]' =: \mathcal{E}'$$

i.e. $\mathcal{E}$ is an integral of motion [4].
As a consequence, the "trajectories" of the solutions in the $(\psi, A)$ plane are (pieces of) the level curves of $\mathcal{E}$.

### 1.5.2 Infinite transition region

For reasons that will become clear, we are particularly interested in the situation of a very broad transition region between the superconductive and the normal phases. To model such situation with the help of the one-dimensional problem described above, we suppose the superconductor to be infinitely-outstretched in the $x$ direction and with the following b.c.:

$$\begin{aligned} \lim_{x\to-\infty} A(x) &= 0 & \lim_{x\to-\infty} \psi(x) &= 1 \\ \lim_{x\to+\infty} A'(x) &= h_0 & \lim_{x\to+\infty} \psi(x) &= 0 \end{aligned} \tag{1.23}$$

i.e. the system is asimptotically in the superconductive-bulk phase at $x \to -\infty$ and in the normal phase as $x \to +\infty$. The transition region is, thus, smoothly

[4] It is not extremely surprising because $\mathcal{E}$ is basically, up to some harmless constant, the Jacobi integral for a system with Lagrangian density given by the Gibbs free energy density.

stretched across the entire $x$ axis.
Such choice of boundary conditions selects a trajectory in the $(\psi, A)$ plane, i.e. a value of $\mathcal{E}$. The value of $\mathcal{E}$ can be evaluated by imposing e.g. the asymptotic behaviour as $x \to -\infty$, which gives[5] $\mathcal{E} = \frac{\kappa^2}{2}$. The trajectory is, thus, described by the condition:

$$(\psi')^2 + (A')^2 = \frac{\kappa^2}{2}(1-\psi^2)^2 + (A\psi)^2 \tag{1.24}$$

Provided that $\psi$ decays quickly enough as $x \to +\infty$ to guarantee $\psi A \to 0$ (i.e. faster than linearly, since $A'$ has finite limit), then such tells us that there exists only one value for the applied magnetic field compatible with such configuration: $h_0 = \frac{\kappa}{\sqrt{2}}$, which corresponds to $H_0 = H_C$ if one restores the physical units.

An important result is the following:

**Proposition 2.** *There exists a unique solution $(\psi, A)$ of the G-L equations (1.22) with b.c. (1.23) and it satisfies the following properties:*

1. *$\psi$ is a monotonically decreasing function of $x$, i.e. $\psi'(\mathbb{R}) \subseteq (-\infty, 0)$ and $\psi(\mathbb{R}) \subseteq [0, 1]$.*
2. *$A$ is a monotonically increasing function of $x$, i.e. $A'(\mathbb{R}) \subseteq (0, +\infty)$.*

*Proof.* See [6]. □

Before going on to finally introduce the classification of superconductors in such picture, it is necessary to at least determine the asymptotic behavious of the solutions to such problem as $|x| \to \infty$.

1. As $x \to -\infty$ we set $\psi \to 1$, so the leading term can be found by setting $\psi = 1 + f$; since $A \to 0$ in the same limit, then the leading terms can be found by linearization around the solutions $f = A = 0$ of the system (1.22:
$$\begin{cases} f'' = 2\kappa^2 f \\ A'' = A \end{cases}$$
This gives:
$$\begin{aligned} \psi(x) - 1 &\underset{x\to-\infty}{\sim} e^{\sqrt{2}\kappa x} \\ A(x) &\underset{x\to-\infty}{\sim} e^{x} \end{aligned} \tag{1.25}$$
2. As $x \to +\infty$, since $A' \to \frac{\kappa}{\sqrt{2}}$, $A$ has linear behaviour: $A \sim x\frac{\kappa}{\sqrt{2}}$. Substituting into the first equation of the system (1.22) and linearizing around $\psi = 0$:
$$\psi'' = \frac{\kappa^2 x^2}{2}\psi$$

[5] Obviously, if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function which has finite limit as $x \to \infty$, then $f' \to 0$ as $x \to \infty$.

The equation is not analytically solvable but that does not matter because we only need an approximate solution as $x \to +\infty$, and that is clearly given by $x \mapsto \exp\left[-\frac{\kappa}{2\sqrt{2}}x^2\right]$:

$$\begin{aligned} \psi(x) &\underset{x\to+\infty}{\sim} \exp\left[-\frac{\kappa}{2\sqrt{2}}x^2\right] \\ A(x) &\underset{x\to+\infty}{\sim} \frac{\kappa}{\sqrt{2}}x \end{aligned} \tag{1.26}$$

Notice that the condition $\psi A \to 0$ as $x \to +\infty$ that we had naively assumed is actually satisfied. Furthermore, since $A'' = \psi^2 A$, then $A''$ is doomed to decay, as $x \to +\infty$, at least as fast as a gaussian function. As a consequence, all additional terms one may add to find a finer approximation of $A$ as $x \to +\infty$ are either $\mathcal{O}\left(e^{-Kx^2}\right)$ for some $K > 0$, or have second derivative equal to zero (which means they are constant, since the linear term is known).

### 1.5.3 Surface energy

We are now in the condition to introduce and study the quantity upon which our classification is based: the difference between the Gibbs free energy in the superconductive and in the normal phase for the model just discussed:

$$\begin{aligned} \sigma &:= \int_{\mathbb{R}} \mathrm{d}x \, \left[g_s(h_0) - g_n(h_0)\right]\Big|_{h_0=\kappa/\sqrt{2}} \\ &= \int_{\mathbb{R}} \mathrm{d}x \left[\frac{\kappa^2}{2}(1-\psi^2)^2 + (\psi')^2 + (A\psi)^2 + (A')^2 - \sqrt{2}\kappa A'\right] \end{aligned} \tag{1.27}$$

$\sigma$ has clearly the dimensions of a surface energy density.
A careful usage of the integral of motion (1.24), the ODEs (1.22) and the asymptotical behaviours just evaluated allow us to rewrite the surface energy as:

$$\begin{aligned} \sigma =& 2\int_{\mathbb{R}} \mathrm{d}x \left[(\psi')^2 + (A')^2 - \frac{\kappa}{\sqrt{2}}A'\right] \\ =& 2\int_{\mathbb{R}} \mathrm{d}x \left[(\psi')^2 + \left(AA' - \frac{\kappa}{\sqrt{2}}A\right)' - AA''\right] \\ =& 2\int_{\mathbb{R}} \mathrm{d}x \left[(\psi')^2 - (\psi A)^2\right] \end{aligned} \tag{1.28}$$

The classification of the superconductors in the G-L theory is basically founded on this result:

**Proposition 3.** *The sign of the surface energy $\sigma$ is entirely determined by the parameter $\kappa$. Precisely:*

- $\sigma > 0$ *when* $\sqrt{2}\kappa < 1$.

- $\sigma = 0$ *when* $\sqrt{2}\kappa = 1$.
- $\sigma < 0$ *when* $\sqrt{2}\kappa > 1$.

*Proof.* According to Eq. (1.28), the surface energy can be written as:

$$\sigma = 2\int_{\mathbb{R}} \mathrm{d}x \left(\psi' + \psi A\right)\left(\psi' - \psi A\right)$$

Since $\psi' < 0, \psi > 0$ and $A > 0$, the second factor in the integrand is always negative. Hence the sign of $\sigma$ is uniquely determined by $F := \psi' + \psi A$.
First of all let us notice that, thanks to Eq. (1.24), we can equivalently rewrite the integrand as:

$$\begin{aligned}(\psi')^2 - (\psi A)^2 =& \frac{\kappa^2}{2}(1-\psi^2)^2 - (A')^2\\ (\psi' - \psi A)F =& \left[\frac{\kappa}{\sqrt{2}}(1-\psi^2) + A'\right]\left[\frac{\kappa}{\sqrt{2}}(1-\psi^2) - A'\right]\end{aligned}$$

Since $\psi < 1$ and $A' > 0$, $F$ has the same sign of the term $G := A' - \frac{\kappa}{\sqrt{2}}(1-\psi^2)$. Using the characteristic system of ODEs (1.22), we have:

$$\begin{aligned}F' =& \psi'' + \psi' A + \psi A'\\ =& \kappa^2\psi(\psi^2 - 1) + \psi A^2 + \psi' A + \psi A'\\ =& \psi\left[A' - \kappa^2(1-\psi^2)\right] + AF\\ G' =& A'' + \sqrt{2}\kappa\psi\psi' - A'' = \psi(A\psi + \sqrt{2}\kappa\psi')\end{aligned}$$

Let us fix $\sqrt{2}\kappa < 1$, then, as $\psi' < 0$, we have the following inequalities:

$$\begin{aligned}F' >& \psi\left[A' - \frac{\kappa}{\sqrt{2}}(1-\psi^2)\right] + AF = \psi G + AF\\ G' >& \psi(\psi A + \psi') = \psi F\end{aligned}$$

We want to show that, in such case, $F < 0$ (and, as a consequence, $G < 0$). Let us, then, suppose there exists $x_0 \in \mathbb{R}$ such that $F(x_0) \geq 0$. Then $G(x_0) \geq 0$ and, from the above inequalities, $F'(x_0), G'(x_0) > 0$. As $x$ increases above $x_0$, two things can happen: $F'$ and $G'$ can either maintain their sign or change it. However, neither is possible.
If the latter is true, i.e. there exists at least one $x > x_0$ such that $G'(x) = 0$; let $x_1$ be the first of such values. Then $F(x_1) < 0$ by the second of the above inequalities, but this implies $G(x_1) < 0$ as well since $F$ and $G$ share their sign point by point. But as a consequence of the continuity of $G'$, there is some $x_2 \in (x_0, x_1)$ such that $G'(x_2)$ is still positive, hence $G(x_2) > G(x_0) > G(x_1)$. By the intermediate value Theorem there must be some $x_3 \in (x_2, x_1)$ such that $G(x_3) = G(x_0)$ and, by Rolle's Theorem, $x_4 \in (x_0, x_3)$ such that $G'(x_4) = 0$. This is a contradiction because we had chosen $x_1$ to be the first point after

$x_0$ such that $G'(x_1) = 0$, but $x_4$ has the same property and, by construction, $x_4 < x_1$.
If $G'$ does not change sign, then $G \to +\infty$ as $x \to +\infty$ but this is absurd because such limit can be evaluated thanks to the b.c. (1.23) and is zero.
Hence there is no $x_0 \in \mathbb{R}$ such that $G(x_0) \geq 0$ and the Proposition is proved.
For the cases $\sqrt{2}\kappa \geq 1$, one just has to change the inequalities involving $F', G', F$ and $G$. □

The Proposition gives us a reasonable value of $\kappa$ that allows us to distinguish between different types of superconductors:

**Definition 1.1** (Type I and type II superconductors)**.** In the G-L theory, we classify superconductors into two families, according to the *Ginzburg-Landau parameter* $\kappa$:

1. If $\sqrt{2}\kappa < 1$, we say that the superconductor is *of type I.*
2. If $\sqrt{2}\kappa > 1$ we call the superconductor *of type II.*

## 1.6 Interpretation of the scaling factors

We have now developed enough information to give a physical interpretation to the remaining scaling factors introduced in (1.2.3).
As for the G-L parameter $\kappa$, we have just seen that it determines the type of the superconductor.
Given the aim of this section, the arguments we will bring forth are rather heuristic and neither completely satisfactory nor beyond any criticism for we will try to simplify the problems as much as we can in order to be left with functions that only depend parametrically with the scaling factor we want to highlight. This might of course lead to an abrupt drop in some of the so far blindly-assumed regularities of the functions.

### 1.6.1 Penetration depth

To give an interpretation to the penetration length let us consider the one dimensional problem introduced in (1.22). In the bulk region (i.e., if we consider the b.c. 1.23, the limit $x \to -\infty$), we have seen that $A \sim e^x$. Restoring units this becomes:

$$A(x) \underset{x \to -\infty}{\sim} \sqrt{2} H_C \xi \, e^{x/\lambda} \tag{1.29}$$

This equation gives the physical interpretation for $\lambda$: it defines the typical variation-rate for the vector potential (and consequently for the magnetic field) in the bulk region of the superconductor. In other words, the Meissner effect gets more and more abrupt as $\lambda$ increases.
For this reason $\lambda$ is called *penetration depth.*

### 1.6.2 Coherence length

The significance of the other length $\xi$ defined in Eq. (1.7) can be easily inferred in the one-dimensional model we have introduced. Let us fix, for the sake of symplicity, a type I superconductor, i.e. we expect that as soon as $\psi > 0$, $A = 0$ due to Meissner effect. This of course breaks the continuity of $A$, so we should expect $\psi$ not to be as regular as we may naively wish.
Let $x_0 \in \mathbb{R}$ such that $\psi(x) > 0$ forall $x < x_0$ and $\psi(x) = 0$ for $x > x_0$. In the region $x < x_0$ the system (1.22) reduces to:

$$\begin{cases} \psi'' = \kappa^2(\psi^2 - 1)\psi \\ \psi(-\infty) = 1 \\ \psi(x_0) = 0 \end{cases} \tag{1.30}$$

This problem is exactly solvable, but it is easier to use the first integral of motion Eq. (1.24) which gives a first-order ODE with the first b.c. identically satisfied thanks to the way we fixed $\mathcal{E}$.
Since $A = 0$ and $\psi' < 0$ in the region $x < x_0$:

$$\begin{cases} \psi' = \frac{\kappa}{\sqrt{2}}(\psi^2 - 1) \\ \psi(x_0) = 0 \end{cases} \tag{1.31}$$

The ODE is easily solved by separation of variables:

$$\int \frac{\psi'(x)\,\mathrm{d}x}{\psi^2(x) - 1} \underset{u=\psi(x)}{=} -\int \frac{\mathrm{d}u}{u^2 - 1} \underset{u=\tanh v}{=} -\int \mathrm{d}v = C - v$$

Hence:

$$\psi(x) = \tanh\left[-\frac{\kappa}{\sqrt{2}}x + C\right]$$

Imposing the b.c. at $x_0$ gives $C = \frac{\kappa}{\sqrt{2}}x_0$.
In order to highlight how the expression depends on $\xi$, one needs to restore the physical units:

$$\psi(x) = \psi_0 \tanh\left[\frac{x_0 - x}{\sqrt{2}\xi}\right] \tag{1.32}$$

Hence, the physical significance of $\xi$ is clarified: it determines the variation-rate of the order parameter in the transition region.

