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The Action of the Axisymmetric and Stationary Symmetry Group of General Relativity on a Static Black Hole

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Chapter 1

Introduction

General relativity is a theory of gravitation developed by Albert Einstein in 1915. Beside its mathematical beauty, General Relativity is a mile stone achievement in the understanding of our universe that changed entirely our perception of the notions of space and time. The theory can be summarize in the phrase attributed to the American physicist John Wheeler "*Space-time tells matter how to move; matter tells space-time how to curve*". This concept can be expressed through a mathematical formulation and indeed is embodied by Einstein's equations of General Relativity:

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

These are 10 equations in 4 variables of nonlinear partial differential equations, the task of solving these equation is rather tiresome even for simple cases, to complicate things in general doesn't exist a method to solve non linear partial differential equations hence it becomes cumbersome to solve this equations explicitly even using some ansatz.

One may object that Einstein's equations can be written in the more general form:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Including the cosmological constant Λ. This equation, even though holds very interesting proprieties, it will not be considered in this thesis, consequently henceforward the cosmological constant will be zero, the reason why will be explained later on.

Many different methods for generating new solutions without solving explicitly Einstein equations have been implemented. One of these methods is Ernst's generating technique. It consists in the observation that, in the case of electro-vacuum stationary axisymmetric spacetime, two complex scalar potentials can be defined, a gravitational potential $\mathcal{E}$ and a electromagnetic potential Φ so that the task of finding the solution of Einstein's is reduced to a coupled system of two complex partial differential equations, given an ansatz regarding the two potentials the problem can be reduced to a single differential equation. This equations hold true if and only if the cosmological constant is $\Lambda = 0$, indeed this is the case the Ernst studied and later efforts to expand his method to contain the cosmological constant have failed as shown in [7].

Ernst's technique can be used also to to generate new solution from already known one by observing that Ernst's equations have some sort of symmetry. Five transformations (3.12) can be generated by these symmetries and then applied to a know metric, called seed metric, in order to generate a new solution which in some cases is not different from the previous and can be transformed back to the original solution by coordinate transformation. In this thesis we analyze Ernst's generating technique starting from the Schwarzschild solution as seed metric. We analyze the electro-vacuum case, therefore $T_{\mu\nu}$ in this thesis is going to be the stress tensor of a sourceless, stationary and axisymmetric electromagnetic field. The reason is that on a macroscopic scale the weak and strong nuclear forces play a secondary role.

We will use natural units (c=G=1) throughout all this thesis, the signature is everywhere (-,+,+,+) except for Appendix B where is (+,-,-,-).

Chapter 2

Ernst's method

In this chapter we proceed by giving a definition of stationary and axisymmetric spacetime and we show how the line element can be expressed in a general form. We than summarize the theoretical basis behind [2].

2.1 Definition of axisymmetric and stationary spacetime

A spacetime is said to be stationary if there exist a timelike Killing vector ξ^{μ} whose orbits are complete. In a similar manner a spacetime is axisymmetric if there exists a spacelike Killing vector Ψ^{μ} whose integral curves are closed.
We call a spacetime stationary and axisymmetric if it possesses both these symmetries and if, in addition the two killing vectors commute:

$$[\xi, \Psi] = 0$$

These spacetimes are interesting in the frame of the theory of general relativity inasmuch as they describe equilibrium configurations of axisymmetric rotating bodies.
The commutativity of ξ^{μ} and Ψ^{μ} implies that we can choose coordinate like $(x^0 = t,\, x^1 = \phi,\, x^2,\, x^3)$ so that $\xi^{\mu} = \partial_t$ and $\Psi^{\mu} = \partial_{\phi}$ are coordinate vector fields. By doing so we set ourselves in such a coordinate system that the metric components will be independent of t an ϕ.
Thus far we still have a set of ten equations but in two coordinates.
To simplify things we can lean on the following theorem.

Theorem 2.1.1 *Let ξ^{μ} and Ψ^{μ} be two commuting Killing fields such that:*
I) $\xi_{[\mu}\Psi_{\nu}\nabla_{\sigma}\xi_{\lambda]}$ and $\xi_{[\mu}\Psi_{\nu}\nabla_{\sigma}\Psi_{\lambda]}$ each vanishes at at least one point of the spacetime.
II) $\xi^{\mu}R_{\mu}^{[b}\xi^{\sigma}\Psi^{\lambda]} = \Psi^{\mu}R_{\mu}^{[b}\xi^{\sigma}\Psi^{\lambda]} = 0$
Then the 2-planes orthogonal to the killing fields are integrable.

The proof of the theorem can be found in [13].
The meaning and the usefulness of this theorem is that the two-dimensional subspaces of the tangent space at each point which are spanned by the vectors orthogonal to the two killing vectors are integrable i.e. tangent to two-dimensional surfaces.
This theorem has two main consequences the first is that it simplifies the metric, indeed if a tangent space is generated by the coordinate vectors of t and ϕ then the one tangent to this space will be spanned by x^2, x^3 i.e. the mixed components of the two sets of components are null.The metric reduces to six unknown functions.
The second consequence is that the hypotheses of the theorem has to be satisfied in order for the theorem to be true, this trivial consideration is actually very important because it gives a method to select which spacetimes are candidates to be axisymmetric and stationary and furthermore gives a constrain to the stress-energy tensor , for example a perfect fluid satisfies condition II) as well as a stationary axisymmetric electromagnetic field without sources or electromagnetic currents. Condition II) is also satisfied by vacuum spacetimes where $R_{\mu\nu}$=0. Hypotheses I) is satisfied when a space time is asymptotically flat and there has to be an axis of rotation on which Ψ^μ vanishes.
The functions of the metric might be defined as V = -g_{00} = -$\xi^\mu \xi_\mu$, W = g_{12} = $\xi^\mu \Psi_\mu$, X = $\Psi^\mu \Psi_\mu$, the remaining non-zero elements are g_{22}, g_{23} and g_{33}.
Thus far the coordinates x^2 and x^3 haven't been defined. The task ahead can be further simplified by an educated choice of coordinates. We define the x^2 coordinate as ρ, that satisfies:

$$\rho^2 = VX + W$$

This is minus the determinant of t-ϕ part of the metric. Assuming that $\nabla_\mu \rho \neq 0$. x^3 is chosen as z, so that $\nabla_\mu z$ is orthogonal to $\nabla_\mu \rho$ (this implies that z can be redefined as z'= h(z)). In these coordinates the metric takes form:

$$ds^2 = -V(dt - \omega)^2 + V^{-1}\rho^2 d\phi^2 + \Omega^2(d\rho^2 + \Lambda dz^2)$$

where $\omega = W/V$. This is the general form of the stationary axisymmetric spacetime that satisfies hypothesis of Theorem 2.1.
The metric can be furthermore simplified under the assumption $R_{\mu\nu}$=0. In this case yields that:

$$D^\mu D_\mu \rho = 0$$

Where D is the covariant derivative on the two-dimensional surface spanned by ρ and z. The former equations has the consequence that Λ is a function of z alone, hence given the degree of freedom on z we can always choose Λ=1 simplifying furthermore the metric that takes form (defining V = f and $\gamma = \frac{1}{2}\ln(V\Omega^2)$):

$$ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}\left(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right)$$

This metric is the most general one not only in the vacuum case, but also in the eventuality of the existence of an axisymmetric and stationary electromagnetic field as shown in [12], which is the case we are going to analyze.
Except for the trivial constant re-scaling or shift of the origin of the coordinates t, z, ρ and ϕ, we have completely specified our coordinate system and the metric assumes a remarkably simple form.

2.2 Ernst's method

In a series of papers [1]-[2], Ernst showed that stationary axisymmetric space-times both in vacuum and electro-vacuum admit a formulation in terms of potentials, called Ernst potentials.
The general metric for a stationary axisymmetric space-time is given by the Lewis-Weyl-Papapetrou (or LWP) line element, that was found in the former section, namely:

$$ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}\left(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right) \tag{2.1}$$

Where (t,ρ,z,ϕ) are Weyl coordinates and f,ω and γ depend only on ρ and z due to the symmetries.
Now we focus on analyzing Einstein's theory of general relativity coupled with Maxwell's electromagnetism.
The field equation can be derived from the principle of least action by the variation of the following action

$$S(g_{\mu\nu}, A_\mu) = \frac{1}{16\pi}\int \left(R - F^{\mu\nu}F_{\mu\nu}\right)\sqrt{-g}\, d^4x \tag{2.2}$$

With Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ where A_μ is the U(1) gauge 4-potential.
The field equation for the metric an the vector potential A_μ can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 2\left(F_{\mu\alpha}F_\nu^{\ \alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{2.3}$$

$$\nabla^\mu F_{\mu\nu} = \partial^\mu(\sqrt{-g}F_{\mu\nu}) = 0 \tag{2.4}$$

The equations of the electromagnetic field are complete with

$$\nabla^\mu \star F_{\mu\nu} = 0$$

Where the star operator indicates to take the Hodge dual of the Faraday tensor, this last relation can be equivalently written as

$$\nabla_{[\alpha}F_{\beta\gamma]} = 0$$

These equations once solved under some assumption give a specific form to the line element (2.1) and to the four potential. The task is quite cumbersome

hence the need for a generating technique such as the one proposed by Ernst that we are going to introduce.
Henceforward the differentials operators $\nabla, \triangle$ are just the flat gradient and Laplacian in cylindrical Weyl coordinate (ρ,z,ϕ).
Given the effective Lagrangian density proposed by Ernst in [1] and [2]:

$$\begin{aligned}\mathcal{L} = &-\frac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f + \frac{1}{2}\rho^{-1}f^{2}\nabla\omega\cdot\nabla\omega + 2\rho f^{-1}\nabla A_t\cdot\nabla A_t - \\ &-2\rho^{-1}f(\nabla A_\phi - \omega\nabla A_t)\cdot(\nabla A_\phi - \omega\nabla A_t)\end{aligned}$$

where A_t and A_ϕ are the only non-null components of the 4-potential, (again due to the symmetries involved). Some calculation regarding the fields: f,ω, A_t, A_ϕ are located in A.3. Two sets of two equations, one set for the electromagnetic potential and the other for the gravitational potential, can be obtained trough manipulation of the Euler-Lagrange equations deriving from the effective Lagrangian density, readily:

$$\nabla\cdot[\rho^{-2}f(\nabla A_\phi - \omega\nabla A_t)] = 0 \tag{2.5}$$

$$\nabla\cdot[f^{-1}\nabla A_t + \rho^{-2}f\omega(\nabla A_\phi - \omega\nabla A_t)] = 0 \tag{2.6}$$

And for the gravitational potential:

$$\nabla\cdot[\rho^{-2}f^{2}\nabla\omega - 4\rho^{-2}fA_t(\nabla A_\phi - \omega\nabla A_t)] = 0 \tag{2.7}$$

$$\begin{aligned}f\triangle f = &\nabla f\cdot\nabla f - \rho^{-2}f^{4}\nabla\omega\cdot\nabla\omega + 2f\nabla A_t\cdot\nabla A_t + \\ &+2\rho^{-2}f^{3}(\nabla A_\phi - \omega\nabla A_t)\cdot(\nabla A_\phi - \omega\nabla A_t)\end{aligned} \tag{2.8}$$

Actually these equations can be found from (2.3) and (2.4) after some manipulations, than the effective Lagrangian can be defined so that these field equations can be easily found .
Given the equations Ernst noticed that they can be simplified by defining a set of scalar functions.
Indeed equation (2.5) may be regarded as the integrability condition for the existence of a magnetic potential $\bar{A}_\phi$, such as:

$$\hat{\phi}\times\nabla\bar{A}_\phi = \rho^{-1}f(\nabla A_\phi - \omega\nabla A_t) \tag{2.9}$$

So it seems advantageous to introduce the complex scalar potential

$$\Phi = A_t + i\bar{A}_\phi$$

Equation (2.7) may be regarded as the integrability condition for the existence of a new potential χ, such that:

$$\hat{\phi}\times\nabla\chi = \rho^{-1}f^{2}\nabla\omega - 2\hat{\phi}\times\mathrm{Im}(\Phi^{*}\nabla\Phi) \tag{2.10}$$