As a consequence, the G-L parameter $\kappa = \lambda/\xi$ is an "measure" of the relative order of magnitude between the variation-rates of the magnetic field and the order parameter. In the classification we have introduced, we can qualitatively say that the superconductor is of type I if the coherence length is much bigger that the penetration length while it is of type II if the magnetic field is able to soak through the supercondutor for a distance not negligeble if compared with the typical size of variation of the order parameter.

| Element | $\xi$ [nm] | $\lambda$ [nm] | $\kappa$ |
|---|---|---|---|
| Sn | 2300 | 34 | 0.015 |
| Al | 1600 | 16 | 0.010 |
| Pb | 83 | 37 | 0.446 |
| Cd | 760 | 110 | 0.145 |
| Nb | 38 | 35 | 0.921 |

**Table 1.2:** *Coherence length and penetration length for some elements.*

### 1.6.3 Critical field

Let us fix the temperature of the superconductor and vary the external field. As a first approximation, if one neglects the fluctuations in the order parameter, i.e. the kinetic term in the Gibbs free energy density, as we have seen the minimizers are:

1. The bulk-superconductive solution $\psi = 1$ and $\operatorname{curl} \boldsymbol{A} = 0$. The Gibbs free energy density of such phase is, in natural units:
$$g_{\text{bsc}} = g_n + h_0^2$$
2. The normal solution $\psi = 0$ and $\operatorname{curl} \boldsymbol{A} = h_0$. Its Gibbs free energy density is:
$$g_{\text{n}} = g_n + \frac{\kappa^2}{2}$$

The only parameter is, of course, the external magnetic field and one immediately sees that the normal solution is energetically favoured (i.e. $g_{\text{n}} < g_{\text{bsc}}$) as long as $\sqrt{2}h_0 > \kappa$, i.e., restoring the units, $H_0 > H_C$.
Thus, for every fixed temperature, $H_C$ (itself a function of $T$, as its expression depends on the parameters $\alpha$ and $\beta$) is the value of the external field at which the normal-superconductive transition occurs in a type-I superconductor.

| Material | $\mu_0 H_C$ [mT] | Material | $\mu_0 H_C$ [mT] |
|---|---|---|---|
| Al | 10.49 | Sn | 30.55 |
| Cd | 2.805 | Ta | 82.9 |
| Ga | 5.93 | Th | 16.0 |
| Hg | 41.1 | Tl | 17.65 |
| In | 28.15 | U | 10.0 |
| Pb | 80.34 | V | 140 |
| Re | 20.1 | Zn | 5.41 |

**Table 1.3:** *Critical field for some elements measured at* 0 K.

# Chapter 2

# Abrikosov lattice solutions

From now on, unless specifically stated otherwise, we are going to consider the case $\sqrt{2}\kappa > 1$ and $\Omega = \mathbb{R}^2$ to avoid surface issues and difficulties due to the shape of the superconductor.
We are going to carry out a general introduction to 2-dimensional lattices and what we mean by invariance of a state $(\psi, \boldsymbol{A})$ wrt a lattice. Then we will focus on solutions of the GL equations for type-II superconductors near the critical field $H_{C2}$ (whose expression we are going to derive). In this part we will review the original results from Abrikosov [1, 2] in a more precise mathematical setting, following the works [23, 29, 28]. Finally, we will find an approximate expression for the lowest critical field $H_{C1}$.

Before moving on let us once again consider the operator $\mathrm{curl}^\dagger$ and notice that it is equivalent to a linear operator acting on the gradient of a given function $f$. In particular, since $\mathrm{curl}^\dagger f := (f_y, -f_x)$ it is natural to introduce:

**Definition 2.1.** We call $\mathfrak{I} : \mathbb{R}^2 \to \mathbb{R}^2$ the unique $\mathbb{R}$-linear operator whose action on the canonical basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is given by:

$$\mathfrak{I}(\boldsymbol{e}_1) := -\boldsymbol{e}_2 \qquad\qquad \mathfrak{I}(\boldsymbol{e}_2) := \boldsymbol{e}_1 \tag{2.1}$$

It is easy to check that:

**Proposition 4.** *$\mathfrak{I}$ is bijective, skew-symmetric and isometric (wrt the canonical scalar product).*

Furthermore, from the properties of the previous Proposition, it follows that $\mathfrak{I}^2(\boldsymbol{x}) = -\boldsymbol{x}$:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \mathfrak{I}(\boldsymbol{x}) \cdot \mathfrak{I}(\boldsymbol{y}) = -\mathfrak{I}^2(\boldsymbol{x}) \cdot \boldsymbol{y}$$

Since the equation must hold for all $\boldsymbol{y} \in \mathbb{R}^2$, then the statement is true.
The geometrical meaning of the operator $\mathfrak{I}$ follows from the obvious identity $\boldsymbol{x} \cdot \mathfrak{I}(\boldsymbol{y}) \equiv (\boldsymbol{x} \wedge \boldsymbol{y}) \cdot \boldsymbol{e}_3$ where the vector product $\wedge$ is to be interpreted with the non-canonical embedding of $\mathbb{R}^2$ in $\mathbb{R}^3$ ($x\boldsymbol{e}_1 + y\boldsymbol{e}_2 \mapsto x\boldsymbol{e}_1 + y\boldsymbol{e}_2 + 0\boldsymbol{e}_3$), hence

$\boldsymbol{x} \cdot \mathfrak{I}(\boldsymbol{y})$ is the signed area of the parallelogram spanned by the two vectors $\boldsymbol{x}, \boldsymbol{y}$. Finally, from the property $\mathfrak{I}^2(\boldsymbol{x}) = -\boldsymbol{x}$ it also follows that:

$$\boldsymbol{x} \cdot \mathfrak{I}(\boldsymbol{x}) = \mathfrak{I}(\boldsymbol{x}) \cdot \mathfrak{I}^2(\boldsymbol{x}) = -\boldsymbol{x} \cdot \mathfrak{I}(\boldsymbol{x})$$

Hence $\mathfrak{I}(\boldsymbol{x}) \in \big(\mathrm{Span}\{\boldsymbol{x}\}\big)^{\perp}$, as it could also be derived by observing that the matrix that represents $\mathfrak{I}$ in the canonical basis is the rotation matrix of $-\frac{\pi}{2}$ (or equivalenlty, if one identifies $\mathbb{R}^2$ with $\mathbb{C}$, $\mathfrak{I}$ is simply the multiplication by $-i$).

## 2.1 2-dimensional lattices

**Definition 2.2** (Lattice)**.** A lattice $\Lambda$ is a finitely-generated, free $\mathbb{Z}$-submodule of $(\mathbb{R}^2, +)$, i.e. a set of the form:

$$\Lambda := \{m_1 \boldsymbol{a}_1 + m_2 \boldsymbol{a}_2, \quad (m_1, m_2) \in \mathbb{Z}^2\}$$

for some linearly indipendent set $\{\boldsymbol{a}_1, \boldsymbol{a}_2\} \subset \mathbb{R}^2$ called *lattice basis*.

The lattice basis is called positively (risp. negatively) oriented if the determinant of the endomorphism that maps the canonical basis of $\mathbb{R}^2$ to $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is positive (risp. negative).
We call a *lattice cell* a set of the kind:

$$\mathcal{L} := \big\{\boldsymbol{x} + p\boldsymbol{\lambda} + q\boldsymbol{\lambda}', \quad (p, q) \in [0, 1]^2\big\}$$

for $\boldsymbol{\lambda}, \boldsymbol{\lambda}' \in \Lambda$ linearly indipendent. The area of the cell is defined in the obvious way:

$$|\mathcal{L}| := |\boldsymbol{\lambda} \cdot \mathfrak{I}(\boldsymbol{\lambda}')| \equiv |\mathfrak{I}(\boldsymbol{\lambda}) \cdot \boldsymbol{\lambda}'|$$

Each lattice $\Lambda$ has a minimal cell of non-zero surface, we call its area $|\Lambda|$.

Obviously, for a fixed lattice, there exists more than one lattice basis. Namely, if $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is a lattice basis for $\Lambda$, then $\{\boldsymbol{b}_1 := b_{11}\boldsymbol{a}_1 + b_{12}\boldsymbol{a}_2, \boldsymbol{b}_2 := b_{21}\boldsymbol{a}_1 + b_{22}\boldsymbol{a}_2\}$ is still a lattice basis iff for all $n_1, n_2 \in \mathbb{Z}$, the vector $\boldsymbol{v} := n_1\boldsymbol{b}_1 + n_2\boldsymbol{b}_2$ is still in $\Lambda$. Plugging in the expression for $\boldsymbol{b}_1, \boldsymbol{b}_2$ wrt to the old basis vectors:

$$\boldsymbol{v} = (n_1 b_{11} + n_2 b_{21})\boldsymbol{a}_1 + (n_1 b_{12} + n_2 b_{22})\boldsymbol{a}_2$$

Hence all the coefficients $b_{ij}$ must be integers. This is, however, not enough because we need the area of cell spanned by $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ to be left invariant by the transformation. Using the definition of $\mathfrak{I}$ it is easy to see[1] that, if we let $M := \left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right]$, then the area tranforms as:

$$|\mathfrak{I}(\boldsymbol{b}_1) \cdot \boldsymbol{b}_2| = |\det M||\mathfrak{I}(\boldsymbol{a}_1) \cdot \boldsymbol{a}_2|$$

[1] Let $\boldsymbol{a}_i = x_i\boldsymbol{e}_1 + y_i\boldsymbol{e}_2$ for $i = 1, 2$, then $\boldsymbol{b}_k = (b_{k1}x_1 + b_{k2}x_2)\boldsymbol{e}_1 + (b_{k1}y_1 + b_{k2}y_2)\boldsymbol{e}_2$. Hence:

$$\begin{aligned}\mathfrak{I}(\boldsymbol{b}_1) \cdot \boldsymbol{b}_2 =& (b_{11}y_1 + b_{12}y_2)(b_{21}x_1 + b_{22}x_2) - (b_{11}x_1 + b_{12}x_)(b_{21}y_1 + b_{22}y_2) \\ =& (b_{11}b_{22} - b_{12}b_{21})(x_2y_1 - x_1y_2) \equiv (\det M)(\boldsymbol{a}_1 \cdot \mathfrak{I}(\boldsymbol{a}_2))\end{aligned}$$

Hence we also require $|\det M| = 1$.
At last, if we ask for the orientation of the basis to be preserved (the reason will be clear in a second), we should restrict to matrices with positive determinant. In other words, for a given lattice all its basis of a chosen orientation can be computed from a starting one via action of the special linear group $\mathbf{SL}(2, \mathbb{Z})$.

Intuitively, a key concept in the study of lattices is the shape. Having reduced the problem to two dimension allows us to treat such notion extremely easy thanks to the fact that $\mathbb{R}^2$ is isomorphic to $\mathbb{C}$ (we will use the non-canonical mapping $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \mapsto z(\boldsymbol{x}) \equiv x := x_1 + ix_2$). The complex moltiplication does, indeed, represent the operations that we expect the lattice shape to be invariant underneath: rotation of the basis vector and rescaling. In other words, given a lattice $\Lambda$, for every complex number $\lambda$ intuition tells us that the rescaled and rotated lattice $\lambda\Lambda := \{z^{-1}(\lambda v), \quad \boldsymbol{v} \in \Lambda\}$ has the same "shape" of $\Lambda$. As a consequence, a necessary condition for a number $\tau \in \mathbb{C}$ to represent the shape of the lattice $\Lambda$ is it being blind to complex multiplication (i.e. $\tau(\Lambda) = \tau(\lambda\Lambda)$ for all $\lambda \in \mathbb{C}$). Since, of course, given a basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ of $\Lambda$, the corresponding basis vectors of $\lambda\Lambda$ are rotated by $\arg\lambda$ and rescaled of $|\lambda|$ (or, in the complex plane, they result from the moltiplication of $a_1, a_2$ by $\lambda$), then it is customary to take:

$$\tau := \frac{a_2}{a_1} \tag{2.2}$$

However, this is not a good definition because it is strongly basis-dependent:

1. First of all, its imaginary part depends on the orientation:

$$\tau = \frac{a_{21} + ia_{22}}{a_{11} + ia_{12}} = \frac{a_{11}a_{21} + a_{12}a_{22}}{|a_1|} + i\frac{a_{11}a_{22} - a_{12}a_{21}}{|a_1|}$$

   In particular, $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ is positively-oriented iff $\operatorname{Im}\tau > 0$. This is customarily solved by restricting to positively-oriented basis, hence we take the mapping $\Lambda \mapsto \tau$ to have value in the upper half-plane $\mathbb{H}$.

2. Even if restricted to basis of a given orientation, $\tau$ is not well-defined because given two basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ and $\{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ of the same lattice, then it is easy to see that:

$$\tau(\{\boldsymbol{b}_1, \boldsymbol{b}_2\}) = \frac{b_{22}\tau(\{\boldsymbol{a}_1, \boldsymbol{a}_2\}) + b_{21}}{b_{12}\tau(\{\boldsymbol{a}_1, \boldsymbol{a}_2\}) + b_{11}}$$

   where $\left[\begin{smallmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{smallmatrix}\right] \in \mathbf{SL}(2, \mathbb{Z})$ is the basis change matrix. Hence, the two $\tau$-s are related through a Möbius map with integer coefficients.
   Introducing, thus, an action of $\mathbf{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ in the following fashion[2]:

$$Mz := \frac{az + b}{cz + d} \qquad M = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \in \mathbf{SL}(2, \mathbb{Z}) \tag{2.3}$$

[2]Actually, since $Mz = (-M)z$ for all $z \in \mathbb{H}$, then the action should be defined of the quotient group $\mathbf{SL}(2, \mathbb{Z})/\{\pm 1\}$. It is easy to check that, for all $z \in \mathbb{H}$, $Mz \in \mathbb{H}$ and $(MN)z = M(Nz)$, hence the action is well-defined.

According to our definitions, for every lattice $\Lambda$, the image $\tau(\Lambda)$ is a whole orbit in the upper half-complex plane $\mathbb{H}$ of the newly-introduced action of $\mathbf{SL}(2,\mathbb{Z})$. Such action is neither free[3] nor transitive[4](had the action been transitive, there would have been only one lattice shape) but it is faithful[5] .This makes the search for a *fundamental domain* (a connected subset of $\mathbb{H}$ that containes one and only one point from each orbit) a non-trivial task. It can be shown (see e.g. [4]) that a fundamental domain for the action of $\mathbf{SL}(2,\mathbb{Z})$ on $\mathbb{H}$ is given by:

$$\mathbb{F} := \left\{ z \in \mathbb{H} : \qquad |z| \geq 1, \quad 2\,\mathrm{Re}\, z \in [-1,1] \right\} \tag{2.4}$$

For our purposes, this tells us that we can wlog choose a basis for any lattice $\Lambda$ such that $\tau(\Lambda) \in \mathbb{F}$.

Two useful examples of lattice shapes are the square and triangular lattices. The former is generated e.g. by the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, hence $\tau_{\mathrm{sq}} = i \equiv e^{i\pi/2}$. As for the latter, we can pick as a lattice basis $\left\{\boldsymbol{e}_1, \frac{1}{2}\boldsymbol{e}_1 + \frac{\sqrt{3}}{2}\boldsymbol{e}_2\right\}$, hence $\tau_{\mathrm{tr}} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \equiv e^{i\pi/3}$. In both cases the "natural" basis choice also guarantees $\tau_{\mathrm{sq}}, \tau_{\mathrm{tr}} \in \mathbb{F}$.

## 2.2 Abrikosov lattice states

With this ideas in mind, we are now going on to introduce lattice-periodic solutions of the GL equations. A first rough attempt would be the requirement of "exact" periodicity of both functions $(\psi, \boldsymbol{A})$ on the lattice $\Lambda$ (sometimes called *double periodicity*), i.e. for all $\boldsymbol{\lambda} \in \Lambda$, $\psi(\boldsymbol{x}+\boldsymbol{\lambda}) = \psi(\boldsymbol{x})$ (the same goes for $\boldsymbol{A}$). This is, however, too strong a requirement for we only need the physical quantities to be blind to lattice translations, hence we can allow for a change in $(\psi, \boldsymbol{A})$ provided that it leads to gauge-equivalent state:

**Definition 2.3** (Abrikosov lattice states)**.** Let $\Lambda$ be a lattice. We say that $(\psi, \boldsymbol{A}) \in \mathcal{X}$ is an *Abrikosov* $\Lambda$*-lattice state* iff there exists a group homomorphism $(\Lambda, +) \ni \boldsymbol{\lambda} \mapsto \chi_\lambda \in (H^2(\Omega, \mathbb{R}), +)$ such that the actions of the two groups on $(\psi, \boldsymbol{A})$ are equivariant wrt such homomorphism, i.e.:

$$\begin{cases} (\mathcal{T}_\lambda \psi)(\boldsymbol{x}) = (\mathcal{G}_{\chi_\lambda} \psi)(\boldsymbol{x}) \\ (\mathcal{T}_\lambda \boldsymbol{A})(\boldsymbol{x}) = (\mathcal{G}_{\chi_\lambda} \boldsymbol{A})(\boldsymbol{x}) \end{cases} \tag{2.5}$$

for all $\boldsymbol{x} \in \mathbb{R}^2$.

It is easy to see that the gauge, lattice translation-equivariance is not only sufficient to preserve the physical quantities but it is also necessary (from the conservation of the magnetic field and the superconductive current one gets the

[3]The equation $Mz = z$ for fixed $M$ has, in general, at least one solution.

[4]E.g. no matrix can send $i$ to a point $w$ with $\mathrm{Im}\, w > 1$, since $\mathrm{Im}\{Mi\} = (c^2 + d^2)^{-1} \leq 1$.

[5]Thanks to the identity of polynomials, the only way for the equation $Mz = z$ to hold for all $z \in \mathbb{H}$ is to choose $c = b = 0$ and $a = c$; futhermore, the requirement $\det M = 1$ fixes $c = a^{-1}$, hence $a = \pm 1$ because they are the only integers to have multplicative inverse in $\mathbb{Z}$.

same conditions).
Notice also that the same transformations on the functions that leave the Gibbs free energy invariant map Abrikosov states to Abrikosov states. In particular:

1. The gauge and translational symmetries map $\Lambda$-lattice states to $\Lambda$-lattice states.

2. The rotation matrix $R \in \mathcal{O}(2)$ maps $\Lambda$-lattice states to $U_R\Lambda$-lattice states.

Notice that, although the lattice may change due to the application of a symmetry transformation, the shape is invariant in each of the three cases.

On a given Abrikosov state, due to the periodicity of the physical quantities, their integral over the whole superconductive volume (which we have chosen to be the whole $\mathbb{R}^2$) diverges. However, it is more natural to focus on the following quantities:

**Definition 2.4** (Average over a lattice)**.** Let $\Lambda$ be a lattice. For every function $f : \mathbb{R}^2 \to \mathbb{R}$ periodic wrto the given lattice, we define its average over $\Lambda$ the integral:

$$\langle f \rangle_\Lambda := \frac{1}{|\mathcal{L}|} \int_{\mathcal{L}} f(\boldsymbol{x}) \mathrm{d}^2 x \tag{2.6}$$

where $\mathcal{L}$ is a $\Lambda$-lattice cell.

Notice that we named the integral "average over the lattice" and not "over the lattice *cell*" because it is completely indipendent of the choice of $\mathcal{L}$. Indeed, it is clear that the area of each cell is a multiple of $|\Lambda|$ (say $|\mathcal{L}| = k|\Lambda|$) and, since $f$ is doubly periodic, the integral can also be written as a sum of $k$ integrals on the $k$ minimal cells contained in $\mathcal{L}$, hence:

$$\langle f \rangle_\Lambda = \frac{1}{k|\Lambda|} k \int_{\mathcal{L}_0} f(\boldsymbol{x}) \mathrm{d}^2 x$$

only depends on the values of $f$ in the minimal cell $\mathcal{L}_0$ (which is a property of the lattice).

An obvious and useful example is the average Gibbs free energy:

$$G_\Lambda[\psi, \boldsymbol{A}] := \left\langle |\mathcal{D}_A \psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 + |\mathrm{curl}\boldsymbol{A} - h_0|^2 \right\rangle_\Lambda \tag{2.7}$$

### 2.2.1 Quantization of the magnetic flux

An important feature of the Abrikosov lattice states is the so-called quantization of the magnetic flux. In order to see what one means by that, let us fix a cell $\mathcal{L} := \{\boldsymbol{x} + p\boldsymbol{\lambda} + q\boldsymbol{\lambda}', \quad p, q \in [0, 1]\}$ and let us compute:

$$\Phi(\mathcal{L}) := \int_{\mathcal{L}} (\mathrm{curl}\boldsymbol{A})(\boldsymbol{x}) \mathrm{d}^2 x = \int_{\partial\mathcal{L}} (\boldsymbol{A} \cdot \boldsymbol{\tau}) \, \mathrm{d}\ell$$

Now, the boundary of $\mathcal{L}$ is of course made up of the four segments one gets by fixing $p \in \{0,1\}$, letting $q$ vary and then doing the converse. The orientations of the two curves with the same parameter fixed is of course oppsite, hence we have two terms of the following kind:

$$\begin{aligned} I_q &= \int_0^1 \mathrm{d}q\, \boldsymbol{A}(\boldsymbol{x}+\boldsymbol{\lambda}+q\boldsymbol{\lambda}')\cdot\frac{\boldsymbol{\lambda}'}{\|\boldsymbol{\lambda}'\|} - \int_0^1 \mathrm{d}q\, \boldsymbol{A}(\boldsymbol{x}+q\boldsymbol{\lambda}')\cdot\frac{\boldsymbol{\lambda}'}{\|\boldsymbol{\lambda}'\|} \\ &= \int_0^1 \mathrm{d}q\, \frac{\boldsymbol{\lambda}'}{\|\boldsymbol{\lambda}'\|}\cdot(\nabla\chi_\lambda)(\boldsymbol{x}+q\boldsymbol{\lambda}') = \chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_\lambda(\boldsymbol{x}) \end{aligned}$$

Furthermore, the curve with $q=0$ has the same orientation as the curve with $p=1$ and vice versa, hence:

$$\Phi(\mathcal{L}) = \chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_\lambda(\boldsymbol{x})-\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda})+\chi_{\lambda'}(\boldsymbol{x}) \tag{2.8}$$

Thanks to the algebraic properties of lattice states, such quantity has some significant features:

**Lemma 4.** *Let $(\psi,\boldsymbol{A})$ be an Abrikosov $\Lambda$-lattice state such that $\psi\neq 0$, then for all $\boldsymbol{\lambda},\boldsymbol{\lambda}'\in\Lambda$ the function:*

$$K_{\lambda,\lambda'}(\boldsymbol{x}) := \chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_\lambda(\boldsymbol{x})-\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda})+\chi_{\lambda'}(\boldsymbol{x})$$

*is indipendent of $\boldsymbol{x}$ and there exists $n\in\mathbb{Z}$ such that:*

$$K_{\lambda,\lambda'} = n2\pi\frac{\boldsymbol{\lambda}\cdot\mathfrak{J}(\boldsymbol{\lambda}')}{|\Lambda|} \tag{2.9}$$

*Proof.* Thanks to the gauge, lattice translation-equivariance of the state, one has:

$$((\mathcal{T}_\lambda\circ\mathcal{T}_{\lambda'})\psi)(\boldsymbol{x}) = e^{i\chi_{\lambda+\lambda'}(\boldsymbol{x})}\psi(\boldsymbol{x}) = e^{i\chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')}e^{i\chi_{\lambda'}(\boldsymbol{x})}\psi(\boldsymbol{x})$$

Where $\psi(\boldsymbol{x})\neq 0$, it must be: $e^{i[\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x})]}=1$, hence:

$$\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x}) = 2\pi n_{\lambda,\lambda'}(\boldsymbol{x})$$

for some integer $n_{\lambda,\lambda'}(\boldsymbol{x})\in\mathbb{Z}$ depending in general on each of the three vectors $\boldsymbol{x},\boldsymbol{\lambda},\boldsymbol{\lambda}'$.
However, the gauge-invariance of the vector potential forces to drop the $\boldsymbol{x}$-dependence as it implies $\nabla(\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x}))=\boldsymbol{0}$. We, thus, say:

$$\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x})\in 2\pi\,\mathbb{Z}$$

Thanks to the abelian property of the translational group, switching $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ does not change anything, hence also $\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda})-\chi_\lambda(\boldsymbol{x})\in 2\pi\,\mathbb{Z}$ (even though in general $n_{\lambda,\lambda'}\neq n_{\lambda',\lambda}$).
Hence we can conclude that:

$$\begin{aligned} K_{\lambda,\lambda'} =& \chi_\lambda(\boldsymbol{x}+\boldsymbol{\lambda}')+\chi_{\lambda'}(\boldsymbol{x})-\chi_\lambda(\boldsymbol{x})-\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda}) \\ =& \chi_{\lambda+\lambda'}(\boldsymbol{x})-2\pi n_{\lambda,\lambda'}-\chi_{\lambda'+\lambda}(\boldsymbol{x})+2\pi n_{\lambda',\lambda}\in 2\pi\,\mathbb{Z} \end{aligned}$$

Let us now go through the properties of the map $(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mapsto K_{\lambda,\lambda'}$:

1. It is skew-symmetric as the definition clearly shows.

2. It is bilinear:

$$\begin{aligned}K_{\lambda+\lambda',\lambda''} =&\chi_{\lambda+\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda}'')-\chi_{\lambda+\lambda'}(\boldsymbol{x})-\chi_{\lambda''}(\boldsymbol{x}+\boldsymbol{\lambda}+\boldsymbol{\lambda}')+\chi_{\lambda''}(\boldsymbol{x})\\ =&2\pi n_{\lambda,\lambda'}+\chi_{\lambda}(\boldsymbol{x}+\boldsymbol{\lambda}''+\boldsymbol{\lambda}')+\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda}'')+\\ &-2\pi n_{\lambda,\lambda'}-\chi_{\lambda}(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x})-\chi_{\lambda''}(\boldsymbol{x}+\boldsymbol{\lambda}+\boldsymbol{\lambda}')+\chi_{\lambda''}(\boldsymbol{x})\\ =&\chi_{\lambda}(\boldsymbol{x}+\boldsymbol{\lambda}'+\boldsymbol{\lambda}'')-\chi_{\lambda}(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda''}(\boldsymbol{x}+\boldsymbol{\lambda}'+\boldsymbol{\lambda})+\chi_{\lambda''}(\boldsymbol{x}+\boldsymbol{\lambda}')+\\ &+\chi_{\lambda'}(\boldsymbol{x}+\boldsymbol{\lambda}'')-\chi_{\lambda''}(\boldsymbol{x}+\boldsymbol{\lambda}')-\chi_{\lambda'}(\boldsymbol{x})+\chi_{\lambda''}(\boldsymbol{x})\\ \equiv&K_{\lambda,\lambda''}+K_{\lambda',\lambda''}\end{aligned}$$

It follows that $K_{\lambda,\lambda'}$ shares the same properties of the application $(\boldsymbol{\lambda},\boldsymbol{\lambda}')\mapsto 2\pi\frac{\boldsymbol{\lambda}\cdot\jmath(\boldsymbol{\lambda}')}{|\Lambda|}$ (the area of each lattice cell is an integer multiple of the minimal cell), hence the two functions are proportional and the coefficient lies in $\mathbb{Z}$. □

Thanks to the Lemma and the previous calculation, we can collect the results of this section in the following

**Theorem 3** (Quantization of the magnetic flux)**.** *Let $(\psi,\boldsymbol{A})$ be a non-conductive (i.e. $\psi\neq 0$) Abrikosov $\Lambda$-lattice state, then for every $\Lambda$-lattice cell $\mathcal{L}$ there exists an integer $n\in\mathbb{Z}$ such that:*

$$\int_{\mathcal{L}}(\mathrm{curl}\boldsymbol{A})(\boldsymbol{x})\mathrm{d}^2x=2\pi n \tag{2.10}$$

*Equivalently, one can say that the average magnetic field on the lattice $\Lambda$ is given by:*

$$\langle\mathrm{curl}\,\boldsymbol{A}\rangle_{\Lambda}=\frac{2\pi n}{|\Lambda|} \tag{2.11}$$

Restoring units, one finds that the flux of the magnetic field over a minimal cell $\mathcal{L}$ is:

$$\int_{\mathcal{L}}(\mathrm{curl}\boldsymbol{A})(\boldsymbol{x})\mathrm{d}^2x=n2\pi\sqrt{2}H_C\frac{\lambda^2}{\kappa}\equiv n\frac{2\pi\hbar c}{e^*}$$

For this reason, we may call the quantity:

$$\Phi_0:=\frac{2\pi\hbar c}{e^*} \tag{2.12}$$

*fluxon* or magnetic flux quantum.

## 2.3 Perturbative approach

### 2.3.1 Gauge fixing

Now that we have explored some general properties of the Abrikosov lattice states, we are going to discuss such states within the GL theory.