A gravitational potential can be introduce as:

$$\mathcal{E} = f - |\Phi|^2 + i\chi$$

Substituting the magnetic and gravitational potential in the Lagrangian density one can rewrite the effective action as:

$$S = \int \left[\frac{(\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot(\nabla\mathcal{E}^* + 2\Phi\nabla\Phi^*)}{(\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2)^2} - 2\frac{\nabla\Phi\cdot\nabla\Phi^*}{\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2} \right] d\rho dz \quad (2.11)$$

Which is a real action, the correspondent Euler-Lagrange equations are:

$$\nabla \frac{\delta\mathcal{L}}{\delta(\nabla\mathcal{E})} = \frac{\delta\mathcal{L}}{\delta\mathcal{E}}$$

$$\nabla \frac{\delta\mathcal{L}}{\delta(\nabla\Phi)} = \frac{\delta\mathcal{L}}{\delta\Phi}$$

Which yield the Ernst equations:

$$\left(Re(\mathcal{E}) + |\Phi|^2\right) \triangle\mathcal{E} = (\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\mathcal{E} \quad (2.12)$$

$$\left(Re(\mathcal{E}) + |\Phi|^2\right) \triangle\Phi = (\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot\nabla\Phi \quad (2.13)$$

Therefore these two equation simplify the problem of finding solutions of an axisymmetric and stationary spacetime. These equations actually carry out the same role as the field equations derived from action (2.2). Indeed Ernst found out that, in the case of axisymmetric and stationary spacetimes the field equations derived from action (2.2) can be reduced to a coupled system of complex vectorial differential equations.
By solving these two equations, given some boundary conditions and the previous considerations, one can retrieve the f and ω functions of the metric (2.1). Two other first order partial differential equations for $\gamma(\rho, z)$, can be obtained, indeed it has to be noticed that in Ernst's formulation through the effective Lagrangian there is no mention of $\gamma(\rho, z)$. This function is clearly present in action (2.2) and therefore can be retrieved from its field equations through various manipulations (is not a straight forward calculation) and then write the new equations using the $\mathcal{E}, \Phi$ potentials. The equations are:

$$\begin{aligned}\partial_\rho\gamma(\rho,z) =& \frac{\rho}{4(Re(\mathcal{E}) + |\Phi|^2)^2}\Big[\left(\partial_\rho\mathcal{E} + 2\Phi^*\partial_\rho\Phi\right)\left(\partial_\rho\mathcal{E}^* + 2\Phi\partial_\rho\Phi^*\right) - \\ & \left(\partial_z\mathcal{E} + 2\Phi^*\partial_z\Phi\right)\left(\partial_z\mathcal{E}^* + 2\Phi\partial_z\Phi^*\right)\Big] - \frac{\rho}{Re(\mathcal{E}) + |\Phi|^2}\cdot \\ & \left(\partial_\rho\Phi\partial_\rho\Phi^* - \partial_z\Phi\partial_z\Phi^*\right)\end{aligned} \quad (2.14)$$

$$\begin{aligned}\partial_z\gamma(\rho,z) =& \frac{\rho}{4(Re(\mathcal{E}) + |\Phi|^2)^2}\Big[\left(\partial_\rho\mathcal{E} + 2\Phi^*\partial_\rho\Phi\right)\left(\partial_z\mathcal{E}^* + 2\Phi\partial_z\Phi^*\right) + \\ & \left(\partial_z\mathcal{E} + 2\Phi^*\partial_z\Phi\right)\left(\partial_\rho\mathcal{E}^* + 2\Phi\partial_\rho\Phi^*\right)\Big] - \frac{\rho}{Re(\mathcal{E}) + |\Phi|^2}\cdot \\ & \left(\partial_\rho\Phi\partial_z\Phi^* + \partial_z\Phi\partial_\rho\Phi^*\right)\end{aligned} \quad (2.15)$$

Therefore $\gamma(\rho, z)$ can be obtained by simple quadrature. The equations for $\gamma(\rho, z)$ are completely uncoupled from the ones of f and ω, consequently they can be solved separately.

Chapter 3

Symmetry group

Here we explain the general method adopted to find the infinitesimal generators of the action and give a definitions of the these infinitesimal generators. More information can be found in [15].

3.1 Symmetry generators of the action

A Lie point symmetry is a change of variables of the form (if the equation is a partial differential equation of a function u in p variables x^i):

$$\tilde{x}^n = \tilde{x}^n(x^1, x^2, \dots, u; \varepsilon_N) \tag{3.1}$$

$$\tilde{u} = \tilde{u}(x^1, x^2, \dots, u; \varepsilon_N) \tag{3.2}$$

That given

$$H(x^1, x^2; \dots, u, u_{,n}, u_{,nm}, \dots) = 0$$

satisfies the condition:

$$H(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{u}, \tilde{u}_{,n}, \tilde{u}_{,nm}, \dots) = 0$$

i.e. a Lie point symmetry is a change of variables mediated by r parameters ε_N that leaves an ordinary differential equation or a partial differential equation unchanged, this definition implies that a symmetry is a transformation that maps a solution into a solution.
A simple example is given by the unitary harmonic oscillator, intended as equation:

$$y''(x) = -y(x)$$

Where the apostrophes represent the derivative of the variable x. It's rather easy to verify via substitution that the change of variables $\tilde{y} = ay$ and $\tilde{x} = x + b$ leave the former differential equation unchanged, consequently these are two

symmetries of said equation.
Usually symmetries are not that easy to be spotted for example (with y(x)≡ y):

$$y'' = xy'y$$

It might have symmetries but they are not as explicit to find as the previous example. The concept of generator of a symmetry or infinitesimal generator arises from this difficulty.
An infinitesimal generator X is a vector field that acting on a differential equation leaves it unchanged. If the equation is a partial differential equation of a function u in p variables x^n :

$$H(x^1, x^2, \ldots, u, u_{,n}, u_{,nm}, \ldots) = 0$$

than if X is a symmetry generator with $H = H(x^1, x^2, \ldots, u, u_{,n}, u_{,nm}, \ldots)$:

$$X(H) = 0$$

holds. The general form for this operator is (in the case of Lie point symmetries):

$$X_N = \xi_N^n \frac{\partial}{\partial x^n} + \eta_N \frac{\partial}{\partial u}$$

where ξ and η are functions of u and x^n alone.
Roughly speaking, a Lie point symmetry of a system is a local group of transformations that maps every solution of the system to another solution of the same system. In other words, it maps the solution set of the system to itself.
So if we already know a (special) solution of a partial differential equation, we can apply a finite symmetry transformation to obtain a (possibly) new solution. This new solution will depend on at most as many new parameters as there are in the symmetry transformation we have used.
The name "symmetry generator" comes from the fact that ξ and η are actually defined through the symmetry i.e.:

$$\xi^i = \left.\frac{\partial \tilde{x}^i}{\partial \varepsilon_N}\right|_{\varepsilon_N=0}$$

$$\eta = \left.\frac{\partial \tilde{u}}{\partial \varepsilon_N}\right|_{\varepsilon_N=0}$$

Hence pointing out the role that the symmetries play in the symmetry generator and why they might be called infinitesimal generator, given these definition one can find a method to find the ξ and η, that has to be stressed, depends only on the independent variables and the function of the independent variables and not its derivatives (in the case of Lie point symmetries), once these function are found one can than solve them with respect of the parameters ε_N and finally find the symmetries.
It can be noted a similarity between the definition of a generator of a symmetry

generator and a Killing vector. Indeed the definition of a Killing vector is that the Lie derivative of the metric over a vector field is null:

$$\pounds_{\xi}(g_{\mu\nu}) = 0$$

Meaning that ξ, that is called a Killing vector field, leaves the metric invariant. This is the analogous of the definition given previously for the symmetry generator X for a differential equation. The previous equation yields Killing equation:

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = \nabla_{(\mu}\xi_{\nu)} = 0$$

Which will come in handy later on.
Taking this analogy into account is not illogical to find a way to transform equation (2.11) into a sort of metric and find its killing vectors, therefore finding its symmetry generators.
Rewriting equation (2.11) for simplicity

$$S = \int \frac{(\nabla\mathcal{E} + 2\Phi^*\nabla\Phi)\cdot(\nabla\mathcal{E}^* + 2\Phi\nabla\Phi^*)}{(\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2)^2} - 2\frac{\nabla\Phi\cdot\nabla\Phi^*}{\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2} dx^4$$

It can be noticed that by the substitution:

$$\nabla \to d$$

the equation now looks like a metric, furthermore by explicitly define the complex potentials as complex variables i.e.: $\mathcal{E} = x + iy$ and $\Phi = z + iw$, the action is transformed into a metric of the form:

$$ds^2 = \frac{(d\mathcal{E} + 2\Phi^* d\Phi)(d\mathcal{E}^* + 2\Phi d\Phi^*)}{(\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2)^2} - 2\frac{d\Phi d\Phi^*}{\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2}$$

and then:

$$\begin{aligned} ds^2 =& \frac{1}{4(w^2 + x + z^2)^2}[dx^2 + dy^2 - 4x(dz^2 + dw^2) + 4dy(-dzw + dwz) \\ &+ 4dx(dww + dzz)] \end{aligned} \tag{3.3}$$

From here Killing equation can be used to find the symmetries generators and then the symmetries transformations. It has to be said that the maximum number of Killing vectors that a metric can posses is $\frac{N(N+1)}{2}$, here N is the dimension of the manifold considered. The number of Killing vectors is maximal when the metric has constant curvature.

3.1.1 The vacuum case

The vacuum case is considered to illustrate the technique.
It happens when Φ=0, in this case the action (2.11) becomes:

$$S = \int \frac{\nabla\mathcal{E}\cdot\nabla\mathcal{E}^*}{(\mathcal{E} + \mathcal{E}^*)^2} dx^4$$

Transforming it into a metric yields:

$$ds^2 = \frac{d\mathcal{E}d\mathcal{E}^*}{(\mathcal{E}+\mathcal{E}^*)^2}$$

and then:

$$ds^2 = \frac{1}{4x^2}(dx^2 + dy^2)$$

This metric is clearly conformally flat consequently it has 3 killing vectors.
As already said these vectors can be found by $\nabla_{(\mu}\xi_{\nu)} = 0$. Therefore one has to find the Christoffel symbols for the covariant derivative and solve the resulting differential equations. In this case we have:

$$\begin{cases} \dfrac{\partial \xi_y}{\partial y} - \dfrac{\xi_x}{x} = 0 \\ \dfrac{\partial \xi_x}{\partial x} + \dfrac{\xi_x}{x} = 0 \\ \dfrac{\partial \xi_x}{\partial y} + \dfrac{\partial \xi_y}{\partial x} + \dfrac{2\xi_y}{x} = 0 \end{cases} \tag{3.4}$$

The solutions of these equation are

$$\xi_x = \frac{1}{x}(a + by)$$

$$\xi_y = \frac{1}{x^2}\left(ay + \frac{b}{2}y^2 + c - b\frac{x^2}{2}\right)$$

which are the covariant components of the Killing vector, with the three real parameters $a, b, c \in \mathbb{R}$, the contravariant components are:

$$\xi^x = 4x\,(a + by)$$

$$\xi^y = 4\left(ay + \frac{b}{2}y^2 + c - b\frac{x^2}{2}\right)$$

Therefore inasmuch as we have three parameters we can define three Killing vectors by choosing a different values of a, b, c for each vector.
By choosing a =1 ,b=c=0 the first Killing vector can be obtained

$$\xi^1_\mu = 4x\,\partial_x + 4y\,\partial_y$$

The second by b=1,a=c=0

$$\xi^2_\mu = 4xy\,\partial_x + 2\left(x^2 - y^2\right)\partial_y$$

The third by c=1, a=b=0

$$\xi^3_\mu = 2\partial_y$$

Given these vector fields and the Lie parentheses we can identify a Lie algebra, the commutators are:

$$[\xi^1, \xi^2] = 4\xi^2$$

$$[\xi^1, \xi^3] = -4\xi^3$$
$$[\xi^2, \xi^3] = -4\xi^1$$

This relations can be re-conduct the the $\mathfrak{su}$(1,1) algebra, actually these commutator represent the $\mathfrak{sl}(2, \mathbb{R})$ algebra or Special linear Lie algebra, but this algebra is isomorphic to $\mathfrak{su}$(1,1).
Now that we have the symmetries generators and identifies the algebra, we can proceed to find the symmetries.
Since we know that in the coordinates (x, y):

$$X(H) = 0$$

is to be true, it has to be true also if we apply the coordinate transformation $x'(x, y; \epsilon)$,$y'(x, y; \epsilon)$, that is a symmetry transformation. Considering the action of the symmetry generator as a first integral of the parametrizing coordinate in this case ϵ, it can be stated that:

$$\frac{dH'}{d\epsilon} = X'(H') = 0$$

where the apostrophe highlights the fact that we are in the x', y' coordinate now. This equation gives us a new set of differential equations in the new variables, namely:

$$\begin{cases} \frac{\mathrm{d}x'}{\mathrm{d}\epsilon} = f(x', y') \\ \frac{\mathrm{d}y'}{\mathrm{d}\epsilon} = g(x', y') \end{cases} \tag{3.5}$$

where f and g are real functions that depends on the explicit form of the infinitesimal generator. These equations, along with the boundaries conditions $x'(x, 0) = x$ and $y'(y, 0) = y$, once integrated give back the symmetries of the system.
In our case taking the first Killing vector yield:

$$\begin{cases} \frac{\mathrm{d}x'}{\mathrm{d}\epsilon} = x' \\ \frac{\mathrm{d}y'}{\mathrm{d}\epsilon} = y' \end{cases} \tag{3.6}$$

The solution is trivial

$$\begin{cases} x' = e^{\epsilon} x \\ y' = e^{\epsilon} y \end{cases} \tag{3.7}$$

now setting $\mathcal{E}' = x' + iy'$ yields:

$$\mathcal{E}' = e^{\epsilon}(x + iy) = e^{\epsilon}\mathcal{E}$$

Now setting $|\lambda|^2 = e^{\epsilon}$ gives back the first of the transformations (3.12). It has to be noted that on this case there is no real reason to define λ as a complex number which means it yields two parameters, albeit it makes sense to do so

in the electro-vacuum case, in this case λ has to be considered real.
From the second Killing vector the following equations hold:

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}\epsilon} = x'y' \\ \dfrac{\mathrm{d}y'}{\mathrm{d}\epsilon} = \dfrac{x'^2 - y'^2}{2} \end{cases} \tag{3.8}$$

These equations are a set of coupled first order ordinary differential equations and can be resolved by means of the Lie point symmetries. The solution is:

$$\begin{cases} x' = \dfrac{x}{1 + 4y\epsilon + 4\epsilon^2(x^2 + y^2)} \\ y' = \dfrac{y + 2(x^2 + y^2)\epsilon}{1 + 4y\epsilon + 4\epsilon^2(x^2 + y^2)} \end{cases} \tag{3.9}$$

Then

$$\mathcal{E}' = x' + iy' = \frac{x + iy}{1 + i\,2\epsilon(x + iy)}$$

Defining $2\epsilon = c$ we find back Ehlers transformation The last case for the third Killing vector is quite trivial. The system is now:

$$\begin{cases} \dfrac{\mathrm{d}x'}{\mathrm{d}\epsilon} = 0 \\ \dfrac{\mathrm{d}y'}{\mathrm{d}\epsilon} = 1 \end{cases} \tag{3.10}$$

with solution

$$\begin{cases} x' = x \\ y' = \epsilon + y \end{cases} \tag{3.11}$$

Hence defining ϵ = b, we have:

$$\mathcal{E}' = x + i(y + b)$$

Which is the second of the (3.12).

3.1.2 The electro-vacuum case

We repeat the line element for this case

$$ds^2 = \frac{(d\mathcal{E} + 2\Phi^* d\Phi)(d\mathcal{E}^* + 2\Phi d\Phi^*)}{(\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2)^2} - 2\frac{d\Phi d\Phi^*}{\mathcal{E} + \mathcal{E}^* + 2|\Phi|^2}$$

The maximum number of killing vectors we can find here is 10, albeit this metric is not flat nor conformally flat, therefore less than 10 killing vectors are expected.
Solving Killing equation$\nabla_{(\mu}\xi_{\nu)} = 0$, and raising the indexes:

$$\xi^x = 4a_1xy + 2a_2(wx + zy) + a_3(-2wy + 2xz) + 4a_2x + 2a_6z - 2a_7w$$

$$\xi^{y} = 2a_1\left(y^2 - x^2\right) + 2a_2(wy - xz) + 2a_3(wx + yz) + 4a_4 y + 2a_6 w \\ + 2a_7 z + 4a_8$$

$$\xi^{z} = 2a_1(wx + yz) + 4a_2 wz - a_2 y + 2a_3\left(x + z^2 - w^2\right) + 2a_4 z - a_5 w - a_6$$

$$\xi^{w} = 2a_1(wy - xz) + 2a_2\left(w^2 - z^2\right) + a_2 x + 4a_3 zw + a_3 y + 2a_4 w + a_5 z + a_7$$

Consequently the Killing vectors are:

$$\xi_1^{\mu} = 4xy\,\partial_x + 2\left(y^2 - x^2\right)\,\partial_y + 2(wx + yz)\,\partial_z + 2(wy - zx)\,\partial_w$$

$$\xi_2^{\mu} = 2(wx + yz)\,\partial_x + 2\,(wy - zx)\,\partial_y + (4wz - y)\,\partial_z + \left(2\left(w^2 - z^2\right) + x\right)\,\partial_w$$

$$\xi_3^{\mu} = 2(zx - wy)\,\partial_x + 2(wx + yz)\,\partial_y + (x + 2(z^2 - w^2))\,\partial_z + (4wz + y)\partial_w$$

$$\xi_4^{\mu} = 4x\,\partial_x + 4y\,\partial_y + 2z\,\partial_z + 2w\,\partial_w$$

$$\xi_5^{\mu} = -w\,\partial_z + z\,\partial_w$$

$$\xi_6^{\mu} = 2z\,\partial_x + 2w\,\partial_y - \,\partial_z$$

$$\xi_7^{\mu} = -2w\,\partial_x + 2z\,\partial_y + \,\partial_w$$

$$\xi_8^{\mu} = 4\,\partial_w$$

Having the Killing vectors, now we proceed to find the relative Lie algebra.

$$[\xi_1, \xi_4] = -4\xi_1; \quad [\xi_1, \xi_6] = -2\xi_2; \quad [\xi_1, \xi_7] = -2\xi_3$$

$$[\xi_1, \xi_8] = -4\xi_4; \quad [\xi_2, \xi_3] = -2\xi_1; \quad [\xi_2, \xi_4] = -2\xi_2$$

$$[\xi_2, \xi_5] = -\xi_3; \quad [\xi_2, \xi_6] = -6\xi_5; \quad [\xi_2, \xi_7] = -\xi_4$$

$$[\xi_2, \xi_8] = -4\xi_6; \quad [\xi_3, \xi_4] = -2\xi_3; \quad [\xi_3, \xi_5] = \xi_2$$

$$[\xi_3, \xi_6] = \xi_4; \quad [\xi_3, \xi_7] = -6\xi_5; \quad [\xi_3, \xi_8] = -4\xi_7$$

$$[\xi_5, \xi_7] = -\xi_6; \quad [\xi_6, \xi_7] = -\xi_8; \quad [\xi_4, \xi_6] = -2\xi_6$$

$$[\xi_4, \xi_7] = -2\xi_7; \quad [\xi_4, \xi_8] = -4\xi_8; \quad [\xi_5, \xi_6] = \xi_7$$

these commutators form to the $\mathfrak{sl}(3, \mathbb{R})$ algebra that is isomorphic to $\mathfrak{su}(2,1)$. By integrating these killing vectors with the same method of the previous section the finite transformations (3.12) of the action (2.11) can be obtained.

3.2 Lie symmetries and invariants

Therefore action (2.11) has some symmetries, which means that there exists a number of transformations of $\mathcal{E}$ and Φ which leave the effective actions unchanged. These transformations where found by Ernst, Ehlers and Harrison, and are:

$$\begin{aligned}
I)&\quad \mathcal{E} \to \mathcal{E}' = |\lambda|^2\mathcal{E} && \Phi \to \Phi' = \lambda\Phi \\
II)&\quad \mathcal{E} \to \mathcal{E}' = \mathcal{E} + ib && \Phi \to \Phi' = \Phi \\
III)&\quad \mathcal{E} \to \mathcal{E}' = \frac{\mathcal{E}}{1+ic\mathcal{E}} && \Phi \to \Phi' = \frac{\Phi}{1+ic\mathcal{E}} \\
IV)&\quad \mathcal{E} \to \mathcal{E}' = \mathcal{E} - 2\beta^*\Phi - |\beta|^2 && \Phi \to \Phi' = \Phi + \beta \\
V)&\quad \mathcal{E} \to \mathcal{E}' = \frac{\mathcal{E}}{1-2\alpha^*\Phi - |\alpha|^2\mathcal{E}} && \Phi \to \Phi' = \frac{\Phi + \alpha\mathcal{E}}{1-2\alpha^*\Phi - |\alpha|^2\mathcal{E}}
\end{aligned} \tag{3.12}$$

where $b, c \in \mathbb{R}$ and $\alpha, \lambda, \beta \in \mathbb{C}$. Some of these transformation are just gauge symmetries and can be reabsorbed by a coordinate transformation, while others actually have non-trivial physical effects. Indeed transformation I) II) and IV) are gauge symmetries while III) and V) are not. Transformation III) is called Ehlers transformation, while transformation V) is called Harrison transformation.

Transformation I) is actually a generalization of the duality transformation in electromagnetism, that would be given by $\lambda = e^{i\alpha}$, hence the electromagnetic potential Φ has a duality rotation consisting in the replacement $\Phi \to \Phi e^{i\alpha}$.

The corresponding Lie group for equation (2.11) is SU(2,1), in the case where Φ=0 the Lie group is SU(1,1).

These transformations can be combined to produce new transformation, for example II) $\circ$ I) produces

$$\mathcal{E}' = |\lambda|^2\mathcal{E} + ib \quad \Phi' = \lambda\Phi$$

Which is a 2-parameter group and it still is a gauge transformation therefore is trivial, but the concept is clear, and by fixing a parameter discrete transformation can be achieved as shown in chapter 6.

3.3 Symmetries of the effective action

It's interesting to evaluate the symmetries of the effective action

$$\begin{aligned}
\mathcal{L} = &-\frac{1}{2}\rho f^{-2}\nabla f\cdot\nabla f + \frac{1}{2}\rho^{-1} f^2 \nabla\omega\cdot\nabla\omega + 2\rho f^{-1}\nabla A_t\cdot\nabla A_t - \\
&- 2\rho^{-1} f\left(\nabla A_\phi - \omega\nabla A_t\right)\cdot\left(\nabla A_\phi - \omega\nabla A_t\right)
\end{aligned}$$

in the fields f, ω, A_t and A_ϕ. The transformations that leave the action invariant are:

$$f' = f, \quad \omega' = \omega, \quad A_t' = A_t, \quad A_\phi' = A_\phi + d_1;$$

$$f' = f, \quad \omega' = \omega + d_2, \; A'_t = A_t, \; A'_\phi = A_\phi + d_2 A_t;$$
$$f' = f, \quad \omega' = \omega, \quad A'_t = A_t + d_3, \quad A'_\phi = A_\phi;$$
$$f' = fe^p, \quad \omega' = \omega e^{-p}, \quad A'_t = A_t e^{\frac{p}{2}}, \quad A'_\phi = A_\phi e^{-\frac{p}{2}};$$

$$f' = -\frac{d_4^2\rho^2}{f} + f2d_4 f\omega + d_4^2 f\omega^2, \quad \omega' = \frac{-\rho^3 d_4 - \rho f^2\omega + d_4 f^2\omega^2}{\rho[f + d_4(\rho - f\omega)][-f - d_4(\rho + f\omega)]},$$
$$A'_t = A_t + d_4 A_\phi, \quad A'_\phi = A_\phi;$$