For every fixed lattice shape $\tau \in \mathbb{F}$, we can obviously build a family of lattices that share the same shape but have bases:

$$\{\boldsymbol{a}_1 := \ell \boldsymbol{e}_1, \boldsymbol{a}_2 := \ell \boldsymbol{\tau}\} \tag{2.13}$$

where $\boldsymbol{\tau}$ is the element of $\mathbb{R}^2$ associated to $\tau$ by the isomorphism $\mathbb{R}^2 \leftrightarrow \mathbb{C}$ and $\ell$ is a positive real number. As a consequence, the cell $\mathcal{L}$ spanned by $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ has area $|\mathcal{L}| = \ell^2 \operatorname{Im} \tau$.
We are going to use the following Theorem to fix the gauge:

**Theorem 4.** *Let $(\psi, \boldsymbol{A})$ be an Abrikosov $\Lambda$-lattice state with average magnetix field per cell $h$ and basis $\{\boldsymbol{a}_1, \boldsymbol{a}_2\}$ as in Eq. (2.13). There exists an Abrikosov $\Lambda$-lattice state $(\hat{\psi}, \boldsymbol{A}_0^h + \hat{\boldsymbol{A}})$ gauge-equivalent to a translation of $(\psi, \boldsymbol{A})$ such that:*

1. $\boldsymbol{A}_0^h(\boldsymbol{x}) := -\frac{h}{2}\mathfrak{J}(\boldsymbol{x})$.
2. $\hat{\boldsymbol{A}}$ *is doubly periodic wrt* $\Lambda$.
3. $\langle \hat{\boldsymbol{A}} \rangle_\Lambda = 0$.
4. $\operatorname{div} \hat{\boldsymbol{A}} = 0$
5. $(\mathcal{T}_{a_k} \hat{\psi})(\boldsymbol{x}) = e^{i\frac{h}{2}\boldsymbol{a}_k \cdot \mathfrak{J}(\boldsymbol{x})} \hat{\psi}(\boldsymbol{x})$ *for* $k = 1, 2$.

*Proof.* See [28]. □

The vector potential $\boldsymbol{A}_0^h$ depends linearly on the average magnetic field $h$. Since we know from the previous subsection that there is a direct relation between such value, the lattice area and the number of quanta per cell, we wish to get rid of such dependence. Namely we know that:

$$h := \langle \operatorname{curl} \boldsymbol{A} \rangle_\Lambda = \frac{2\pi n}{\ell^2 \operatorname{Im} \tau}$$

where $\tau$ is the shape of the lattice $\Lambda$ and $(\psi, \boldsymbol{A})$ is an Abrikosov state on such lattice.
Notice that, since $\tau \in \mathbb{H}$, $h$ and $n$ always share the same sign, hence we introduce the real quantity:

$$\zeta := \sqrt{\frac{n}{h}} \equiv \ell \sqrt{\frac{\operatorname{Im} \tau}{2\pi}}$$

and rescale the state accordingly:

$$\begin{cases} \phi(\boldsymbol{x}) := \zeta \psi(\zeta \boldsymbol{x}) \\ \boldsymbol{\alpha}(\boldsymbol{x}) := \zeta \boldsymbol{A}(\zeta \boldsymbol{x}) \end{cases} \tag{2.14}$$

The lhs reduce to the rhs when $\zeta = 1$, hence if $(\psi, \boldsymbol{A})$ is a $\Lambda$-lattice state (where $\Lambda$ is spanned by $\{\ell \boldsymbol{e}_1, \ell \boldsymbol{\tau}\}$), then $(\phi, \boldsymbol{\alpha})$ is a $\Lambda_\tau$-lattice state, where the normalized lattice $\Lambda_\tau$ is $\Lambda_\tau := \sqrt{\frac{2\pi}{\operatorname{Im} \tau}} \Lambda \equiv \ell_\tau \Lambda$.
Notice that the area of the cell $\mathcal{L}_\tau$ spanned by $\{\ell_\tau \boldsymbol{e}_1, \ell_\tau \boldsymbol{\tau}\}$ is $|\mathcal{L}_\tau| = 2\pi$ and the

two lattices have of course the same shape.

As for the energy per lattice cell:

$$\begin{aligned} G_{\mathcal{L}_\tau}[\psi, \boldsymbol{A}] =& \frac{1}{2\pi}\int_{\mathcal{L}_\tau}\left[|\mathcal{D}_A\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2-1)^2 + |\mathrm{curl}\boldsymbol{A} - h_0|^2\right]\mathrm{d}^2x \\ =& \frac{1}{2\pi}\int_{\mathcal{L}_\tau}\left[\left|\frac{i}{\zeta^2}\nabla\phi + \frac{1}{\zeta^2}\phi\boldsymbol{\alpha}\right|^2 + \frac{\kappa^2}{2}\left(\frac{|\phi|^2}{\zeta^2}-1\right)^2 + \left|\frac{1}{\zeta^2}\mathrm{curl}\boldsymbol{\alpha} - h_0\right|^2\right]\mathrm{d}^2x \\ =& \frac{1}{2\pi\zeta^4}\int_{\mathcal{L}_\tau}\left[|\mathcal{D}_\alpha\phi|^2 + \frac{\kappa^2}{2}(|\phi|^2-\zeta^2)^2 + \left|\mathrm{curl}\boldsymbol{\alpha} - \zeta^2 h_0\right|^2\right]\mathrm{d}^2x \end{aligned}$$

Hence:

$$G_{\mathcal{L}_\tau}[\phi, \boldsymbol{\alpha}] = \frac{1}{2\pi\zeta^4}\int_{\mathcal{L}_\tau}\left[|\mathcal{D}_\alpha\phi|^2 + \frac{\kappa^2}{2}(|\phi|^2-\zeta^2)^2 + \left|\mathrm{curl}\boldsymbol{\alpha} - \zeta^2 h_0\right|^2\right]\mathrm{d}^2x \quad (2.15)$$

The rescaled GL equations follow immediately:

$$\boxed{\begin{cases} \triangle_\alpha\phi + \kappa^2(|\phi|^2-\zeta^2)\phi = 0 \\ \mathrm{curl}^\dagger\,\mathrm{curl}\,\boldsymbol{\alpha} + \mathrm{Re}\{\phi^*\mathcal{D}_\alpha\phi\} = \boldsymbol{0} \end{cases}} \quad (2.16)$$

Finally, if $(\psi, \boldsymbol{A})$ is chosen in the form prescribed by Theorem (4), then the resulting $(\phi, \boldsymbol{\alpha})$ have the following properties:

1. $\boldsymbol{\alpha} = \boldsymbol{A}_0^n + \boldsymbol{a}$ with $\boldsymbol{A}_0^n := -\frac{n}{2}\mathfrak{I}(\boldsymbol{x})$.
2. $(\phi, \boldsymbol{a})$ satisfy the properties 2.-5. of the Theorem in the lattice $\Lambda_\tau$

Notice that the parameter $\zeta$ contributes to fixing the state $(\phi, \boldsymbol{\alpha})$ as it enters the fist PDE (to different $\zeta$-s will in general correspond different states).

From now on, unless stated otherwise, we will work in the gauge fixed by the Theorem and in the normalized lattice $\Lambda_\tau$. For clarity of notation, we will go back to calling $(\phi, \boldsymbol{\alpha})$ with the standard $(\psi, \boldsymbol{A}_0^n + \boldsymbol{A})$.

### 2.3.2 Normal-state perturbations and Abrikosov function

We will follow a perturbative approach similar to the usual one in QM to account for small variations around a known state (which is going to be the conductive one). Before blindly writing formal expansions for each of the three quantities that determine a state, let us have a look at the equations (2.16). The physical situation we are going to account for is the following: we fix the temperature of the sample and turn on the magnetic field so that the phase is normal. We then lower the field quasi-statically until we reach the critical value. Just underneath it, the order parameter slighty deviates from the normal-state value ($\psi = 0$). In the perturbative approach, we say that the variation of the order parameter

is of the first order in some parameter $\varepsilon$. According to the second equation, however, since the current density is quadratic in $\psi$, the first non-vanishing term in the vector potential formal expansion is the order $\varepsilon^2$ (in general, only the even-power terms will be non-zero). This last statement is also true for $\zeta$, as it is clear from the first equation. Finally, since the variations in $\boldsymbol{A}$ and $\zeta$ are non-vanishing only for the even powers of $\varepsilon$, it follows that the expansion for $\psi$ only has odd-power terms.
Hence we take:

$$\begin{aligned} \psi_\varepsilon =& \varepsilon\psi_1 + \mathcal{O}\big(\varepsilon^3\big) \\ \boldsymbol{A}_\varepsilon =& \varepsilon^2\boldsymbol{A}_1 + \mathcal{O}\big(\varepsilon^4\big) \\ \zeta_\varepsilon^2 =& n\kappa^{-2} + \varepsilon^2\zeta_1 + \mathcal{O}\big(\varepsilon^4\big) \end{aligned} \tag{2.17}$$

The zero-th order term $\zeta_0$ in the $\zeta^2$ expansion is of course its value on the verge of the normal phase, i.e. for $h_0 \to h_{C2}$ from above, hence it can be evaluated by requiring the Gibbs free energy per cell (2.15) to be fixed at the value $\kappa^2/2$ as $h_0$ decreases with $(\psi = 0, \boldsymbol{A}_0^n)$:

$$\frac{\kappa^2}{2} = \frac{\kappa^2}{2} + \left|\frac{n}{\zeta_0} - h_{C2}\right|^2 + \mathcal{O}\big(\varepsilon^4\big)$$

Hence $h_{C2}\zeta_0 = n$. As we are going to see in the following subsection (Eq. 2.28), $h_{C2} = \kappa^2$.
The remaining terms give the linearized GL system:

$$\boxed{\begin{aligned} \triangle_{A_0^n}\psi_1 =& n\psi_1 \\ \operatorname{curl}\boldsymbol{A}_1 =& \frac{1}{2}\big[\,\langle|\psi_1|^2\rangle_{\Lambda_\tau} - |\psi_1|^2\big] \end{aligned}} \tag{2.18}$$

The first equation is straightforward if one plugs the perturbative expansion in the first equation of the system (2.16):

$$\begin{aligned} 0 =& \varepsilon\triangle_{A_0^n+\varepsilon^2 A_1}\psi_1 + \varepsilon\kappa^2(\varepsilon^2|\psi_1|^2 - n\kappa^{-2})\psi_1 + \mathcal{O}\big(\varepsilon^3\big) \\ =& \varepsilon\big[\triangle_{A_0^n}\psi_1 - n\psi_1\big] + \mathcal{O}\big(\varepsilon^3\big) \end{aligned}$$

For the second one we are going to need the following:

**Lemma 5** (Ladder operators)**.** *One can write:*

$$L_n := \triangle_{A_0^n} - n \equiv -\eta_n^\dagger\eta_n \tag{2.19}$$

*where $\eta_n, \eta_n^\dagger$ are the annhilation and creation operators:*

$$\eta_n := 2\frac{\partial}{\partial z^*} + \frac{n}{2}z \qquad\qquad \eta_n^\dagger := 2\frac{\partial}{\partial z} - \frac{n}{2}z^* \tag{2.20}$$

*with $z := x_1 + ix_2$. Such operators satisfy the relation $\big[\eta_n^\dagger, \eta_n\big] = 2n$.*

*Proof.* Thanks to the definition $\triangle_A := \mathcal{D}_A^\dagger \circ \mathcal{D}_A$ and the fact that $\operatorname{div} \boldsymbol{A}_0^n = 0$:

$$L_n = -\triangle + in(\boldsymbol{x} \cdot \operatorname{curl}^\dagger) + \frac{n^2}{4}\|\boldsymbol{x}\|^2 - n$$

If we let, like in the statement of the Lemma, $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \mapsto z = x_1 + ix_2$, then of course we have the following identifications: $\mathfrak{I}(\boldsymbol{x}) \mapsto -iz$, $\nabla f \mapsto 2f_{z^*}$, $\triangle f \mapsto 4f_{z^*z}$ and $\boldsymbol{x} \cdot \boldsymbol{y} \mapsto \operatorname{Re}\{z^* w\}$ with $\boldsymbol{x} \mapsto z$ and $\boldsymbol{y} \mapsto w$. Plugging into the expression for $L_n$:

$$\begin{aligned} L_n &= -4\frac{\partial^2}{\partial z^* \partial z} + i2n \operatorname{Im}\left\{ z^* \frac{\partial}{\partial z^*} \right\} + \frac{n^2}{4}|z|^2 - n \\ &= -4\frac{\partial^2}{\partial z^* \partial z} - n\left[ z\frac{\partial}{\partial z} - z^*\frac{\partial}{\partial z^*} \right] + \frac{n^2}{4} z^* z - n \end{aligned}$$

So one could write:

$$\begin{aligned} L_n f &= -4\frac{\partial^2 f}{\partial z^* \partial z} - n\left[ f + z\frac{\partial f}{\partial z} \right] + nz^*\frac{\partial f}{\partial z^*} + \frac{n^2}{4} z^* z f \\ &= -4\frac{\partial^2 f}{\partial z^* \partial z} - n\frac{\partial (zf)}{\partial z} + nz^*\frac{\partial f}{\partial z^*} + \frac{n^2}{4} z^* z f \\ &= -2\frac{\partial}{\partial z}\left[ 2\frac{\partial f}{\partial z^*} + \frac{n}{2} zf \right] + \frac{n}{2} z^*\left[ 2\frac{\partial f}{\partial z^*} + \frac{n}{2} zf \right] \\ &= -\left[ 2\frac{\partial}{\partial z} - \frac{n}{2} z^* \right]\left[ 2\frac{\partial f}{\partial z^*} + \frac{n}{2} zf \right] \end{aligned}$$

Proving the commutation relation is then straightforward:

$$\left[\eta_n^\dagger, \eta_n\right] f = n\frac{\partial (zf)}{\partial z} - nz^*\frac{\partial f}{\partial z^*} + n\frac{\partial (z^* f)}{\partial z^*} - nz\frac{\partial f}{\partial z} = 2nf$$

□

Such situation is well known in QM where the Hamiltonian for the 1-dimensional harmonic oscillator (of unit mass and frequency) $\mathcal{H} := \frac{1}{2}\left[ -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 \right]$ can be rewritten as $\mathcal{H} = a^\dagger a + \frac{1}{2}$ for two proper operators $a, a^\dagger$ satisfying $\left[a^\dagger, a\right] = -1$. The spectrum of the operator $a^\dagger a$ then follows from its algebraic properties and it is $\sigma(a^\dagger a) = \mathbb{N}_0$, hence its secular equation reads $a^\dagger a \psi_\nu = \nu\psi_\nu$ for $\nu \in \mathbb{N}_0$. Thanks to the Lemma, we have just seen that, for fixed number of magnetic flux quanta $n$, the first linearized equation in (2.18) is precisely of such kind with $\nu = 0$ and $a = \eta_n$ (notice that actually the commutator is not exactly the identity but it is proportional to it, this will have consequences in the following). Hence, it follows from such an analogy that the first order term in the order parameter expansion around the normal solution lies in the kernel of the annhilation operator $\eta_n$, i.e. $\eta_n \psi_1 = 0$ (just like the ground state of the harmonic oscillator satisfies $a\psi_0 = 0$):