With constants $d_1, d_2, d_3, d_4, p \in \mathbb{R}$.
As shown by Kinnersley in [17], these equations are gauge transformation together with the transformation in the vacuum case that are:

$$f' = f, \quad \omega' = \omega + a;$$
$$f' = fe^b, \quad \omega' = \omega e^{-b};$$
$$f' = \frac{(\omega c - 2)^2 f^2 - c^2\rho^2}{4f}, \quad \omega' = \frac{(-c\omega^2 + 4\omega)f^2 + 2\rho^2 c}{(\omega c - 2)^2 f^2 - c^2\rho^2};$$

With constants a,b,c $\in \mathbb{R}$.
This is an important digression because it highlights the power of Ernst's method and the power of its symmetries, for the symmetries expressed in terms of Ernst's potentials can produce physically nonequivalent new solutions while the symmetries express in terms of the fields f, ω, A_t and A_ϕ do not, hence are gauge transformations.
Therefore Ernst's potentials are more than just a mere renomination of the fields, they possess additional non-trivial information that makes the invariants of the action expressed in terms of Ernst's potentials possess remarkable proprieties, such as not being all gauge transformations.
In [17] Kinnersley shows a method to produce again the non-trivial transformation (3.12) from the former ones in the vacuum case.
The method consists in analyzing the field equations for f, ω, that is the (2.5) and (2.6) in vacuum, and f, χ (the former are the components of Ernst's potential). These equations are:

$$\boldsymbol{\nabla} \cdot \left(f^{-1}\nabla f + \rho^{-2} f^2 \omega \nabla\omega\right) = 0$$

$$\boldsymbol{\nabla} \cdot \left(\rho^{-2} f^2 \nabla\omega\right) = 0$$

and

$$\boldsymbol{\nabla} \cdot \left(f^{-1}\nabla f + f^{-2}\chi\nabla\chi\right) = 0$$

$$\boldsymbol{\nabla} \cdot \left(f^{-2}\nabla\chi\right) = 0$$

The two can be mapped into each other by the mapping

$$f \to \rho f^{-1} \quad \omega \to i\chi$$

Then the invariants of the effective action produced by this mapping are :

$$f' = f \quad \chi' = \chi - \alpha$$
$$f' = \beta f \quad \chi' = \beta\chi$$
$$f' = \frac{f}{(1-\gamma\chi)^2 + \gamma^2 f^2} \quad \chi' = \frac{\chi - \gamma(f^2+\chi^2)}{(1-\gamma\chi)^2 + \gamma^2 f^2}$$

With α, β and γ as real constants. The first of these is a translation therefore a gauge transformation. The second is a rescaling and the third is the gravitational duality rotation discovered by Ehlers.
Indeed in terms of Ernst's potential the last equation is

$$\mathcal{E}' = \frac{\mathcal{E}}{1 + i\gamma\mathcal{E}}$$

which is Ehlers transformation.

3.4 Electric and magnetic line elements

Another symmetry for the stationary axisymmetric case is the discrete transformation of metric (2.1), by:

$$t \to i\psi$$
$$\phi \to i\tau$$

this transformation is called double Wick rotation and changes the electric metric(2.1) that will be denoted with e as subscript, i.e.:

$$ds_e^2 = -f(dt - \omega d\phi)^2 + f^{-1}\left(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right)$$

to the magnetic metric that will be denoted with m as subscript:

$$ds_m^2 = f_m(d\psi - \omega d\tau)^2 + f_m^{-1}\left(e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 d\tau^2\right) \tag{3.13}$$

This equation can again be solution of Einstein's equations for the stationary asymmetric space-time, but is different from Weyl metric, written as in (2.1), consequently it can be used to produce new solutions using the 1-parameter groups in the previous section that are different from the ones produced from metric (2.1).

Chapter 4

The Schwarzschild solution

In this chapter the Schwarzschild solution is introduced along with some of its proprieties, then we give the explicit representation in the case of electric and magnetic metrics.

4.1 The solution

In Einstein's theory of general relativity, the Schwarzschild metric is the solution to the Einstein field equations that describes the gravitational field outside a spherical mass, on the assumption that the electric charge of the mass, angular momentum of the mass, and universal cosmological constant are all zero. The solution is a useful approximation for describing slowly rotating astronomical objects such as stars and planets, including Earth and the Sun. It was found by Karl Schwarzschild in 1916.

The metric is:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2$$

This metric is peculiar because, by Birkhoff theorem, every spherically symmetric solution of the vacuum field equations must be static and asymptotically flat. This means that an exterior spherical solution must be given by the Schwarzschild metric. Therefore is an interesting solution to analyze and to use as a seed metric.

We have to identify the terms of metrics (2.1) and (3.13) in order to translate them into the Schwarzschild metric. In order to do so we need to change from Weyl coordinates to spherical ones. The transformation is given by:

$$\begin{cases} \rho(r,\theta) = \sin(\theta)\sqrt{r^2 - 2mr} \\ z(r,\theta) = \cos(\theta)(r - m) \end{cases} \tag{4.1}$$

the time coordinate and the azimuth angle are left unchanged. The line element $d\rho^2 + dz^2$ in the new coordinates takes form:

$$d\rho^2 + dz^2 = \left[\frac{r^2 - 2mr + m^2\sin^2(\theta)}{r^2 - 2mr}\right] dr^2 + \left[r^2 - 2mr + m^2\sin^2(\theta)\right] d\theta^2$$

4.2 Schwarzschild in electric LWP form

Taken metric (2.1) the first thing to notice is that Schwarzschild metric is non rotating which translate mathematically in $\omega = 0$. This metric doesn't present an electromagnet field either, hence $\Phi = 0$.
The identification with f is straight forward and we have:

$$f = \left(1 - \frac{2m}{r}\right)$$

Having f and the line element in the previous section gives us the γ function at once:

$$e^{2\gamma} = \frac{r^2 - 2mr}{r^2 - 2mr + m^2\sin^2(\theta)}$$

Therefore we have translated the electric Weyl line element into the Schwarzschild metric and retrieved the functions involved in the transformations in section 3.2., which we will use later on.

4.3 Schwarzschild in magnetic LWP form

Taking in this case metric (3.13) with $\omega = 0$ because the metric is still non rotating and doing the trivial transformation $\psi \to \phi$, $\tau \to$ t than comparing with the Schwarzschild metric the following equations hold:

$$\begin{cases} \dfrac{\rho^2}{f_m} = 1 - \dfrac{2m}{r} \\ \dfrac{e^{2\gamma}}{f_m}(d\rho^2 + dz^2) = \left(1 - \dfrac{2m}{r}\right)^{-1} dr^2 + r^2 d\theta^2 \end{cases} \tag{4.2}$$

Therefore:

$$\begin{cases} f_m = r^2\sin^2(\theta) \\ e^{2\gamma} = \dfrac{r^4\sin^2(\theta)}{r^2 - 2mr + m^2\sin^2(\theta)} \end{cases} \tag{4.3}$$

Therefore we have translated the magnetic Weyl line element into the Schwarzschild metric and retrieved the functions involved in the transformations in section 3.2., which we will use later on.

Chapter 5

Electric and Magnetic solutions

In this Chapter we will apply the transformations (3.12) to the Schwarzschild metric in both electric and magnetic case and we will analyze each transformation, we will start from the former one. Having no rotation nor electric charge the Ernst potentials are:

$$\mathcal{E} = f \quad ; \quad \Phi = 0$$

5.1 Electric metric

5.1.1 Transformation I

This case the transformation is trivial i.e., as already said is a gauge transformation as we are about to show for the Schwarzschild metric.
Having already found all functions needed, it can be stated that:

$$\mathcal{E} = f = \left(1 - \frac{2m}{r}\right)$$

Therefore:

$$\mathcal{E}' = |\lambda|^2 \left(1 - \frac{2m}{r}\right) = |\lambda|^2 f$$

This translates in metric (2.1) as:

$$ds^2 = -|\lambda|^2 f (dt - \omega d\phi)^2 + \frac{1}{|\lambda|^2 f}\left(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right)$$

now substituting the obtained value for γ and changing coordinates, the line element becomes:

$$ds^2 = -f|\lambda|^2 dt^2 + |\lambda|^{-2} f dr^2 + |\lambda|^{-2} r^2 d\theta^2 + |\lambda|^{-2} r^2 \sin^2(\theta) d\phi^2$$

Hence by the substitution $|\lambda|^2 t \to \hat{t}$ and $|\lambda|^{-2} r \to \hat{r}$ the metric simplify but the latter transformation affects the function f that becomes:

$$f = \left(1 - \frac{2m}{|\lambda|^2 \hat{r}}\right)$$

which is not invariant but since m is just a constant it can be redefined as $m \to |\lambda|^2 \hat{m}$ that simplifies in the previous equations and yields a new f function, say $\hat{f}$, that is a complete analogous of the old f, the same can be said of the metric. Therefore the transformation can be undone by a change of coordinates, hence a gauge transformation.

5.1.2 Transformation II

Here a trivial case is again analyzed, and is showed that is a matter of choosing the right coordinates.
The transformation is:

$$\mathcal{E}' = f + ib$$

Hence the f remain unchanged while ω might change.
Applying equation (2.10) with χ' = b:

$$0 = \nabla \omega$$

So ω = k, where k $\in \mathbb{R}$ is a constant. Therefore substituting in metric (2.1):

$$ds^2 = -f(dt - kd\phi)^2 + \frac{1}{f}\left(e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right)$$

By doing the transformation $\hat{t} \to t - k\phi$, which is trivial because the metric doesn't depend on t and neither on ϕ, one gets back to Schwarzschild metric after a change of coordinates and after one has written explicitly f and γ.
This previous can be interpreted physically as a change of coordinates from rotating to non-rotating frame of reference.

5.1.3 Transformation III

This is the first non trivial transformation and as previously said it's called Ehlers transformation. Since the we have no electromagnetic potential the transformation simply becomes:

$$\mathcal{E}' = \frac{\mathcal{E}}{1 + ic\mathcal{E}} = \frac{\mathcal{E} - ic\mathcal{E}^2}{1 + c^2\mathcal{E}^2} = \frac{r^2 - 2mr}{r^2 + c^2(r-2m)^2} - i\frac{c(r-2m)^2}{r^2 + c^2(r-2m)^2}$$

therefore

$$f' = \frac{r^2 - 2mr}{r^2 + c^2(r-2m)^2}; \quad \chi' = \frac{c(r-2m)^2}{r^2 + c^2(r-2m)^2}$$

It can be easily noticed that this metric adds a rotation since there is the χ' field. Indeed using equation (2.10) the ω can be found as:

$$\omega = 4cm\cos(\theta)$$

For the γ field it can be easily verified that substituting $\mathcal{E}$ with $\mathcal{E}'$ in equations (2.14)-(2.15) yields the relation $\gamma' = \gamma$. Now we have all the fileds necessary to write the metric, because $\Phi' = 0$ since $\Phi = 0$. The metric is:

$$\begin{aligned} ds^2 = &- \frac{r^2 - 2mr}{r^2 + c^2(r-2m)^2}[dt - 4mc\cos(\theta)d\phi]^2 + \frac{r^2 + c^2(r-2m)^2}{r^2 - 2mr}dr^2 \\ &+ (r^2 + c^2(r-2m)^2)d\theta^2 + (r^2 + c^2(r-2m)^2)\sin^2(\theta)d\phi^2 \end{aligned} \tag{5.1}$$

It can be easily verified that for c=0 the metric is again Schwarzschild metric. Now the real problem is identifying the metric and trying to make sense out of it. For m=0 the background metric can be obtained and it is not a Minkowski spacetime even though for c=0 and m=0 it does become a Minkowski line element.

A way to identify this metric is through classifications, for example using the Petrov classification which is based on the analysis of the Weyl tensor. We will use this type of classification. After calculations it can be found that this metric is a type D metric and in particular seems to be a Taub-NUT sort of metric i.e. it might exist a coordinate transformations that map this metric into the well known Taub-NUT metric. The hint to this assumption is given by the singular points of the metric which are r=0 and r=2m like Schwarzschild but unlike Schwarzschild the Kretschmann scalar doesn't present a discontinuity at r=0, actually it doesn't present a discontinuity at all,which means that there is no curvature singularity, and this is a characteristic of the Taub-NUT solutions.

Further suspicion is raised by the values of the Komar mass and dual Komar mass indeed the dual Komar mass is not null:

$$M = -\frac{1}{8\pi}\int_{S^2_\infty} \star dk = \frac{m(1-c^2)}{1+c^2+c^4}$$

$$\tilde{M} = -\frac{1}{8\pi}\int_{S^2_\infty} dk = \frac{mc}{1+c^2}$$

Where k is the 1-form belonging to the timelike Killing vector and S^2_∞ is a spacelike 2-surface evaluated at spacelike infinity. The presence of a non-zero dual Komar integral is important because usually the meaning of the NUT parameter is that of the dual of the mass. The NUT or gravomagnetic parameter is the analogous of the magnetic charge in electromagnetism, indeed as the magnetic charge is the dual of the electric charge the NUT parameter might be seen as the dual of the mass. Consequently the second integral states that a dual mass exists and it might be regarded as a NUT parameter.

The Taub-NUT metric is interesting and quite peculiar. It's a solution of Einstein equations without a curvature singularity indeed in his common form it's

easy to realize that it has at most two event horizons, within these horizons the spacetime does not show a singular behavior. The Taub-NUT metric has other interesting features like close timelike curves.
The canonical Taub-NUT metric (as presented in [10]) has form:

$$\begin{aligned} ds^2 = &- \frac{R^2 - 2m'R - l^2}{R^2 + l^2} \left(dt' - 2l\cos(\theta)d\phi\right)^2 + \frac{R^2 + l^2}{R^2 - 2m'R - l^2} dR^2 \\ &+ \left(R^2 + l^2\right)\left(d\theta^2 + \sin^2(\theta)d\phi^2\right) \end{aligned} \tag{5.2}$$

where m' is the mass and l is the Taub-NUT parameter. The coordinates transformation from metric (5.1) to metric (5.2) is given by:

$$r \to \frac{1}{\sqrt{1+c^2}}\left(R + \frac{2c^2 m}{\sqrt{1+c^2}}\right), \quad t \to t'\sqrt{1+c^2}$$

$$m \to -\frac{l\sqrt{1+c^2}}{2c}, \quad c \to \frac{m' - \sqrt{m'^2 + l^2}}{l}$$

As previously said in this form is easy to verify that this metric has no curvature singularity and two event horizons for $R = m' \pm \sqrt{m'^2 + l^2}$.

5.1.4 Transformation IV

This transformation is again a gauge transformation.
For $\mathcal{E} \neq 0$ but $\Phi = 0$ we have:

$$\mathcal{E}' = \mathcal{E} - |\Phi|^2$$

and

$$\Phi' = \beta$$

Hence even though this transformation is trivial it adds a constant potential.
The first equation can be seen as:

$$\mathcal{E}' = \mathcal{E} - \left|\Phi'\right|^2 = f - \left|\Phi'\right|^2$$

Consequently the metric remains almost untouched since only the complex potential is changed.
Actually taking equation (2.10) is easy to verify that $\nabla\omega' = 0$ therefore ω=k with k constant $\in \mathbb{R}$, hence a rotation is added, this rotation is analogous to the one seen in case II indeed it can be removed by the coordinate transformation $\hat{t}= t + k\phi$, it conserves the same physical meaning too.
By defining explicitly the complex constant β as $\beta = a + ib$ (with $a, b \in \mathbb{R}$) We can than define:

$$A_t = a$$

$$\hat{A}_\phi = b$$

Then by the means of equations (2.9) we find that:

$$A_\phi = b$$

Where b in this case b is the constant of integration that emerges from equation (2.9) and, being just a constant, doesn't create confusions nor one looses generality to set it with the same letter as $\mathrm{Im}(\Phi)$.
Hence having all the potential equal to a, b the equation of motion are unchanged, indeed the electric and magnetic fields depend on the derivatives of A_μ therefore remain unchanged if the potential is constant, hence transformation IV is gauge transformation.

5.1.5 Transformation V

Here we have again a non trivial transformation called Harrison transformation. By just looking at it, it's easy to notice that it can add a not trivial potential even if the initial complex potential equals zero. This observation can be easily observed in our case, indeed we have:

$$\mathcal{E}' = \frac{\mathcal{E}}{1 - |\alpha|^2 \mathcal{E}} = f' - |\Phi'|^2$$

which gives us a way to determine f′ and for the complex potential:

$$\Phi' = \frac{\alpha \mathcal{E}}{1 - |\alpha|^2 \mathcal{E}} = (a + ib) \frac{r - 2m}{r - |\alpha|^2 (r - 2m)}$$

Where $\alpha = a + ib$. Hence f′ is readily obtained:

$$f' = \frac{r(r - 2m)}{\left(2m|\alpha|^2 + r\left(1 - |\alpha|^2\right)\right)^2}$$

For the γ field it can be easily verified that substituting $\mathcal{E}$ with $\mathcal{E}'$ and Φ with Φ' in equations (2.14)-(2.15) yields the relation $\gamma' = \gamma$.
We proceed now to find the components of the vector potential.
From the equation of Φ it can be obtained:

$$A'_t = a \frac{r - 2m}{r - |\alpha|^2 (r - 2m)}$$

$$\hat{A}'_\phi = b\, \frac{r - 2m}{r - |\alpha|^2 (r - 2m)}$$

Consequently by applying equation (2.9) A'_ϕ can be obtained:

$$A_\phi = 2\mathrm{bm} \cos(\theta) + k$$

Where k ∈ ℝ the constant of integration.
Thus the metric an the vector potential now read:

$$\begin{aligned} ds^2 = & -\frac{r^2-2mr}{(2m|\alpha|^2+r(1-|\alpha|^2))^2}dt^2+\frac{(2m|\alpha|^2+r(1-|\alpha|^2))^2}{r^2-2mr}dr^2 \\ & +\left[(2m|\alpha|^2+r(1-|\alpha|^2))^2\right]d\theta^2+\left[(2m|\alpha|^2+r(1-|\alpha|^2))^2\right]d\phi^2 \end{aligned} \tag{5.3}$$

$$A_\mu = \left(a\frac{r-2m}{r(1-|\alpha|^2)+2m|\alpha|^2}, 0, 0, 2\mathrm{bm}\cos(\theta)+k\right)$$

The first thing that has to be said is that the rotation here is not be taken into account because it can be eliminated by simple gauge transformation. It is to verify that in this case equation (2.10) yields:

$$\omega' = const.$$

Consequently it can be eliminate by the coordinate transformation of the former paragraph.
Another fact that stands out is that the terms containing r in front of the angular variables (θ, ϕ) resemble the square of a distance, therefore defining $\bar{r} = 2|\alpha|^2 + r(1-|\alpha|^2)$ the line element simplifies and the electric potential changes into a more readable form. Indeed defining the magnetic monopole as $p = 2bm$ and substituting the inverse relation of the previous equation i.e.

$$r = \frac{\bar{r}-2|\alpha|^2}{1-|\alpha|^2}$$

Into the four potential yields

$$A_t = a\left[\frac{2m}{1-|\alpha|^2}-\frac{2m}{(1-|\alpha|^2)\bar{r}}\right] \tag{5.4}$$

$$A_\phi = p\cos(\theta) \tag{5.5}$$

The constant k is irrelevant to our end and doesn't affect the equation of motion hence it can be eliminated. The same can be said of the constant that appears in A_t hence we are going to ignore this constant.
By defining $q = -\frac{2am}{1-|\alpha|^2}$ the potential assumes the well known form:

$$A_t = \frac{q}{\bar{r}} \tag{5.6}$$

Here q is the electric charge.
And the azimuthal component remains unchanged. Hence the four potential takes form

$$A_\mu = \left(\frac{q}{\bar{r}}, 0, 0, p\cos(\theta)\right) \tag{5.7}$$

This is the four potential for the Reissner–Nordström metric, so it can be expected that the former coordinate transformation will yield the notorious Reissner–Nordström metric when applied to metric (5.3). We are going to do that. Noticing that

$$dr = \frac{d\bar{r}}{1 - |\alpha|^2}$$

The metric transforms in

$$\begin{aligned} ds^2 = &- \frac{(-2m + \bar{r})(-2|\alpha|^2 m + \bar{r})}{(-1 + |\alpha|^2)^2 \bar{r}^2} dt^2 + \frac{\bar{r}^2}{(-2m + \bar{r})(-2|\alpha|^2 m + \bar{r})} d\bar{r}^2 + \\ &\bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2(\theta) d\phi^2 \end{aligned}$$

it's easy to see that by changing $t = t'(-1 + |\alpha|^2)$ the metric simplifies a lot. We focus our attention on the terms in front of dt^2 by expanding them we obtain:

$$1 - \frac{2m(1 + |\alpha|^2)}{\bar{r}} + \frac{4|\alpha|^2 m^2}{\bar{r}^2}$$

Defining the mass as $M = m(1 + |\alpha|^2)$ and the square of the electromagnetic charge as $Q^2 = q^2 + p^2 = 4m^2|\alpha|^2$, this statement sets a constrain on the constants a and b.
The metric than becomes:

$$ds^2 = \left(1 - \frac{2M}{\bar{r}} + \frac{Q^2}{\bar{r}^2}\right) dt'^2 + \left(1 - \frac{2M}{\bar{r}} + \frac{Q^2}{\bar{r}^2}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2(\theta) d\phi^2$$

That is the Reissner–Nordström metric with four potential (5.7).
The metric is well studied and has a curvature singularity for r=0, and two event horizons for $r_\pm = M \pm \sqrt{M^2 - Q^2}$ when $M^2 > Q^2$.

5.2 Magnetic metric

5.2.1 Transformation I

Also for the case of the magnetic metric this transformation is trivial as is going to be proved.
The transformation reads:

$$\mathcal{E}' = |\lambda|^2 f_m$$

In the metric (3.13) takes form

$$ds^2 = |\lambda|^2 f_m d\psi^2 + \frac{e^{2\gamma_m}}{|\lambda|^2 f_m} \left(d\rho^2 + dz^2\right) - \frac{\rho^2}{|\lambda|^2 f_m} d\tau^2$$

By changing the coordinates to spherical ones

$$ds^2 = |\lambda|^2 r^2 \sin^2(\theta) d\psi^2 + \frac{1}{|\lambda|^2} \left[\frac{r}{r - 2m} dr^2 + r^2 d\theta^2\right] - \frac{1}{|\lambda|^2} \frac{r - 2m}{r} d\tau^2$$

by changing the coordinates with $\tilde{\tau} = \frac{\tau}{|\lambda|^2}$, $r = |\lambda|^2 \tilde{r}$, $|\lambda|^2 \psi = \tilde{\psi}$ and $m = |\lambda|^2 \tilde{m}$ the metric becomes Schwarzschild again i.e. is a gauge transform, in fact the metric in these ne coordinate is:

$$ds^2 = \tilde{r}^2 \sin^2(\theta) \tilde{\psi}^2 + \left[\frac{\tilde{r}}{\tilde{r} - 2\tilde{m}} d\tilde{r}^2 + \tilde{r}^2 d\theta^2 \right] - \frac{\tilde{r} - 2\tilde{m}}{\tilde{r}} d\tau$$

5.2.2 Transformation II

This transformation is completely analogous to the one of the electric case.
The new gravitational potential reads:

$$\mathcal{E}' = f_m + ib$$

hence using equation (2.10)

$$\nabla \chi' = 0 = \nabla \omega$$

hence

$$\omega = k$$

where k $\in \mathbb{R}$ is a constant. Similarly to the electric case the transformation $\hat{\psi} = \psi - k\tau$ diagonalize the metric, giving again Schwarzschild. Also for the magnetic metric transformation II is like putting the system into a uniformly rotating frame, therefore by a change of coordinates it can be changed to a non-rotating system.

5.2.3 Transformation III

This transformation is non trivial and leads to a peculiar result.
We have that:

$$\mathcal{E}' = \frac{f_m}{1 + icf_m} = \frac{r^2 \sin^2(\theta)}{1 + c^2 r^4 \sin^4(\theta)} + ic \frac{r^4 \sin^4(\theta)}{1 + c^2 r^4 \sin^4(\theta)}$$

Therefore:

$$f'_m = \frac{r^2 \sin^2(\theta)}{1 + c^2 r^4 \sin^4(\theta)}$$

and

$$\chi' = c \frac{r^4 \sin^4(\theta)}{1 + c^2 r^4 \sin^4(\theta)}$$

then plugging the last relation into equation (2.10) gives:

$$\omega' = -4c(r - 2m) \cos(\theta)$$

Therefore Ehlers's transformations adds a rotation to the space-time but leaves it without an electromagnetic potential.