**Theorem 5.** *Let $\psi$ satisfy the first equation in (2.18). Then $\psi \in \ker \eta_n$, with $\eta_n$ defined in the Lemma (5).*

We are now ready to derive the equation for $\boldsymbol{A}_1$: first of all let us plug the perturbative ansantz (2.17) in the second equation from (2.16):

$$0 = \varepsilon^2 \big[ \mathrm{curl}^\dagger \, \mathrm{curl} \, \boldsymbol{A}_1 + \mathrm{Re}\big\{ \psi_1^* \mathcal{D}_{A_0^n} \psi_1 \big\} \big] + \mathcal{O}\big(\varepsilon^4\big) \tag{2.21}$$

If one writes out the annhilation operator explicitely, one realizes that it has the following relation with the operator $\mathcal{D}_{A_0^n}$:

$$\eta_n = i\partial_2 + \frac{n}{2} x_1 - i \left[ i\partial_1 - \frac{n}{2} x_2 \right] \equiv (\mathcal{D}_{A_0^n})_2 - i(\mathcal{D}_{A_n})_1$$

Since $\eta_n \psi_1 = 0$, then:

$$\begin{aligned}
0 =& \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_2 \psi_1 \big\} + \mathrm{Im}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_1 \psi_1 \big\} \\
=& \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_2 \psi_1 \big\} + \frac{1}{2} \big[ \psi_1^* \partial_1 \psi_1 + \psi_1 \partial_1 \psi_1^* \big] = \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_2 \psi_1 \big\} + \frac{1}{2} \partial_1 |\psi_1|^2 \\
0 =& \mathrm{Im}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_2 \psi_1 \big\} - \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_1 \psi_1 \big\} \\
=& \frac{1}{2} \big[ \psi_1^* \partial_2 \psi_1 + \psi_1 \partial_2 \psi_1^* \big] - \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_1 \psi_1 \big\} = \frac{1}{2} \partial_2 |\psi_1|^2 - \mathrm{Re}\big\{ \psi_1^* \big(\mathcal{D}_{A_0^n}\big)_1 \psi_1 \big\}
\end{aligned}$$

Hence $\mathrm{Re}\big\{ \psi_1^* \mathcal{D}_{A_0^n} \psi_1 \big\} = \frac{1}{2} (\partial_2 |\psi_1|^2, -\partial_1 |\psi_1|^2) \equiv \frac{1}{2} \, \mathrm{curl}^\dagger \, |\psi_1|^2$.
Since $\mathrm{curl}^\dagger \, f \equiv \mathfrak{I}(\nabla f)$ and $\mathfrak{I}$ is a linear, bijective operator, then there exists a constant $C$ such that:

$$\mathrm{curl} \, \boldsymbol{A}_1 + \frac{1}{2} |\psi_1|^2 = C$$

Since the total vector potential (to the second order in $\varepsilon$) is $\boldsymbol{A}_0^n + \varepsilon^2 \boldsymbol{A}_1$ and we have required that the flux of the magnetic field through the cell $\mathcal{L}_\tau$ of the normalized lattice is $2\pi n$, it follows that:

$$0 = \int_{\mathcal{L}_\tau} \mathrm{curl} \boldsymbol{A}_1 \mathrm{d}^2 x = 2\pi C - \frac{1}{2} \int_{\mathcal{L}_\tau} |\psi_1|^2 \mathrm{d}^2 x$$

this fixes the constant and gives the second equation in (2.18).

As for the energy, we first of all prove the following relation:

**Lemma 6.** *The first order expansion term of the order parameter around the normal solution satisfies the following integral equation:*

$$\left[ \frac{1}{2} \left\langle |\psi_1|^{\,2} \right\rangle_{\Lambda_\tau} - \kappa^2 \zeta_1 \right] \left\langle |\psi_1|^{\,2} \right\rangle_{\Lambda_\tau} + \left[ \kappa^2 - \frac{1}{2} \right] \left\langle |\psi_1|^{\,4} \right\rangle_{\Lambda_\tau} = 0 \tag{2.22}$$

*Proof.* If one multiplies the first equation in (2.16) scalarly in $L^2(\mathcal{L}_\tau)$ by $\psi_1$ and substitutes the perturbative expansions (2.17), one gets:

$$0 = \varepsilon(\psi_1, \triangle_{A_0^n + \varepsilon^2 A_1} \psi_1) - \varepsilon n(\psi_1, \psi_1) + \varepsilon^3 \kappa^2 \big[ (\psi_1, |\psi_1|^2 \psi_1) - \zeta_1(\psi_1, \psi_1) \big] + \mathcal{O}\big(\varepsilon^4\big)$$

Now notice that, by definition of $\triangle_A := \mathcal{D}_A^\dagger \circ \mathcal{D}_A$, it follows the non-linear relation:

$$\begin{aligned}\triangle_{A+B} f =& i \operatorname{div} \mathcal{D}_{A+B} f + \langle \boldsymbol{A}+\boldsymbol{B},\, \mathcal{D}_{A+B} f\rangle \\ =& -\triangle f + i \operatorname{div}\big[f(\boldsymbol{A}+\boldsymbol{B})\big] + i\langle \boldsymbol{A}+\boldsymbol{B},\, \nabla f\rangle + f\|\boldsymbol{A}+\boldsymbol{B}\|^2 \\ =& \triangle_A f + if \operatorname{div} \boldsymbol{B} + 2i\langle \boldsymbol{B}, \mathcal{D}_A f\rangle + f\|\boldsymbol{B}\|^2\end{aligned}$$

In our case, the last term is of the order $\varepsilon^4$, hence it can be neglected. As for the other ones, since we chose $\operatorname{div}\boldsymbol{A}_1 = 0$:

$$\begin{aligned}0 =& \varepsilon(\psi_1, (\triangle_{A_0^n} - n)\psi_1) + \varepsilon^3\big(\psi_1, 2\langle \boldsymbol{A}_1, \mathcal{D}_{A_0^n}\psi_1\rangle + \kappa^2|\psi_1|^2\psi_1 - \kappa^2\zeta_1\psi_1\big) + \mathcal{O}\big(\varepsilon^4\big) \\ =& \varepsilon^3\big[2\big(\psi_1, \langle \boldsymbol{A}_1, \mathcal{D}_{A_0^n}\psi_1\rangle\big) + \kappa^2\|\psi_1\|^4_{L^4(\mathcal{L}_\tau)} - \kappa^2\zeta_1\|\psi_1\|^2_{L^4(\mathcal{L}_\tau)}\big] + \mathcal{O}\big(\varepsilon^4\big)\end{aligned}$$

where the first order term is of course zero if $\psi_1 \in \ker L_n$ as the first equation in (2.18) requires.
Now, the second and third term in the rhs of the previous equation are both real, so for the equality to hold at the thid order in $\varepsilon$, the first term must also be real (i.e. one could replace $\big(\psi_1, \langle \boldsymbol{A}_1, \mathcal{D}_{A_0^n}\psi_1\rangle\big)$ with its real part as they must coincide). By Eq. (2.21), the second Equation in (2.18) and thanks to the Lemma (3), one could rewrite the term as:

$$\begin{aligned}\big(\psi_1, \langle \boldsymbol{A}_1, \mathcal{D}_{A_0^n}\psi_1\rangle\big) =& \int_{\mathcal{L}_\tau} \langle \boldsymbol{A}_1,\, \mathrm{Re}\big\{\psi_1^* \mathcal{D}_{A_0^n}\psi_1\big\}\rangle \mathrm{d}^2x = \\ =& \mathcal{O}\big(\varepsilon^4\big) - (\boldsymbol{A}_1, \operatorname{curl}^\dagger \operatorname{curl}\boldsymbol{A}_1)_{L^2(\mathcal{L}_\tau, \mathbb{R}^2)} \\ =& \mathcal{O}\big(\varepsilon^4\big) - \|\operatorname{curl}\boldsymbol{A}_1\|^2_{L^2(\mathcal{L}_\tau)} \\ =& \mathcal{O}\big(\varepsilon^4\big) - \frac{1}{4}\big[2\pi\,\langle|\psi_1|^2\rangle^2_{\Lambda_\tau} - 2\,\langle|\psi_1|^2\rangle_{\Lambda_\tau}\|\psi_1\|^2_{L^2(\mathcal{L}_\tau)} + \|\psi_1\|^4_{L^4(\mathcal{L}_\tau)}\big] \\ =& \mathcal{O}\big(\varepsilon^4\big) - \frac{1}{4}\big[\|\psi_1\|^4_{L^4(\mathcal{L}_\tau)} - \langle|\psi_1|^2\rangle_{\Lambda_\tau}\|\psi_1\|^2_{L^2(\mathcal{L}_\tau)}\big]\end{aligned}$$

Hence, plugging back into the original equation gives the desired relation:

$$0 = \varepsilon^3\left[\left(\kappa^2 - \frac{1}{2}\right)\|\psi_1\|^4_{L^4(\mathcal{L}_\tau)} + \left(\frac{1}{2}\langle|\psi_1|^2\rangle_{\Lambda_\tau} - \kappa^2\zeta_1\right)\|\psi_1\|^2_{L^2(\mathcal{L}_\tau)}\right] + \mathcal{O}\big(\varepsilon^4\big)$$

□

The Lemma helps us to prove:

**Proposition 5** (Perturbative expansion of the Gibbs free-energy)**.** *If the Eqs (2.16) have a solution of the form (2.17), then we have the following expansion for the average Gibbs free-energy of the supercondutor:*

$$G_{\zeta_\varepsilon}[\psi_\varepsilon, \boldsymbol{A}_0^n + \boldsymbol{A}_\varepsilon] = \frac{\kappa^2}{2} + \varepsilon^4(n\zeta_1)^2\left[1 - \frac{1}{1 + (2\kappa^2 - 1)\beta(\psi_1, \tau)}\right] + \mathcal{O}\big(\varepsilon^6\big) \quad (2.23)$$

*where:*

$$\beta(\psi,\tau) := \frac{\langle|\psi|^4\rangle_{\Lambda_\tau}}{\langle|\psi|^2\rangle^2_{\Lambda_\tau}} \tag{2.24}$$

*is the so-called* Abrikosov function.

*Proof.* Let us go back to the original system (2.16) in the chosen gauge and repeat more or less the same procedure of the Lemma: let us scalarly multiply the first equation by $\psi$ and use the definition of $\triangle_A$:

$$0 = \|\mathcal{D}_\alpha\psi\|^2_{L^2(\mathcal{L}_\tau,\mathbb{C}^2)} + \kappa^2\|\psi\|^4_{L^4(\mathcal{L}_\tau)} - (\kappa\zeta)^2\|\psi\|^2_{L^2(\mathcal{L}_\tau)}$$

where $\boldsymbol{\alpha}$ is short for $\boldsymbol{A}^n_0 + \boldsymbol{A}$.
It follows that the average Gibbs free energy in every lattice (2.15) can be rewritten as:

$$\begin{aligned} G_{\zeta_\varepsilon}[\psi_\varepsilon,\boldsymbol{\alpha}_\varepsilon] =& \frac{1}{2\pi\zeta^4}\int_{\mathcal{L}_\tau}\left[\zeta^2|\psi|^2 - |\psi|^4 + \frac{\kappa^2}{2}(|\psi|^2-\zeta^2)^2 + \left|\operatorname{curl}\boldsymbol{\alpha} - \zeta^2 h_{C2}\right|^2\right]\mathrm{d}^2x \\ =& \frac{1}{2\pi\zeta^4}\int_{\mathcal{L}_\tau}\left[\frac{\kappa^2}{2}\zeta^4 - \frac{\kappa^2}{2}|\psi|^4 + \left|\operatorname{curl}\boldsymbol{\alpha} - (\zeta\kappa)^2\right|^2\right]\mathrm{d}^2x \\ =& \frac{\kappa^2}{2} + \frac{1}{2\pi\zeta^4}\left[\left\|\operatorname{curl}\boldsymbol{\alpha} - (\zeta\kappa)^2\right\|^2_{L^2(\mathcal{L}_\tau)} - \frac{\kappa^2}{2}\|\psi\|^4_{L^4(\mathcal{L}_\tau)}\right] \end{aligned}$$

Plugging in the perturbative expansions (2.17) gives:

$$G_{\zeta_\varepsilon}[\psi_\varepsilon,\boldsymbol{\alpha}_\varepsilon] = \frac{\kappa^2}{2} + \frac{\varepsilon^4}{2\pi\zeta^4}\left[\left\|\operatorname{curl}\boldsymbol{A}_1 - \zeta_1\kappa^2\right\|^2_{L^2(\mathcal{L}_\tau)} - \frac{\kappa^2}{2}\|\psi_1\|^4_{L^4(\mathcal{L}_\tau)}\right] + \mathcal{O}\big(\varepsilon^6\big)$$

Recall that $\langle\operatorname{curl}\boldsymbol{A}_1\rangle_{\Lambda_\tau} = 0$ and substitute the second equation in (2.18) in the last term of the order $\varepsilon^4$:

$$\begin{aligned} \Delta G =& \varepsilon^4\frac{\kappa^4\zeta_1^2}{\zeta^4} - \frac{\varepsilon^4}{4\pi\zeta^4}\left[\left(\kappa^2 - \frac{1}{2}\right)\|\psi_1\|^4_{L^4(\mathcal{L}_\tau)} + \frac{1}{2}\left\langle|\psi_1|^2\right\rangle_{\Lambda_\tau}\|\psi_1\|^2_{L^2(\mathcal{L}_\tau)}\right] + \mathcal{O}\big(\varepsilon^6\big) \\ =& \varepsilon^4\frac{\kappa^4\zeta_1^2}{\zeta^4} - \frac{\varepsilon^4}{2\zeta^4}\kappa^2\zeta_1\left\langle|\psi_1|^2\right\rangle_{\Lambda_\tau} + \mathcal{O}\big(\varepsilon^6\big) \end{aligned}$$

where $\Delta G$ denotes the difference from the normal-phase free energy and we have also used the previous Lemma.
It is now only a metter of rewriting the last term as a function of $\beta$ thanks to the Lemma:

$$\begin{aligned} \kappa^2\zeta_1\left\langle|\psi_1|^2\right\rangle_{\Lambda_\tau} =& \frac{1}{2}\left\langle|\psi_1|^2\right\rangle^2_{\Lambda_\tau} + \left(\kappa^2 - \frac{1}{2}\right)\left\langle|\psi_1|^4\right\rangle_{\Lambda_\tau} \\ \frac{\kappa^2\zeta_1}{\left\langle|\psi_1|^2\right\rangle_{\Lambda_\tau}} =& \frac{1}{2} + \left(\kappa^2 - \frac{1}{2}\right)\beta(\psi_1,\tau) \end{aligned}$$

This clearly gives the expansion for the energy if plugged back in (and thanks to the obvious expansion $\zeta^{-4} = \kappa^4 n^{-2} + \mathcal{O}\big(\varepsilon^2\big)$). □