The metric becomes:

$$
\begin{aligned}
ds^2 =& \frac{r^2 \sin^2(\theta)}{1 + c^2 r^4 \sin^4(\theta)} \left[d\psi + 4c(r-2m)\cos(\theta) d\tau\right]^2 + r^2 \left(1 + c^2 r^4 \sin^4(\theta)\right) d\theta^2 \\
&+ \left(\frac{r}{r-2m}\right) \left(1 + c^2 r^4 \sin^4(\theta)\right) dr^2 - \left(\frac{r-2m}{r}\right) \left(1 + c^2 r^4 \sin^4(\theta)\right) d\tau^2
\end{aligned}
\tag{5.8}
$$

To us the metric is unknown, meaning that we haven't found it in our references, therefore we are not aware if it a known solution without any physical meaning or hasn't been studied for other reasons. Beside these consideration giving a physical meaning to this metric is quite cumbersome, therefore ,as a start, is better to study its invariants, its background metric and determine the Petrov type.

The Petrov type is I-G meaning that this solution is one of the most general (and it can actually degenerate into type II or D solutions). Giving our references [9]-[10] here no vacuum solution of this type that takes this form has been found. The analysis of Kretschmann scalar shows that the metric has a singularity for r$\rightarrow$ 0.

A way to try and make sense of a metric is to analyze the background inasmuch as it tells us how the spacetime behaves when the mass is null.

For m=0 the metric assumes form:

$$
\begin{aligned}
ds^2 =& \frac{r^2 \sin^2(\theta)}{1 + c^2 r^4 \sin^4(\theta)} \left[d\psi + 4cr\cos(\theta) d\tau\right]^2 + r^2 (1 + c^2 r^4 \sin^4(\theta)) d\theta^2 \\
&+ (1 + c^2 r^4 \sin^4(\theta)) dr^2 - (1 + c^2 r^4 \sin^4(\theta)) d\tau^2
\end{aligned}
\tag{5.9}
$$

Now it seems quite clear that the metric can be expressed in cylindrical coordinates, by defining the new coordinates

$$
\begin{cases} R = r\sin(\theta) \\ z = r\cos(\theta) \end{cases}
$$

The background metric takes form

$$
ds^2 = \frac{R^2}{1 + c^2 R^4} [d\psi + 4czd\tau]^2 + (1 + c^2 R^4)(-d\tau^2 + dR^2 + dz^2) \tag{5.10}
$$

In this coordinates the metric is much more clear and easier to read, the symmetry cannot be cylindrical because of the explicit dependence on z, though it can be easily seen that there is a rotation that depends on z, which it might be a hunch that this spacetime might represent some sort of vortex.

Analyzing the Petrov type it can be found that this background metric is type D (one of the possible expected ways that a type I metric can degenerate). It can be prove that all type D metric are part of the Plebański–Demiański family i.e. it exists a change of coordinate into the Plebański–Demiański general metric. In our case the background metric is a sub-case called Kundt's space time which is a non-expanding solution with line element:

$$ds^2 = \frac{1}{(1-\alpha\gamma p)^2}\left[\rho^2(-Qdt^2 + \frac{dq^2}{Q}) + \frac{P}{\rho^2}(d\psi + 2\gamma q dt)^2 + \frac{\rho^2}{P}dp^2\right] \quad (5.11)$$

where

$$\rho^2 = \gamma^2 + P^2$$

$$P = k + 2np - \epsilon p^2 + 2\alpha m p^3 + [\alpha^2(k + e^2 + g^2) + \frac{1}{3}\Lambda]p^4$$

$$Q = \epsilon_0 - \epsilon_2 q^2$$

and

$$\epsilon = -\epsilon_2 + 6n\alpha\gamma + 2\gamma^2(3\alpha^2 k + \Lambda)$$

In addition, there are two constraints

$$3m + \gamma(\epsilon_2 + 2\epsilon) + 3\alpha\gamma^2 = 0$$

$$k + e^2 + g^2 - \gamma m + \frac{1}{6}(\epsilon + \epsilon_2)\gamma^2 = \kappa\epsilon_0$$

taken in the limit $\kappa \to 0$, that defines γ and k. And the parameters are real some with a physical significance with two discrete parameters ϵ_0 and ϵ_2 that can assume values 0, ± 1. Out of the six independent continuous parameters m, n, e, g, Λ and α. Only three have a physical meaning, e is the electric charge, g is the magnetic charge and Λ is the cosmological constant, the other three lack a physical meaning therefore the significance has to be given afterwards following the analysis of the metric.
In our case it can be stated that there is no cosmological constant nor an electromagnetic field, consequently e=g=Λ= 0.
The next step is to compare (5.10) with (5.11) to find if really there is a change of coordinates the can transform one into the other, and verify that the parameters satisfy all the previous conditions imposed by the Kundt's line element. It might help to perform a trivial coordinates transformation in (5.11), t$\to$ δt and $q \to \lambda q$, hence adding two extra parameters by a time dilatation and a dilatation of the q coordinate. The metric (5.11) thakes now form:

$$ds^2 = \frac{1}{(1-\alpha\gamma p)^2}\left[\rho^2(-Q\delta^2 dt^2 + \lambda^2\frac{dq^2}{Q}) + \frac{P}{\rho^2}(d\psi + 2\gamma q\delta\lambda dt)^2 + \frac{\rho^2}{P}dp^2\right] \quad (5.12)$$

Now we can proceed with the comparison, the first thing to do is to compare the mixed terms, its easy to see that:

$$2\gamma\delta q = 4cz$$

needs to hold, $q = z$ and $2\gamma\delta\lambda = 2c$, the meaning of a vertical coordinate is then given to q.
The metric can be simplified more by assuming that the coordinate p takes role

a sort of radial coordinate, than P would be a fourth order polynomial in p which it cannot be in our case (as it can be verified explicitly), therefore α =0.
Having this is mind, it can be noticed that in metric (5.10) there are three squared differentials multiplied by a field, if it's to be true in equation (5.11) Q needs to be 1, which holds the relations $\epsilon_2 = 0$ and $\epsilon_1 = 1$.
With the same concept in mind to other relations can be stated:

$$\delta^2\rho^2 = 1 + c^2R^4$$

$$\lambda^2\rho^2 = 1 + c^2R^4$$

obviously $\lambda = \delta$, and therefore $\gamma\lambda^2$ = 2c. Another two relations that can be obtained are:

$$\frac{P}{\gamma^2 + p^2} = \frac{R^2}{1 + c^2R^4}$$

$$\frac{\gamma^2 + p^2}{P}\, dp^2 = \left(1 + c^2R^4\right) dR^2$$

By substituting the first equation into the second and simplifying

$$dp = RdR$$

Consequently, neglecting the arbitrary constant

$$p = \frac{R^2}{2}$$

Plugging this result into the previous relations holds ϵ=0, n = $\frac{1}{4c^2}$, $\delta = 2c$ and $\gamma = \frac{1}{2c}$.
Now we have the ingredients to transform metric (5.10) into metric (5.11), the two metrics can indeed be mapped into each other. The last thing to verify is that the conditions on Kundt metric are satisfied.
The condition on ϵ impose that is null, indeed we have stated that previously.
The first constrain imposes that m=0. The second condition imposes that k = 0 as $\kappa \to 0$. Since the constants m, k do not appear explicitly in our change of coordinate they can be choose arbitrarily, in this case they can be choose to satisfy the imposed constrained.
Therefore it has been proven that metric (5.10) can be mapped into metric (5.11).
Kundt's metric describes a non expanding space-time where generally the derived parameter γ is formally the analogue of the NUT parameter in these spacetimes. Specifically, if $\gamma \neq 0$, these solutions have no curvature singularities.
This identification might help to investigate the significance of metric (5.9) inasmuch as we know its background and the physical significant of it, further investigation on the meaning of (5.9) has to be done.
Equation (5.10) is formally similar to Melvin's magnetic universe, even though

one is a rotating one and the other is a magnetic non-rotating universe, luckily it exists a transformation called Perjes's (Appenix B.1) transformations that does this job i.e. mapping a static solution with electromagnetic field into a stationary solution without an electromagnetic field, sadly this transformation doesn't give the wanted result. The results are also in the appendix.

5.2.4 Transformation IV

This transformation is again a gauge transformation.
The transformation reads:

$$\mathcal{E}' = f_m - |\beta|^2 = f_m - |\Phi|^2$$

and

$$\Phi' = \beta$$

Consequently as for the electric case the vector potentials are constant and do not act on the equation of motion and again it can be easily seen that the metric can be left invariant. therefore this transformation is again trivial even for the magnetic case.

5.2.5 Transformation V

The Harrison transformation for the magnetic metric is given by

$$\mathcal{E}' = \frac{f_m}{1 - |\alpha|^2 f_m} = f'_m - \left|\Phi'\right|^2$$

and for the complex electromagnetic potential:

$$\Phi' = \frac{\alpha f_m}{1 - |\alpha|^2 f_m} = A'_t + i\hat{A}'_\phi$$

Hence the field f′ can be easily found:

$$f'_m = \frac{r^2 \sin^2(\theta)}{[1 - |\alpha|^2 r^2 \sin^2(\theta)]^2}$$

It can be easily verified that the transformation doesn't add any rotation, therefore the field ω is equal to zero.
Now the complex electromagnetic potential can be tackled by defining α = a+ib.

$$A'_t = a \frac{r^2 \sin^2(\theta)}{1 - |\alpha|^2 r^2 \sin^2(\theta)}$$

$$\hat{A}'_\phi = b \frac{r^2 \sin^2(\theta)}{1 - |\alpha|^2 r^2 \sin^2(\theta)}$$

And by applying (2.9) then A'_ϕ can be found:

$$A'_\phi = 2b(r-2m)\cos(\theta)$$

Now we have all the elements to write down the metric and the vector potential.

$$ds^2 = -S_\alpha\left[\frac{r-2m}{r}d\tau^2 + \frac{r}{r-2m}dr^2 + r^2 d\theta^2\right] + \frac{r^2\sin^2(\theta)}{S_\alpha}d\psi^2$$

Where:

$$S_\alpha = \left[1-|\alpha|^2 r^2\sin^2(\theta)\right]^2$$

and the vector potential is given by

$$A_\mu = \left(2b(r-2m)\cos(\theta), 0, 0, a\frac{r^2\sin^2(\theta)}{1-|\alpha|^2 r^2\sin^2(\theta)}\right)$$

It has been taken into account that the components of the four-potential have to be switched since the discrete transformation t $\rightarrow i\psi$ and $\phi \rightarrow i\tau$ changes time with angular component and vice-versa therefore why the components during calculation remain the same their vectorial position into the four-potential changed.
This solution is quite general since it has an electro-magnetic component, it can be simplified by choosing b=0. This simplification is done with a purpose, indeed defining the constant $\alpha = B_0/2$ and a = $-B_0/2$ one finds a solution which is a Schwarzschild solution embedded in a magnetic field, actually a magnetic universe. Indeed the background metric m=0 holds exactly Melvin's solution which describes a magnetic universe that is a static, non-singular, cylindrical symmetric spacetime in which there exists an axial magnetic field aligned with the z-axis. It describes a universe containing a parallel bundle of electromagnetic flux held together by its own gravitational field. The Melvin magnetic universe is usually written as:

$$ds^2 = \Xi^2(-dt^2 + dr^2 + r^2 d\theta^2) + \frac{r^2\sin^2(\theta)}{\Xi^2}d\phi^2$$

$$\Xi = \left[1 + \frac{B_0^2}{4}r^2\sin^2(\theta)\right]$$

$$A_\phi = \left(-\frac{B_0}{2}\frac{r^2\sin^2(\theta)}{1+\frac{B_0^2}{4}r^2\sin^2(\theta)}\right)$$

The other components of the four potential are 0.
Everywhere on the axis the magnetic field has the value B_0. The solution that we have found therefore describes a solution that is a static axisymmetic solution in the presence of an external magnetic field. This solution was already

studied by Ernst in [3]. The behavior of the solution is similar to Schwarzschild because it has two critical points at r = 2m, 0 but the Kretschmann scalar gives us $K \propto 1/r^6$ consequently r = 2m is the event horizon and it can be eliminate via Kruskal method while r = 0 is a space-time singularity analogous to the Schwarzschild metric.

Chapter 6

Discrete transformations

As already said in section 3.2 from transformations (3.12) by imposing a condition on the parameters one finds a discrete transformation, i.e. a transformation that doesn't depend on any continuous parameter.
Here we give a peculiar example of discrete transformation, the inversion transformation.
In Appendix B there is another example of discrete transformation, albeit is not derived from any of transformations (3.12) hence is out of context.

6.1 The inversion transformation

This transformation, also called Buchdahl transformation, it is obtained by the combination of transformations I)-II)-III).
By $I) \circ III) \circ II)$ the following equation is obtained:

$$\mathcal{E}' = |\lambda|^2 \frac{\mathcal{E} + ib}{1 + ic(\mathcal{E} + ib)}$$

$$\Phi' = \frac{\lambda \Phi}{1 + ic(\mathcal{E} + ib)}$$

now setting c=b^{-1} and $\lambda = ib^{-1}$ and simplifying then taking the limit for b to infinity:

$$\lim_{b \to \infty} \frac{\mathcal{E} + ib}{ib\mathcal{E}} = \frac{1}{\mathcal{E}}$$

$$\lim_{b \to \infty} \frac{i\Phi}{b(1 + \frac{i}{b}(\mathcal{E} + ib))} = \frac{\Phi}{\mathcal{E}}$$

Hence

$$INV) \quad \mathcal{E}' = \frac{1}{\mathcal{E}} \quad \Phi' = \frac{\Phi}{\mathcal{E}}$$

Here we showed that a discrete transformations can be obtain by an infinite limit of the continuous parameter not necessarily by giving the parameter a

finite value.
The inversion transform has some neat proprieties, it maps gauge transformations in non-gauge ones and vice versa. For example the composition INV)∘II)∘I) gives :

$$\mathcal{E}' = \frac{1}{|\lambda|^2\mathcal{E} + ib}$$

$$\Phi' = \frac{\lambda\Phi}{|\lambda|^2\mathcal{E} + ib}$$

This transformation is quite similar to Ehlers transform, and raises the suspect that a combination of Buchdahl transformation and gauge transformations may give back Ehlers transform as a result.
Indeed by setting $\lambda = 1$ and by the transformation $\mathcal{E} \to 1/\tilde{\mathcal{E}}$ and $\Phi \to \tilde{\Phi}/\tilde{\mathcal{E}}$ one obtains:

$$\mathcal{E}' = \frac{\tilde{\mathcal{E}}}{1 + ib\tilde{\mathcal{E}}}$$

$$\mathcal{E}' = \frac{\tilde{\Phi}}{1 + ib\tilde{\mathcal{E}}}$$

Which is again the Ehlers transformation, but in this case has been derived using I) II) and the inversion transformation.
Harrison transformation can be mapped into the gauge transformation IV) by INV)∘ V) ∘ INV). Starting with $\tilde{\mathcal{E}} \to 1/\mathcal{E}$ and applying it to V) yields

$$\mathcal{E}' = \frac{1}{\mathcal{E} - 2\alpha^*\Phi - |\alpha|^2}$$

$$\Phi' = \frac{\Phi + \alpha}{\mathcal{E} - 2\alpha^*\Phi - |\alpha|^2}$$

Transforming it via the Inversion transform:

$$\mathcal{E}'' = \frac{1}{\mathcal{E}'} = \mathcal{E} - 2\alpha^*\Phi - |\alpha|^2$$

$$\Phi'' = \frac{\Phi'}{\mathcal{E}'} = \Phi + \alpha$$

Which is again transformation IV), therefore we have mapped V) into IV) that is a gauge transformation.
Applying the inversion transformation to IV) yields the non-gauge transformation

$$\mathcal{E}' = \frac{1}{\mathcal{E} - 2\beta^*\Phi - |\beta|^2}$$

$$\Phi' = \frac{\Phi + \beta}{\mathcal{E} - 2\beta^*\Phi - |\beta|^2}$$

It can be noticed that the first equation looks similar to Harrison's transformation. Transformation V) can actually be found from IV) via INV)$\circ$ IV)$\circ$ INV). Indeed from the previous equations applying $\mathcal{E} \to 1/\mathcal{E}$ and $\Phi \to \Phi/\mathcal{E}$, yields:

$$\mathcal{E}' = \frac{\mathcal{E}}{1 - 2\beta^*\Phi - |\beta|^2\mathcal{E}}$$

$$\Phi' = \frac{\Phi + \beta\mathcal{E}}{1 - 2\beta^*\Phi - |\beta|^2\mathcal{E}}$$

That are indeed transformation V).
We now proceed to analyze how the inverse transformation act on Schwarzschild metric.

6.2 Electric case

For non rotating space-times without an electric charge the transformation simplifies a lot. We have:

$$\mathcal{E}' = f' = \frac{1}{\mathcal{E}} = \frac{1}{f}$$

Where f is $1 - 2m/r$. It can be verified that by plugging these results in (2.14) and (2.15) the γ remains unchanged.
Therefore substituting in metric (2.1) and changing coordinates:

$$ds_e^2 = -\frac{r}{r-2m}dt^2 + \frac{r-2m}{r}dr^2 + (r-2m)^2 d\theta^2 + (r-2m)^2\sin^2(\theta)d\phi^2$$

By analyzing the metric is a Petrov type D with Kretschmann scalar K = $\frac{48m^2}{(r-2m)^6}$ this is becomes the same as Schwarzschild with the change R = r-2m there, indeed this metric would have a singularity in r=2m and a event horizon for r=0 and by the former change of coordinates would become again as Schwarschild. Proceeding by changing R = r-2m and M = -m ones obtains:

$$ds_e^2 = -\frac{R-2M}{R}dt^2 + \frac{R}{R-2M}dR^2 + R^2 d\theta^2 + R^2\sin^2(\theta)d\phi^2$$

Then in this case the inversion transformation is trivial and we obtain again Schwarzschild metric, albeit in a different frame of reference.

6.3 Magnetic case

For the magnetic case $f_m = r^2\sin^2(\theta)$ therefore is easy to verify that:

$$\mathcal{E}' = f' = \frac{1}{r^2\sin^2(\theta)}$$

again it can be verified that by plugging these results in (2.14) and (2.15) the γ remains unchanged.
Then applying this transformation to (3.13) one obtains:

$$ds_m^2 = -\left(\left(r^4 - 2mr^3\right)\sin^4(\theta)\right)d\tau^2 + \left(\frac{r^6\sin^4(\theta)}{r^2 - 2mr}\right)dr^2 + \left(r^6\sin^4(\theta)\right)d\theta^2 + \left(\frac{1}{r^2\sin^2(\theta)}\right)d\psi^2$$

This metric is quite peculiar, indeed is a Petrov type I-G hence a general metric and it has a Kretschmann scalar K $\propto \frac{[\csc(\theta)]^{12}}{r^{14}}$ meaning that the metric has a singularity at r=0 and for r$\neq$0 has singularity for $\theta = n\pi$ with n $\in \mathbb{Z}$. To understand this result better we can set $\phi = \phi_0$, therefore posing ourselves on a hyperplane, now the angular singularity appears in this plane as a "singular line" because $\forall r \neq 0$ and $\theta \to 0, \pi$ the metric blows up.

Chapter 7

Conclusions

Ernst's method is nifty and powerful tool not only to simplify the otherwise complicate equations of General Relativity in the stationary and axisymmetric case, but also to generate new ones from previously known one. As in the case of Ehlers transformation of the magnetic line element that generates an unknown metric, at least to us, that might or might not be of physical interest, but it was generated from a rather well know solution of Einstein's equations such as Schwarzschild metric.
The Harrison transformation can be used to add an electric or magnetic potential to a solution, we have showed that Schwarzschild becomes the Reissner–Nordström metric or, in the magnetic metric case, this transformation can be used to embed a spacetime into a magnetic universe.
The parameters of the transformations can be choose to hold discrete transformations such as the inversion transformation that when combined with trivial transformations produces new non-trivial transformations.
If one finds a way to add a parameter that might have (directly or indirectly) a physical interpretation, then it can be applied to a seed metric to produce a new metric enriched of a new propriety, as it is for Harrison transformation of the electric line element that add an electric and magnetic charge to the spacetime.
Although very a nifty tool Ernst's generating technique is still limited to the stationary case, while the axisymmetric actually describes a real physical phenomenon, the stationary case only represent equilibrium situation. Therefore having given a taste of what Ernst's technique can do, I hope it can be expanded to the non-stationary case.

Appendix A

Appendix A

In this appendix various mathematical technicalities are given in order to facilitate the reader in the comprehension of the thesis.

A.1 Differential operators

By writing the metric in Weyl coordinates with the Lewis-Weyl-Papapetrou, we have seen that the Einstein field equations, which usually are written by curved differential operators, can be written in terms of flat differential operators. For this reason we are interested in flat three-dimensional space-time in cylindrical coordinates(ρ,z,ϕ) , whose metric is given by

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

For any scalar function g(ρ,z,ϕ) or vector $\vec{D}$(ρ,z,ϕ)the gradient, the laplacian and the divergence are respectively:

$$\nabla g = \left(\frac{\partial g}{\partial \rho}, \rho^{-1}\frac{\partial g}{\partial \phi}, \frac{\partial g}{\partial z}\right)$$

$$\triangle g = \rho^{-1}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial g}{\partial \rho}\right) + \rho^{-2}\frac{\partial^2 g}{\partial \phi^2} + \frac{\partial^2 g}{\partial z^2}$$

$$\nabla \cdot \vec{D} = \rho^{-1}\frac{\partial}{\partial \rho}\left(\rho D_\rho\right) + \rho^{-1}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z}$$

Note that for the space-times considered these flat differential operators simplify further since all the scalar and vector functions do not depend on the ϕ coordinate, associated to the rotational Killing vector field.
It can be useful for the purposes of this thesis to define the gradient and laplacian in (r,x:= cos(θ)) coordinates:

$$\nabla g(r,x) = \frac{1}{\sqrt{(r-m)^2 - m^2x^2}}\left[\sqrt{r^2 - 2mr}\,\frac{\partial g}{\partial r}\,\hat{e}_r + \sqrt{1-x^2}\,\frac{\partial g}{\partial x}\,\hat{e}_x\right]$$

$$\triangle g(r,x) = \frac{1}{\sqrt{(r-m)^2 - m^2x^2}} \left[\frac{\partial}{\partial r}\left(\left(r^2 - 2mr\right) \frac{\partial g}{\partial r} \right) + \frac{\partial}{\partial x}\left(\left(1 - x^2\right) \frac{\partial g}{\partial x} \right) \right]$$

A.2 Prolate spherical coordinates

Here we define the prolate spherical coordinates (x, y), which are convenient to use for stationary and axisymmetric solutions. They are related to the Weyl coordinates by (ρ,z) the following transformation:

$$\begin{cases} \rho(x,y) = \lambda\sqrt{x^2-1}\sqrt{1-y^2} \\ z(x,y) = \lambda xy \end{cases} \tag{A.1}$$

Where λ is a real positive constant and the two new coordinates are defined for:

$$\rho \geq 0$$

$$-\infty \leq z \leq +\infty$$

These relations determine the domain of (x,y) i.e.:

$$x \geq 1$$

$$-1 \leq y \leq 1$$

The inverse function is:

$$\begin{cases} x(\rho,z) = \dfrac{\Xi^+ + \Xi^-}{2\lambda} \\ \\ y(\rho,z) = \dfrac{\Xi^+ - \Xi^-}{2\lambda} \end{cases} \tag{A.2}$$