### 2.3.3 Spectrum of $\triangle_{A_0^n}$ & upper critical field $H_{C2}$

Thanks to the linearization we have just carried out, we can compute the value of the upper critical field $h_{C2}$ that marks the transition from the normal to the mixed state: it is, by definition, the lowest field such that the first equation in the system (2.18) has a non-trivial solution (i.e. $\psi_1 \neq 0$). Actually, since we have transformed the $h$-dependence away from such equations thanks to the rescaling, one should consider the equation obtained via linearization around $\boldsymbol{A}_0^h$ (so no need to bother about $\zeta$). Such relation is of course:

$$\triangle_{A_0^h}\psi = \kappa^2\psi \tag{2.25}$$

where we have suppressed the subscript for clarity's sake.
Much has already been said in the previous discussion about the secular equation for the operator $\triangle_{A_0^n}$, which is analogous to the previous equation if we identify $h$ with $n$. In particular, we recall the results from Lemma (5) that defines the ladder operators for $L_n := \triangle_{A_0^n} - n$ and provides the commutation relation $\left[\eta_n^\dagger, \eta_n\right] = 2n$. It follows that, if $f_k$ is an eigenfunction for $L_n$ wrt the eigenvalue $k$, then:

$$L_n\eta_n^\dagger f_k = -\eta_n^\dagger\eta_n\eta_n^\dagger f_k = \eta_n^\dagger\big(-\eta_n^\dagger\eta_n + 2n\big)f_k = (k+2n)\eta_n^\dagger f_k$$

Equivalently $L_n\eta_n f_k = (k-2n)\eta_n f_k$ and, by definition of $L_n$, it follows:

$$\triangle_{A_0^n}\eta_n^\dagger f_k = (k+3n)\eta_n^\dagger f_k \qquad ; \qquad \triangle_{A_0^n}\eta_n f_k = (k-3n)\eta_n f_k$$

However, since $\triangle_{A_n^0}$ is positively-defined, such procedure must cease before $k - 3n \leq 0$, hence there must be some $f_0$ such that $L_n f_0 = 0$ (then of course $\triangle_{A_0^n} f_0 = n f_0$) and all the other eigenvectors can be evaluated through action of $\eta_n^\dagger$. Consequently, we get the spectrum $\sigma(L_n) = \{2kn, \quad k \in \mathbb{N}_0\}$ and:

**Theorem 6.** *The operator $\triangle_{A_0^n}$ has spectrum:*

$$\sigma(\triangle_{A_0^n}) = \{(2\nu+1)n \quad \nu \in \mathbb{N}_0\} \tag{2.26}$$

For the equation (2.25) to have a non-trivial solution, then it must be $\kappa^2 \in \sigma(\triangle_{A_0^h})$, i.e.:

$$\kappa^2 = (2\nu+1)h \tag{2.27}$$

for some $\nu \in \mathbb{N}_0$.
For fixed superconductor (i.e. $\kappa$ has a certain, well-defined value), the highest value of $h$ that can satisfy the above equation is $h = \kappa^2$ ($\nu = 0$). This is of course the upper critical field $h_{C2}$ because for every $h > h_{C2}$ the only solution to the equation (2.25) is $\psi = 0$ (the normal phase). Restoring the units:

$$H_{C2} = \sqrt{2}\kappa H_C \tag{2.28}$$

Under our assumption for $\kappa$, $H_{C2}$ is stronger than $H_C$.

We complete this subsection by explicitely compute the functions that made up the space $\ker L_n$. By Theorem (5), of course such space is equal to $\ker \eta_n$, hence we are going to consider the equation:

$$\frac{\partial \psi}{\partial z^*} = -\frac{n}{4} z \psi \tag{2.29}$$

which is solved by $\psi(z, z^*) = e^{-\frac{n}{4}|z|^2}\xi(z)$ for some entire function $\xi$.
However, in our problem we should restrict to $\Lambda_\tau$-quasiperiodic solutions in the sense of property 5. from Theorem (4) with $h = n$ and on the lattice basis $\{\boldsymbol{a}_1 = \ell_\tau \boldsymbol{e}_1,\, \boldsymbol{a}_2 = \ell_\tau \boldsymbol{\tau}\}$. It follows:

$$\begin{aligned}
\psi(z + a_1, z^* + a_1^*) =& e^{-\frac{n}{4}(|z|^2 + 2\ell_\tau \operatorname{Re} z + \ell_\tau^2)} \xi(z + a_1) = e^{i\frac{n}{2}\ell_\tau \operatorname{Im} z - \frac{n}{4}|z|^2}\xi(z) \\
\psi(z + a_2, z^* + a_2^*) =& e^{-\frac{n}{4}(|z|^2 + \ell_\tau^2|\tau|^2 + 2\ell_\tau \operatorname{Re}\{\tau^* z\})} \xi(z + a_2) \\
=& e^{i\frac{n}{2}\ell_\tau \operatorname{Im}\{\tau^* z\} - \frac{n}{4}|z|^2}\xi(z)
\end{aligned}$$

i.e.:

$$\begin{aligned}
\xi(z + \ell_\tau) =& \xi(z) e^{\frac{n}{4}\ell_\tau(\ell_\tau + 2z)} \\
\xi(z + \ell_\tau \tau) =& \xi(z) e^{\frac{n}{4}\ell_\tau(\ell_\tau |\tau|^2 + 2\tau^* z)}
\end{aligned} \tag{2.30}$$

If one observes that the exponent in the first relation (up to the factor $n/4$) can be rewritten as $(z + \ell_\tau)^2 - z^2$, then such equation implies the $\ell_\tau$-periodicity of the function $z \mapsto e^{-\frac{n}{4}z^2}\xi(z)$. Hence we can write the Fourier-like expansion[6]:

$$\xi(z) = e^{\frac{n}{4}z^2} \sum_{\nu \in \mathbb{Z}} \theta_\nu \exp\left[ i 2\nu \frac{\pi}{\ell_\tau} z \right]$$

The second relation fixes the periodicity of the coefficients:

$$\begin{aligned}
\xi(z + \ell_\tau \tau) =& e^{\frac{n}{4}(z^2 + 2\ell_\tau \tau z + \ell_\tau^2 \tau^2)} \sum_{\nu \in \mathbb{Z}} \theta_\nu \exp\left[ i 2\nu \frac{\pi}{\ell_\tau} z + i 2\pi \nu \tau \right] \\
=& e^{\frac{n}{4}(z^2 + 2\ell_\tau \tau^* z + \ell_\tau^2 |\tau|^2)} \sum_{\nu \in \mathbb{Z}} \theta_\nu \exp\left[ i 2\nu \frac{\pi}{\ell_\tau} z \right]
\end{aligned}$$

Hence:

$$\sum_{\nu \in \mathbb{Z}} \theta_\nu \exp\left[ i 2\nu \frac{\pi}{\ell_\tau} z \right] = \sum_{\nu \in \mathbb{Z}} \theta_\nu \exp\left[ i 2 \frac{\pi}{\ell_\tau} (\nu + n) z + i 2\pi \nu \tau + i n \pi \tau \right]$$

[6] Let $f$ be a meromorphic function in $\mathbb{C}$ with period $\omega$, then the function $F$ defined by $F(e^{i2\pi z/\omega}) = f(z)$ is well-defined, meromorphic in $\mathbb{C} \setminus \{0\}$ and, as a consequence, admits a Laurent expansion in every annulus that contains none of its poles: $F(\zeta) = \sum_{\nu \in \mathbb{Z}} a_\nu \zeta^\nu$. Such expansion can be "pulled-back" to $f$ thanks to the definition (i.e. by setting $\zeta(z) = e^{i2\pi\nu z/\omega}$) and leads to a Fourier-like series for $f$.

Shifting $\nu \mapsto \nu + n$ in the lhs gives:

$$\theta_{\nu+n} = \theta_\nu e^{i\pi(n+2\nu)\tau} \tag{2.31}$$

It follows that only $n$ of the $\theta_\nu$-s are indipendent. The results are collected in the following:

**Theorem 7** (Eigenfunctions of the linearized GL equation)**.** *Each and every $\Lambda_\tau$-Abrikosov state function $\psi_\tau$ that also lies in the kernel of $L_n = \triangle_{A_0^n} - n$ can be written as:*

$$\psi_\tau(z, z^*) = \sum_{\nu\in\mathbb{Z}} \theta_\nu e^{i2\nu\frac{\pi}{\ell_\tau}z + \frac{n}{4}(z^2-|z|^2)}, \qquad \theta_{\nu+n} = \theta_\nu e^{i\pi(n+2\nu)\tau} \tag{2.32}$$

*with the isomorphism $\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 \mapsto z = x_1 + ix_2$.*
*In particular, this implies that* $\dim_{\mathbb{C}} \ker L_n|_{\Lambda_\tau} = n$.

## 2.4 Energy-minimizing lattice shape near $H_{C2}$

So far we have worked for general values of $\tau$, now we are going to study how the free energy depends on such parameter and, as a consequence, what lattice shape turns out to be more stable near the critical point $h_{C2}$.
We are going to fix $n = 1$ so that the order parameter $\psi$ is determined (up to a scaling factor) by the choice of the lattice shape according to Theorem (7). Accordingly, $\beta$ is a function only of $\tau \in \mathbb{F}$.
We are going to fix the scaling factor of $\psi_\tau$ so that $\theta_0 = 1$. The other coefficients, then, follow:

$$\theta_\nu = e^{i\pi\nu\tau} \begin{cases} \exp\left[i2\pi\tau \sum_{k=0}^{\nu-1} k\right] & \nu > 0 \\ \exp\left[i2\pi\tau \sum_{k=0}^{-\nu} k\right] & \nu \leq 0 \end{cases} = e^{i\pi\nu^2\tau}$$

Hence the eigenfunction becomes:

$$\begin{aligned} \psi_\tau(z, z^*) &= \sum_{\nu\in\mathbb{Z}} \exp\left[i2\pi\nu\frac{z}{\ell_\tau} + i\pi\nu^2\tau + \frac{1}{4}(z^2 - |z|^2)\right] \\ &\equiv \exp\left[\frac{1}{4}(z^2 - |z|^2)\right] \Theta\left(\frac{z}{\ell_\tau}, \tau\right) \end{aligned} \tag{2.33}$$

where $\Theta(u, w) = \sum_{\nu\in\mathbb{Z}} e^{i\pi\nu(w\nu+2u)}$ (with $(u, w) \in \mathbb{C}\times\mathbb{H}$) is the Jacobi Theta function with characteristics $(a, b) = (0, 0)$ (sometimes also denoted with $\theta_3$).
The vortex cores are of course determined by the lattice shape and are, in particular, those points $z$ such that $\psi = 0$ (superconductor in the normal state). Namely, they are the zeros of the Theta function which, according to [12], are points that satisfy the relation $u = \nu + 1/2 + (\mu + 1/2)w$ for $\mu, \nu \in \mathbb{Z}$, hence the cores are located at:

$$z_{\text{core}} \in \sqrt{\frac{2\pi}{\operatorname{Im}\tau}}\left[\mathbb{Z} + \frac{1}{2} + \left(\mathbb{Z} + \frac{1}{2}\right)\tau\right] \tag{2.34}$$

the notation being $\mathbb{Z}x := \{\nu x, \quad \nu \in \mathbb{Z}\}$ for every $x \in \mathbb{C}$.

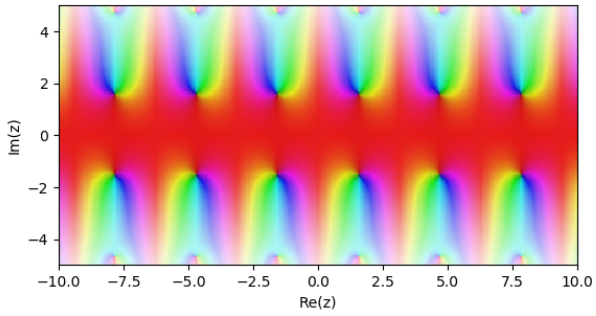


**Figure 2.1:** *Domain coloring of* $\Theta(z/\ell_\tau, \tau)$ *for* $\tau = e^{i\pi/2}$.

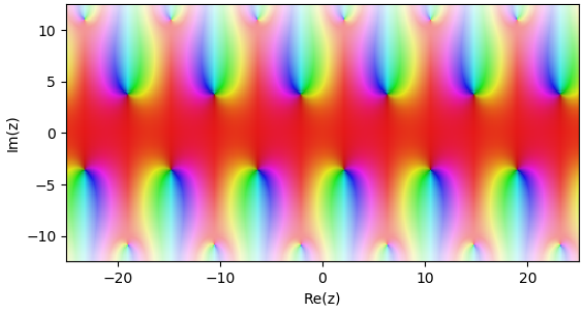


**Figure 2.2:** *Domain coloring of* $\Theta(z/\ell_\tau, \tau)$ *for* $\tau = e^{i\pi/3}$.

According to the perturbative expansion for the Gibbs free energy from Proposition (5), since $2\kappa^2 - 1 > 0$, we see that the leading term is an increasing function of $\beta$. Since all the other parameters (namely $n$ and $\kappa$) are fixed, we can equivalently study the minimization of the Abrikosov function in order to determine the most stable lattice configuration.
According to the definition, we need to evaluate two integrals involving $|\psi_\tau|^2$ which is given by the series representation:

$$|\psi_\tau|^2(z, z^*) = \sum_{(\mu,\nu)\in\mathbb{Z}^2} \exp\left[\frac{(z - z^*)^2}{4} + i2\frac{\pi}{\ell_\tau}(\nu z - \mu z^*) + i\pi(\nu^2\tau - \mu^2\tau^*)\right] \tag{2.35}$$

Such a function is of course doubly-periodic wrt $\Lambda_\tau$ (as it represents the pair density) but such property is not well exploited by the current isomorphism $\mathbb{R}^2 \leftrightarrow \mathbb{C}$ as it uses coordinates wrt the canonical basis instead of the lattice basis $\{\ell_\tau \boldsymbol{e}_1, \ell_\tau \boldsymbol{\tau}\}$. We are, then, going to change our isomorphism and map $\boldsymbol{x} = w_1\ell_\tau\boldsymbol{e}_1 + w_2\ell_\tau\boldsymbol{\tau} \mapsto w := w_1 + iw_2$ so that $|\psi_\tau|^2$ is periodic with period 1 wrto the coordinates $(w_1, w_2)$. The explicit transformation is easily evaluated using the definition $\boldsymbol{\tau} = \operatorname{Re}\tau\boldsymbol{e}_1 + \operatorname{Im}\tau\boldsymbol{e}_2$ since, for all $\boldsymbol{x} \in \mathbb{R}^2$:

$$\boldsymbol{x} = w_1\ell_\tau\boldsymbol{e}_1 + w_2\ell_\tau\boldsymbol{\tau} = \ell_\tau(w_1 + w_2\operatorname{Re}\tau)\boldsymbol{e}_1 + \ell_\tau w_2 \operatorname{Im}\tau\boldsymbol{e}_2$$

Hence, $w$ is given in terms of $z := x_1 + ix_2$ as:

$$\frac{z}{\ell_\tau} = w_1 + \tau w_2 \equiv \frac{1 - i\tau}{2} w + \frac{1 + i\tau}{2} w^* \tag{2.36}$$

Plugging into the series expansion for $|\psi_\tau|^2$ gives (recall $\ell_\tau^2 \operatorname{Im}\tau = 2\pi$):

$$|\psi_\tau|^2(w, w^*) = \sum_{(\mu,\nu)\in\mathbb{Z}^2} \exp\Big[\frac{\pi}{2} \operatorname{Im}\tau (w - w^*)^2 + i\pi a^\tau_{\mu\nu}(w, w^*)\Big] \tag{2.37}$$

with:

$$\begin{aligned} a^\tau_{\mu\nu}(w, w^*) :=& w\big[\nu(1 - i\tau) - \mu(1 - i\tau^*)\big] + w^*\big[\nu(1 + i\tau) - \mu(1 + i\tau^*)\big] + \\ &+ \nu^2\tau - \mu^2\tau^* \end{aligned}$$

The huge gain of all this rewriting is that we can now decompose $|\psi_\tau|^2$ onto the Fourier basis:

$$\Big\{ u_{\nu\nu'}(w_1, w_2) := e^{i2\pi(\nu w_1 + \nu' w_2)} \Big\}_{(\nu,\nu')\in\mathbb{Z}^2} \tag{2.38}$$

and the coefficients $c_{\nu\nu'}$ of such expansion completely determine the integrals we need in order to evaluate $\beta$:

$$\begin{aligned} \|\psi_\tau\|^2_{L^2(\mathcal{L}_\tau)} =& 2\pi\big(u_{00}, |\psi_\tau|^2\big)_{L^2(\mathcal{L}_\tau)} \equiv 2\pi c_{00} \\ \|\psi_\tau\|^4_{L^4(\mathcal{L}_\tau)} =& \big(|\psi_\tau|^2, |\psi_\tau|^2\big)_{L^2(\mathcal{L}_\tau)} \equiv 2\pi \sum_{(\nu,\nu')\in\mathbb{Z}^2} |c_{\nu\nu'}|^2 \end{aligned}$$

where the factor $2\pi \equiv \ell_\tau^2 \operatorname{Im}\tau$ is of course the determinant of the Jacobian of the transformation $(w_1, w_2) \mapsto (x_1, x_2)$.
The evaluation of such coefficients is carried out in [24] with a formal "trick". The result is:

$$c_{\nu\nu'} = \frac{1}{\sqrt{\operatorname{Im}\tau}} (-1)^{\nu\nu'} \exp\Big[-\frac{\pi}{2\operatorname{Im}\tau} |\nu' - \nu\tau|^2\Big] \tag{2.39}$$

The Abrikosov function then follows:

$$\beta(\tau) = |\mathcal{L}_\tau| \left[\frac{\|\psi_\tau\|_{L^4(\mathcal{L}_\tau)}}{\|\psi_\tau\|_{L^2(\mathcal{L}_\tau)}}\right]^4 = \sum_{(\nu,\nu')\in\mathbb{Z}^2} \exp\Big[-\frac{\pi}{\operatorname{Im}\tau} |\nu' - \nu\tau|^2\Big] \tag{2.40}$$

It is easy to check that this expression carries the following symmetries:

**Lemma 7.** *The Abrikosov function $\beta(\tau)$ is invariant under the following transformation:*

1. *The unit translation along the real axis: $\tau \mapsto \tau + 1$.*
2. *The reflection around the imaginary axis: $\tau \mapsto -\tau^*$.*

*3. The inversion and reflection around the imaginary axis:* $\tau \mapsto -\tau^{-1}$.

Thanks to such symmetries the critical points of $\beta$ can be evaluated:

**Theorem 8** (Critical points of the Abrikosov function)**.** *The Abrikosov function* $\beta(\tau)$ *is stationary in* $\mathbb{F}$ *at the points* $\tau_{\rm sq} := e^{i\pi/2}$ *and* $\tau_{\rm tr} := e^{i\pi/3}$.

*Proof.* We need to prove that $\partial_{\tau_1}\beta, \partial_{\tau_2}\beta = 0$ when $\tau \in \{\tau_{\rm sq}, \tau_{\rm tr}\}$.
It is easy to see that the two points are invariant respectively under the transformations:

$$\begin{aligned} \tau_{\rm sq} &= -\tau_{\rm sq}^* & \qquad \tau_{\rm sq} &= -\tau_{\rm sq}^{-1} \\ \tau_{\rm tr} &= 1-\tau_{\rm tr}^* & \qquad \tau_{\rm tr} &= 1-\tau_{\rm tr}^{-1} \end{aligned}$$

Now, thanks to the Lemma, we se that for every $\nu \in \mathbb{N}_0$:

$$\beta(\nu - \tau^*) = \beta((\nu-1) - \tau^*) = \cdots = \beta(-\tau^*) = \beta(\tau)$$

Deriving this relation wrt $\tau_1$: $(\partial_{\tau_1}\beta)(\tau) + (\partial_{\tau_1}\beta)(\nu - \tau^*) = 0$.

For $\nu = 0$, this implies $(\partial_{\tau_1}\beta)(\tau_{\rm sq}) = 0$ while for $\nu = 1$ one can come to the conclusion that $(\partial_{\tau_1}\beta)(\tau_{\rm tr}) = 0$ using the invariances of the two points under the transformations $\tau \mapsto -\tau^*$ and $\tau \mapsto 1-\tau^*$ respectively.
For the derivatives wrt $\tau_2$, with the same procedure one finds that, for all $\nu \in \mathbb{N}_0$, $\beta(\nu - \tau^{-1}) = \beta(\tau)$. Using $\tau^{-1} = \tau^*/|\tau|^2$ and differentiation wrt $\tau_2$ gives:

$$(\partial_{\tau_2}\beta)(\tau) = \frac{\mathrm{Im}\{\tau^2\}}{|\tau|^4}(\partial_{\tau_1}\beta)(\nu - \tau^{-1}) + \frac{\mathrm{Re}\{\tau^2\}}{|\tau|^4}(\partial_{\tau_2}\beta)(\nu - \tau^{-1})$$

We already know that, for the two $\tau$-s at stake, $\partial_{\tau_1}\beta = 0$. Accordingly, for $\nu = 0$ (using $\tau_{\rm sq} = -\tau_{\rm sq}^{-1}$), one finds $(\partial_{\tau_2}\beta)(\tau_{\rm sq}) = -(\partial_{\tau_2}\beta)(\tau_{\rm sq})$ and the same goes with $\nu = 1$ for $\tau_{\rm tr}$ using its invariance under the transformation $\tau \mapsto 1 - \tau^{-1}$. □

It then follows that the energetically-favorable configuration close to the upper critical field must be chosen between the square and the triangular one. Numerical estimations for $\beta(\tau_{\rm sq}), \beta(\tau_{\rm tr})$ have been carried out in [2, 14] respectively with the results:

$$\beta(\tau_{\rm sq}) \approx 1.1803 \qquad \beta(\tau_{\rm tr}) \approx 1.1595 \tag{2.41}$$

It follows that *near the upper critical field, the most stable lattice configuration is the triangular one.*

### 2.4.1 Superconductive current density

Now that we have discussed the lattice shapes, let us briefly study, within the perturbational approach, the superconducting current density $\boldsymbol{j}_{\rm sc}$ defined in Eq.

(1.19). According to the second equation in (2.18), the second order term in the expansion is (restoring units):

$$\boldsymbol{J}_{\rm sc}^{(2)} = -\frac{e^*\hbar}{2m^*}\,\mathrm{curl}^\dagger\,|\psi_\tau|^2 \tag{2.42}$$

Using the complex formalism (in natural units) and Eqs (2.29), (2.33):

$$\begin{aligned} j_{\rm sc}^{(2)} =& i\frac{\partial}{\partial z^*}\big[\psi_\tau^*\psi_\tau\big] \\ =& i\left[\frac{z-z^*}{2}|\psi_\tau|^2(z,z^*) + \psi_\tau(z,z^*)e^{\frac{1}{4}[(z^*)^2-|z|^2]}\frac{\partial}{\partial z^*}\Theta^*(z_\tau,\tau)\right] \\ =& i\left[\frac{z-z^*}{2}|\psi_\tau|^2(z,z^*) + \psi_\tau(z,z^*)e^{\frac{1}{4}[(z^*)^2-|z|^2]}\Theta'^*(z_\tau,\tau)\right] \\ =& i|\psi_\tau|^2(z,z^*)\left[\frac{z-z^*}{2} + \left(\frac{\Theta'(z_\tau,\tau)}{\Theta(z_\tau,\tau)}\right)^*\right] \end{aligned} \tag{2.43}$$

where $z_\tau$ is short for $z/\ell_\tau$ and $'$ denotes the derivative of $\Theta$ wrt $z$.
Notice that the vortex structure around the core is due to the $\mathrm{curl}^\dagger$ operator. Indeed, the leading term in an expansion of $|\psi_\tau|^2$ around any vortex core $z_{\rm c} = x_{1,\rm c} + ix_{2,\rm c}$, is the second (the zero-th and first order terms both vanish as they are proportional to $\psi_\tau$ at the core), i.e. it is a positively-defined quadratic form $\mathfrak{h}$ represented, in coordinates $(x_1, x_2)$ wrt the canonical basis of $\mathbb{R}^2$, by the Hessian matrix (divided by two) $\left[\begin{smallmatrix} C_1 & C_3 \\ C_3 & C_2 \end{smallmatrix}\right]$ of $|\psi_\tau|^2$ at the core. By Sylvester's criterion, the level curves of such quadratic form are concentric ellipses centered in $(x_{1,\rm c}, x_{2,\rm c})$ and with principal axes spanned by:

$$\boldsymbol{u}_\pm := 2C_3\boldsymbol{e}_1 + \big[C_2 - C_1 \pm \Delta\big]\boldsymbol{e}_2$$

with $\Delta := \sqrt{(C_2-C_1)^2+4C_3^2}$.
Then, if we let $(u_+, u_-)$ be the coordinates of a vector $\boldsymbol{x} \in \mathbb{R}^2$ wrt the basis $\{\boldsymbol{u}_\pm\}$, then the quadratic form reduces to the diagonal expression:

$$\begin{aligned} \mathfrak{h}(\boldsymbol{x},\boldsymbol{x}) =& 2\big[C_2\big((C_2-C_1)^2+4C_3^2\big) + \Delta\big(C_2(C_2-C_1)+2C_3^2\big)\big](u_+ - u_{+,\rm c})^2 + \\ &+ 2\big[C_2\big((C_2-C_1)^2+4C_3^2\big) - \Delta\big(C_2(C_2-C_1)+2C_3^2\big]\big](u_- - u_{-,\rm c})^2 \\ =:& \left(\frac{u_+ - u_{+,\rm c}}{K_1}\right)^2 + \left(\frac{u_- - u_{-,\rm c}}{K_2}\right)^2 \end{aligned}$$

The $\mathrm{curl}^\dagger$ operator in the new coordinate system is:

$$\mathrm{curl}^\dagger f = -\frac{1}{4C_3\Delta}\left[\frac{\partial f}{\partial u_-}\boldsymbol{u}_+ - \frac{\partial f}{\partial u_+}\boldsymbol{u}_-\right]$$

As a consequence, the leading term in the expansion of $\boldsymbol{j}_{\rm sc}^{(2)}$ around a core is:

$$\boldsymbol{j}_{\rm sc}^{(2)} \approx -\frac{1}{2}(\mathrm{curl}^\dagger\mathfrak{h})(\boldsymbol{x},\boldsymbol{x}) = \frac{1}{4C_3\Delta}\left[\frac{u_- - u_{-,\rm c}}{K_2^2}\boldsymbol{u}_+ - \frac{u_+ - u_{+,\rm c}}{K_1^2}\boldsymbol{u}_-\right] \tag{2.44}$$

which is the usual simple, non-normalized, asymmetric vortex centered in $(u_{+,\mathrm{c}}, u_{-,\mathrm{c}})$, i.e. in the coordinate system $q_i := u_i - u_{i,\mathrm{c}}$ for $i \in \{+,-\}$, it is the vector field represented by $(q_+, q_-) \mapsto \frac{q_-}{a^2}\boldsymbol{q}_+ - \frac{q_+}{b^2}\boldsymbol{q}_-$.

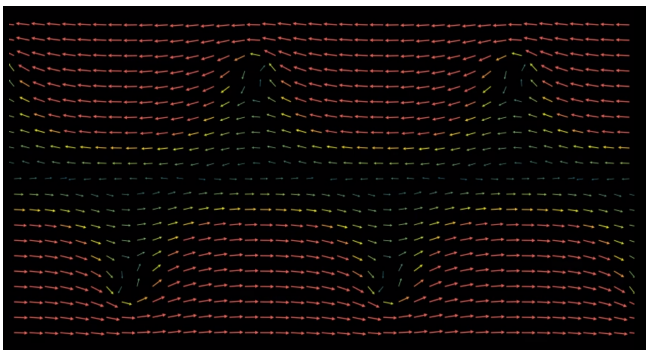

**Figure 2.3:** *Current density for the triangular lattice.*


## 2.5 Lower critical field $H_{C1}$ in the high $\kappa$ limit

We now search for the value of the external field $H_{C1}$ that marks the superconducting-mixed phase transition. Of course, the linearized system we have studied above is no longer a good approximation. However, to roughly predict the $\kappa$-dependence of the external field, it is sufficient to build an approximate solution. In particular, we will "guess" the mathematical shape of a single, isolated and cylindrical symmetric vortex using some "physical" arguments. Such a scenario is reasonable close to the lower critical field as the Meissner effect is annhilated by a gradual penetration of the field.
The cylindrical symmetry of course forces the order parameter and the vector potential to be functions only of the distance $r$ from the vortex core (lengths are still measured in units of $\lambda$). Furthermore, we will fix once again the Coulomb gauge, hence we can set $\boldsymbol{A} \cdot \boldsymbol{e}_r = 0$ to ensure the vanishing of $\mathrm{div}\boldsymbol{A}$:

$$\psi(r,\varphi) = \phi(r)e^{i\varphi} \qquad ; \qquad \boldsymbol{A}(r,\varphi) = a(r)\boldsymbol{e}_\varphi \tag{2.45}$$

Notice that the function $a(r)$ can be given in terms of the field $h(r)$ by inversion of the relation $h = \mathrm{curl}\boldsymbol{A}$:

$$a(r) = \frac{1}{r}\int_0^r h(u)u\,\mathrm{d}u \tag{2.46}$$

which gives as leading term close to the core $a(r) = \frac{h(0)}{2}r + \mathcal{O}\big(r^2\big)$.