Where the two functions $\Xi^{\pm}$ are:

$$\Xi^+ = \sqrt{\rho^2 + (z+\lambda)^2}$$

$$\Xi^- = \sqrt{\rho^2 + (z-\lambda)^2}$$

We are interested in how the metric changes under this coordinate transformation. In general the transformation law of basis dual vectors under a change of coordinates is:

$$dy^\alpha = \frac{\partial y^\alpha}{\partial x^\beta} dx^\beta$$

$$\therefore \quad dy^\alpha dy^\mu = \frac{\partial y^\alpha}{\partial x^\beta}\frac{\partial y^\mu}{\partial x^\nu} dx^\beta dx^\nu$$

In our case, passing from the prolate spheroidal coordinates (x, y) to the Weyl coordinates (ρ,z), we find:

$$d\rho^2 + dz^2 = \left[\left(\frac{\partial \rho}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 \right] dx^2 + \left[\left(\frac{\partial \rho}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \right] dy^2 + \\ + \left(\frac{\partial \rho}{\partial x}\frac{\partial \rho}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \right) dxdy$$

We have also the relations:

$$\frac{\partial\rho}{\partial x} = \lambda \frac{x\sqrt{1-y^2}}{\sqrt{x^2-1}}; \quad \frac{\partial z}{\partial x} = \lambda y$$

$$\frac{\partial\rho}{\partial y} = \lambda \frac{y\sqrt{x^2-1}}{\sqrt{1-y^2}}; \quad \frac{\partial z}{\partial y} = \lambda x$$

Substituting in the line element we find:

$$d\rho^2 + dz^2 = \lambda^2(x^2-y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right)$$

Note that the term dxdy does not appear. This transformation is useful since in (4.1) the block in (ρ,z) is diagonal and $g_{\rho\rho}$ = g_{zz} , so there is the factor $d\rho^2 + dz^2$. We can see how the metric changes also starting from the transformation of the metric components. The general transformation law for the metric is:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta}$$

where the symbol $'$ indicates the components of the coordinate system y^μ.
For the ansatz (2.1) is easy to verify that g_{xy} = 0.
Thus the (ρ,z)-block remains diagonal also in prolate spherical coordinates. Actually this is true only in this particular case where $g_{\rho\rho}$ = g_{zz} ; in general we will find $g_{xy} \neq$0. The prolate spherical coordinates can also be related to the coordinates (r, θ) through the following transformation:

$$\begin{cases} x(r) = \dfrac{r-m}{\lambda} \\ y(\theta) = \cos(\theta) \end{cases}$$

In this case if the original metric is diagonal in (r, θ), it remains diagonal also in (x, y) since the transformation matrix is diagonal.

A.3 Calculations

Taking the Lagrangian density of section 2.2, we can write the action as

$$S = \int \mathcal{L}\, d\rho dz = \int \mathcal{L}'\, \rho d\rho dz$$

Then:

$$\mathcal{L}' = -\frac{1}{2}f^{-2}\nabla f\cdot\nabla f + \frac{1}{2}\rho^{-2}f^2\nabla\omega\cdot\nabla\omega + 2f^{-1}\nabla A_t\cdot\nabla A_t -$$
$$-2\rho^{-2}f(\nabla A_\phi - \omega\nabla A_t)\cdot(\nabla A_\phi - \omega\nabla A_t)$$

As a matter of fact what we need in order to retrieve equations of section (2.2) is $\mathcal{L}'$. Applying Euler-Lagrange equations for the four fields we obtain:

$$\nabla\frac{\delta\mathcal{L}'}{\delta(\nabla A_\phi)} = \frac{\delta\mathcal{L}'}{\delta A_\phi}$$

Produces:

$$\boldsymbol{\nabla}\cdot\Big(\rho^{-2}f(\nabla A_\phi-\omega\nabla A_t)\Big)=0$$

from:

$$\nabla\frac{\delta\mathcal{L}'}{\delta(\nabla A_t)}=\frac{\delta\mathcal{L}'}{\delta A_t}$$

we obtain:

$$\boldsymbol{\nabla}\cdot\Big(f^{-1}\nabla A_t+\omega\rho^{-2}f\left(\nabla A_\phi-\omega\nabla A_t\right)\Big)=0$$

Now we move on to the gravitational fields

$$\nabla\frac{\delta\mathcal{L}'}{\delta(\nabla\omega)}=\frac{\delta\mathcal{L}'}{\delta\omega}$$

yields

$$\boldsymbol{\nabla}\cdot\Big(\rho^{-2}f\omega\nabla\omega\Big)=4\rho^{-2}f\nabla A_t(\nabla A_\phi-\omega\nabla A_t)$$

It can be transformed in :

$$\nabla\cdot[\rho^{-2}f^2\nabla\omega-4\rho^{-2}fA_t(\nabla A_\phi-\omega\nabla A_t)]=0$$

It can be achieved by using :

$$\begin{aligned}\boldsymbol{\nabla}\cdot\Big(\rho^{-2}fA_t(\nabla A_\phi-\omega\nabla A_t)\Big)&=\nabla A_t\left(\rho^{-2}f(\nabla A_\phi-\omega\nabla A_t)\right)\\+A_t\boldsymbol{\nabla}\cdot\Big(\rho^{-2}f(\nabla A_\phi-\omega\nabla A_t)\Big)&=\nabla A_t\left(\rho^{-2}f(\nabla A_\phi-\omega\nabla A_t)\right)\end{aligned}$$

Where in the last passage we have used the equations for A_ϕ.
The last equation for the f is:

$$\nabla\frac{\delta\mathcal{L}'}{\delta(\nabla f)}=\frac{\delta\mathcal{L}'}{\delta f}$$

$$\begin{aligned}&2f^{-3}\nabla f\cdot\nabla f-f^{-}2\triangle f=f^{-3}\nabla f\cdot\nabla f+\rho^{-2}f\nabla\omega\cdot\nabla\omega-2f^{-2}\nabla A_t\cdot\nabla A_t\\&-\rho^{-2}(\nabla A_\phi-\omega\nabla A_t)\cdot(\nabla A_\phi-\omega\nabla A_t)\end{aligned}$$

Simplifying it becomes:

$$\begin{aligned}f\triangle f=\nabla f\cdot\nabla f-\rho^{-2}f^4\nabla\omega\cdot\nabla\omega+2f\nabla A_t\cdot\nabla A_t+\\+2\rho^{-2}f^3(\nabla A_\phi-\omega\nabla A_t)\cdot(\nabla A_\phi-\omega\nabla A_t)\end{aligned}$$

Appendix B

Appendix B

B.1 Perjes's transformations

This transformations are not necessary for the purpose of this thesis, for more information about the theory behind them and how to obtain them see [8]. Here we give the definitions and some basic information about this transformations and then we apply them to the magnetic metric obtained with transformation III).
Perjes's transformations are a set of discrete transformation that map a static spacetime with non-null 4-potential into a stationary spacetime with null 4-potential.
The signature in this appendix is (+,-,-,-).
The line element for the static metric is:

$$ds^2 = -e^{\lambda}(d\rho^2 + dz^2) - e^{-\nu}\rho^2 d\varphi^2 + e^{\nu}dt^2$$

The 4-potential is:

$$A_\mu = (0, 0, \phi, \psi)$$

The line element for the stationary metric is:

$$ds^2 = -e^{\mu}(d\rho^2 + dz^2) - \rho^2 v d\varphi^2 + v^{-1}[dt - w d\varphi]^2$$

Here all the functions depends only on ρ and z.
Prejes found the following transformations,with $\kappa \in \mathbb{R}$ and $\kappa < 0$:

$$\begin{cases} \nu = -2\ln(v) \\ \phi = -\sqrt{-2/\kappa}\, w \\ \lambda = 4\mu - 2\ln(v) \end{cases} \tag{B.1}$$

Starting with Melvin's metric we can identify:

$$\phi = \frac{-2B\rho^2}{4 + B^2\rho^2}$$

$$\nu = \ln\left(\left(1 - \frac{B^2}{4}\rho^2\right)^2\right)$$

$$\lambda = \ln\left(\left(1 - \frac{B^2}{4}\rho^2\right)^2\right)$$

Applying (B.1) holds:

$$\frac{2B\rho^2}{4 + B^2\rho^2} = \sqrt{-2/\kappa}\, w$$

$$v^{-1} = (1 - \frac{B^2}{4}\rho^2)$$

$$\lambda = 0$$

Thus the new stationary metric takes form:

$$ds^2 = -\left(d\rho^2 + dz^2\right) - \frac{4\rho^2}{4 - B^2\rho^2} d\varphi^2 + \left(1 - \frac{B^2}{4}\rho^2\right)\left[dt - \sqrt{\frac{-\kappa}{2}} \frac{2B\rho^2}{4 + B^2\rho^2} d\varphi\right]^2$$

This line element is clearly not equal to (5.10) as already pointed out.
If we transform metric (5.10) of course we will not obtain Melvin's magnetic universe, despite this fact we will apply the transformation (B.1) nonetheless. We first identify the functions of the two metric:

$$v = -\frac{1 + c^2\rho^4}{\rho^2}$$

$$\mu = \ln\left(1 + c^2\rho^4\right)$$

$$w = -4cz$$

Applying (B.1):

$$\nu = -2\ln\left(-\frac{1 + c\rho^4}{\rho^2}\right)$$

$$\phi = \sqrt{-2/\kappa}\, 4cz$$

$$\lambda = \ln\left(\rho^2\left(1 + c^2\rho^4\right)^2\right)$$

Therefore the line element is:

$$ds^2 = -\rho^2\left(1 + c^2\rho^4\right)\left(d\rho^2 + dz^2\right) - \frac{1 + c^2\rho^4}{\rho^2} d\varphi^2 + \frac{\rho^2}{1 + c^2\rho^4} dt^2$$

and the four potential.

$$A_\mu = \left(0, 0, 4c\sqrt{-2/\kappa}\, z, 0\right)$$

Bibliography

[1] F. J. Ernst, *"New Formulation of the Axially Symmetric Gravitational Field Problem. II"*, Phys. Rev. 168 (1968) 1415, [https://doi.org/10.1103/PhysRev.168.1415]

[2] F. J. Ernst, *"New Formulation of the Axially Symmetric Gravitational Field Problem. I"*, Phys. Rev. 167 (1967).

[3] F. J. Ernst, *"Black holes in a magnetic universe"*, J. Math. Phys. 17, 54 (1976), [https://doi.org/10.1063/1.522781]

[4] S. M. Carroll, *Spacetime and geometry: an introduction to General Relativity*, Addison Wesley (2004).

[5] M. Astorino, *Enhanced Ehlers Transformation and the Majumdar-Papapetrou-NUT Spacetime*, JHEP 01 (2020).

[6] M.Astorino,*Embedding hairy black holes in a magnetic universe*, CECS-PHY-13/01.

[7] M.Astorino, *Charging axisymmetric space-times with cosmological constant*, https://arxiv.org/abs/1205.6998.

[8] Z. Prejés, *A method for constructing certain axially symmetrical Einstein-Maxwell fields*, (1968).

[9] H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt, *"Exact solutions of Einstein's field equations"*, (2003), [https://doi.org/10.1017/CBO9780511535185]

[10] J.B.Griffiths, J.Podolsky, *Exact space-times in Einstein's General Relativity*, (2009).

[11] M.A. Melvin, *Pure magnetic and electric geons*, Physics letters Vol. 8 Numb. 1, (1964).

[12] Markus Heusler, *Black hole uniqueness theorems*, Cambridge lecture notes in physics, (1996).

[13] R.M. Wald, *General Relativity*,Chicago Press , (1984).

[14] N. Straumann, *General Relativity With Applications to Astrophysics*,Springer, (2010).

[15] H. Stephani, *Differential equations: their solutions using symmetries*, (1989).

[16] K.S.Thorne, *Absolute stability of Melvin's magnetic universe*, Phys. Rev. 139 (1964) B 224.

[17] W. Kinnersley, *Symmetries of the stationary Einstein-Maxwell field equations*, Journal of Mathematical Physics 18, 1529 (1977), [`http://dx.doi.org/10.1063/1.523458`].

[18] Y.Tanabe, *Generation of the Lewis solutions from the Weyl solutions*, Journal of Mathematical Physics 20, 1486 (1979), [`https://doi.org/10.1063/1.524205`].

[19] P.J. Oliver, Applications of Lie Group to Differential Equations, Spinger, (1998)

[20] J.F.Cornwell, Group Theory in Physics an Introduction,Academy Press, (1997)