First of all let us recall from the end of the first chapter that we expect the two unknown functions to experience large variations over different ranges, namely $\lambda$ for $\boldsymbol{A}$ and $\xi$ for $\psi$ (in our system of units 1 and $\kappa^{-1}$ respectively). Hence,

for large $\kappa$, $\psi$ is basically equal to its bulk value $\psi = 1$ except for a small set whose measure is of the order $\kappa^{-2}$. For the sake of symplicity, we take open neighborhood $B(0, \kappa^{-1})$.
As a consequence, we are only interested in computing a first order expansion of the order parameter $\psi$ around the origin thanks to which we are going to approximate its variation in the ball $B(0, \kappa^{-1})$ and then require continuity for $r = \kappa^{-1}$. Plugging the ansatz into the first GL equation yields:

$$0 = -\phi'' - \frac{1}{r}\phi' + \frac{1}{r^2}\phi - \frac{2}{r}a\phi + a^2\phi + \kappa^2(\phi^2 - 1)\phi \tag{2.47}$$

For $r < \kappa^{-1}$ we may use the first order approximation for $a$ in terms of $h(0)$, which leads to the equation:

$$\begin{aligned} 0 =& -r^2\phi'' - r\phi' + \phi - h(0)r^2\phi + \frac{h^2(0)}{4}r^4\phi + (\kappa r)^2(\phi^2 - 1)\phi \\ \approx& -r^2\phi'' - r\phi' + \phi \end{aligned} \tag{2.48}$$

where we dropped the higher-order terms as $r \to 0$. The resulting ODE is of the type $\sum_{k=0}^{d} r^k \phi^{(k)} = 0$ (Euler equation) for $d = 2$, hence can be solved by a power-law function $\phi(r) = r^k$. Plugging in the ansatz leads to the algebraic equation $0 = k(k-1) + k - 1 = k^2 - 1$ whose solutions are $|k| = 1$. Since we expect the order parameter at the core of the vortex to vanish (i.e. the sample is in the normal state), we are going only to consider the positive solution $k = 1$. The further requirement of continuity as one trespasses the boundary of the ball $B(0, \kappa^{-1})$ leads to the following solution:

$$\phi(r) = \begin{cases} \kappa r & r < \kappa^{-1} \\ 1 & r \geq \kappa^{-1} \end{cases} \tag{2.49}$$

As for $\boldsymbol{A}$, since we are interested in computing the Gibbs free energy which only depends on its curl, we may as well focus on $h := \mathrm{curl}\boldsymbol{A}$.
According to the second GL equation, using $\psi = \rho e^{i\vartheta}$:

$$0 = \mathrm{curl}^\dagger\, h + \rho^2(\boldsymbol{A} - \nabla\vartheta) \tag{2.50}$$

For $r > \kappa^{-1}$ it is $\rho = 1$, so we can find a simple equation for $h$ by taking the curl of both sides (using also the Coulomb gauge condition):

$$-\triangle h + h = 0 \qquad (r > \kappa^{-1}) \tag{2.51}$$

For $r < \kappa^{-1}$, the equation gets of course more complicated because $\rho$ is not constant. However, since the set is small (in the high-$\kappa$ limit), we are going to account for it by averaging over the domain: for large $\kappa$ the ball reduces almost to a points, hence we can treat the global statement as though it were local.

First of all let us notice that:

$$\int\limits_{B(0,\kappa^{-1})} \triangle h \,\mathrm{d}^2x = \int\limits_{\partial B(0,\kappa^{-1})} (\nabla h)\cdot \boldsymbol{n}\,\mathrm{d}\ell$$
$$= \int\limits_{\partial B(0,\kappa^{-1})} \mathfrak{I}(\nabla h)\cdot\mathfrak{I}(\boldsymbol{n})\,\mathrm{d}\ell = -\int\limits_{\partial B(0,\kappa^{-1})} (\mathrm{curl}^\dagger h)\cdot\boldsymbol{\tau}\,\mathrm{d}\ell$$

where $\boldsymbol{\tau} := -\mathfrak{I}(\boldsymbol{n})$ is the tangent unit vector to the circumference $\partial B(0,\kappa^{-1})$ and we used the isometry of $\mathfrak{I}$.
Now, if one requires continuity of $\rho$, it is obvious that $\rho(\partial B(0,\kappa^{-1})) = \{1\}$, hence (using Stokes' Theorem):

$$\int\limits_{B(0,\kappa^{-1})} \triangle h \,\mathrm{d}^2x = \int\limits_{\partial B(0,\kappa^{-1})} (\boldsymbol{A}-\nabla\vartheta)\cdot\boldsymbol{\tau}\,\mathrm{d}\ell = \int\limits_{B(0,\kappa^{-1})} h\,\mathrm{d}^2x - 2\pi$$

The similarity with Eq. (2.51) is crystal clear, the only difference being the additonal factor $2\pi$. In the high-$\kappa$ limit, the integral relation may then be translated into the point-by-point distributional equation:

$$-\triangle h + h = 2\pi\delta_0 \tag{2.52}$$

which accounts for Eq. (2.51) as well since it implies that the integral of the quantity $\triangle h + h$ over a set $E$ vanishes unless the vortex core lies in $E$. It is important to point out that such equation is not at all a direct consequence of the GL system but is a simplification constructed in such a way to ensure that global properties on the small set $B(0,\kappa^{-1})$ are maintained.
Under the assumption of $\boldsymbol{A}$ being a function only of the distance from the vortex core (hence $h$ too), the positive solution to the Eq. (2.51) is known and it is the first modified Bessel function of the second kind $K_0$. We recall the following asymptotics for such function:

$$\begin{aligned} K_0(r) &\underset{r\to 0}{\sim} |\log r| + C \\ K_0(r) &\underset{r\to\infty}{\sim} \sqrt{\frac{\pi}{2r}}e^{-r} \end{aligned} \tag{2.53}$$

The corresponding vector potential can be evaluated by integration of the defining relation $h = \mathrm{curl}\boldsymbol{A}$ over $B(0,r)$, using Stokes' Theorem and plugging in Eq. (2.52):

$$2\pi r a(r) = \int\limits_{B(0,r)} h\,\mathrm{d}^2x = 2\pi + \int\limits_{\partial B(0,r)} (\nabla h)\cdot\boldsymbol{n}\,\mathrm{d}^2x = 2\pi + 2\pi r h'(r)$$

Hence:

$$\boldsymbol{A}(r) = \left[\frac{1}{r} + h'(r)\right]\boldsymbol{e}_\varphi \tag{2.54}$$

One could use the properties of the modified Bessel functions of the second kind to write $\boldsymbol{A}$ more explicitely: $h'(r) = K_0(r) = -K_1(r)$. Since the limiting behaviour of $K_1$ as $r \to 0$ is $-r^{-1} + o(1)$, $a(r)$ is everywhere well-defined and continuous. Furthermore, $K_1$ still decays exponentially at large distances, hence $a$ is bounded.

We are now ready to compute an approximate expression for the energy difference between the single approximate vortex state we have just constructed and the superconductive bulk state. For the sake of symplicity, we are going to consider a superconductor that fills the whole space. However, in such case the total energy of the superconductive state is infinite, so we compute the Gibbs free energy in a neighborhood $B(0,R)$ for $R > \kappa^{-1}$ and make some estimations for $R \to \infty$. Since the order parameter $\psi$ has a different behavious outside and inside the ball $B(0,\kappa^{-1})$, it is natural to consider the two contribution to $\Delta G$ separately:

$$\begin{aligned}(\Delta G)(R) :=& G[\psi, \boldsymbol{A}, B(0,R)] - G[1, \boldsymbol{0}, B(0,R)] \\ \equiv& G[\psi, \boldsymbol{A}, B(0,\kappa^{-1})] + G[\psi, \boldsymbol{A}, B(0,R) \setminus B(0,\kappa^{-1})] - \pi R^2 h_0^2\end{aligned}$$

**Lemma 8.** *Let $G_\kappa$ be the difference between the Gibbs free energies in $B(0,\kappa^{-1})$ and $G_R$ that in $B(0,R) \setminus B(0,\kappa^{-1})$.*

1. *For all $\kappa > 1$, assuming $h_0 \leq \kappa^2$, $G_\kappa$ is bounded indipendently of $\kappa$.*
2. *For large $R$ and $\kappa$ we have the following behaviour:*

$$G_R \sim 2\pi\big[\log\kappa - 2h_0\big]$$

*Proof.* In the order:

1. The term reads:

$$\begin{aligned}G_\kappa &= \int\limits_{B(0,\kappa^{-1})} \mathrm{d}^2x \left[|\mathcal{D}_A\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 + (h-h_0)^2 - h_0^2\right] \\ &= \int\limits_{B(0,\kappa^{-1})} \mathrm{d}^2x \left[|\mathcal{D}_A\psi|^2 + \frac{\kappa^2}{2}(1-|\psi|^2)^2 + h(h-2h_0)\right]\end{aligned}$$

   Thanks to the approximate expression for $\phi$ (Eq. 2.49), we can say that both $\psi$ and $|\nabla\psi|$ are bounded. Furthermore, we have already noticed that the approximate expression for $a(r)$ (Eq. 2.54) is bounded as well. As a consequence there exist two constants $C_1, C_2 > 0$ such that:

$$|\mathcal{D}_A\psi|^2 \leq C_1\kappa^2 \qquad ; \qquad \frac{\kappa^2}{2}(1-|\psi|^2)^2 \leq C_2\kappa^2$$

   As for $h$, the limiting behaviour as $r \to 0$ tells us that the integral of its modulus and its squared modulus over $B(0,\kappa^{-1})$ both converge. Since (recall $h > 0$ by construction): $-2h_0 \leq h(h-2h_0) \leq h^2$ then:

$$|h(h-2h_0)| \leq \max\{2h_0, h^2\}$$

In both cases the integral over $B(0,\kappa^{-1})$ is bounded indipendently of $\kappa$ (if the maximum is $h^2$ is trivial, if it is $2h_0$, just use the hypotesis and the fact that the area of the ball is proportional to $\kappa^{-2}$). Using the triangular inequality, the first result then follows.

2. For $r > \kappa^{-1}$ we have $\phi = 1$, hence the second GL equation reduces to $\boldsymbol{A} = \mathrm{curl}^\dagger h \equiv \mathfrak{I}(\nabla h)$. Using the isometry of $\mathfrak{I}$:

$$\begin{aligned} G_R &= \int\limits_{B_R \setminus B_{1/\kappa}} \mathrm{d}^2x \big[|\boldsymbol{A}|^2 + (h-h_0)^2 - h_0^2\big] \\ &= \int\limits_{B_R \setminus B_{1/\kappa}} \mathrm{d}^2x \big[|\nabla h|^2 + h(h-2h_0)\big] \end{aligned}$$

Thanks to the integration by parts formula (Eq. 1.17) we can write (let $C_{R,1/\kappa} := B_R \setminus B_{1/\kappa}$):

$$\begin{aligned} \int\limits_{C_{R,1/\kappa}} \big[|\nabla h|^2 + h^2\big]\mathrm{d}^2x &= \int\limits_{\partial C_{R,1/\kappa}} h\,(\nabla h)\cdot\boldsymbol{n}\,\mathrm{d}\sigma + \int\limits_{C_{R,1/\kappa}} h\big[h - \triangle h\big]\mathrm{d}^2x \\ &= 2\pi\big[Rh(R)h'(R) - \kappa^{-1}h(\kappa^{-1})h'(\kappa^{-1})\big] \end{aligned}$$

while the other term can be computed with the divergence Theorem:

$$\begin{aligned} \int\limits_{C_{R,1/\kappa}} h &= \int\limits_{C_{R,1/\kappa}} \triangle h = \int\limits_{\partial C_{R,1/\kappa}} (\nabla h)\cdot\boldsymbol{n}\,\mathrm{d}\sigma \\ &= 2\pi\big[Rh'(R) - \kappa^{-1}h'(\kappa^{-1})\big] \end{aligned}$$

Hence, using the asymptotics for $h$ and $h'$:

$$\begin{aligned} G_R &= 2\pi R h'(R)\big[h(R) - 2h_0\big] - 2\pi\frac{h'(\kappa^{-1})}{\kappa}\big[h(\kappa^{-1}) - 2h_0\big] \\ &\sim -\frac{2\pi}{\kappa}\big(-\kappa\big)\big[\log\kappa - 2h_0\big] = 2\pi\big[\log\kappa - 2h_0\big] \end{aligned}$$

□

From the Lemma it follows that:

$$(\Delta G)(R,\kappa) \underset{R,\kappa\to\infty}{\sim} 2\pi\big[\log\kappa - 2h_0\big] + C \tag{2.55}$$

As a consequence, a rough estimation of $h_{C1}$ is given by the value of $h_0$ that changes the sign of the asymptotic for $\Delta G$, i.e.: $h_{C1} \approx (\log\kappa)/2$ up to a constant (which we are not going to write down because it is indipendent of $\kappa$ and our aim was simply to determine an approximate $\kappa$-dependence). Restoring the physical units:

$$H_{C1} \approx \frac{H_C}{\sqrt{2}}\frac{\log\kappa}{\kappa} \tag{2.56}$$

Notice that the product $H_{C1} \cdot H_{C2} = H_C^2 \log \kappa$ has a very mild $\kappa$-dependence, hence the critical field $H_C$ for bulk type-I superconductors (at fixed temperature) is approximately the geometric mean of the two critical fields $H_{C1}, H_{C2}$ at the same temperature.

| Material | $\mu_0 H_{C1}$ [mT] | $\mu_0 H_{C2}$ [mT] | $T_{\text{obs}}$ [K] |
|---|---|---|---|
| CNb | 12 | $1.69\times10^3$ | 4.2 |
| CTa | 22 | 460 | 1.2 |
| $Cd_{0.05}Hg_{0.95}$ | 28 | 31 | 2.16 |
| $In_{0.94}Pb_{0.06}$ | 9.5 | 18 | 3.12 |
| $In_{0.75}Tl_{0.25}$ | 21.6 | 50 | 3.16 |
| $Nb_{0.1}Ta_{0.9}$ | 8.4 | 15.4 | 4.195 |
| $O_3SrTi$ | 0.195 | 42.0 | 0 |
| V | 80 | 340 | 1.79 |

**Table 2.1:** *Critical fields for some type-II superconducting compounds and alloys at the temperature* $T_{\text{obs}}$.

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