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Conserved charges in General Relativity in presence of line singularities

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Introduction

A black hole is a region of spacetime where the gravitational field is so strong that not even light can escape. Objects like this have already been supposed in 1784 by John Michell and, a few years later, by Pierre-Simon Laplace: they noticed the possibility of having a body that would not allow the light to escape. In fact, from the classical concept of escape velocity, that is the minimum velocity of an object to escape from the gravitational attraction of a massive body, we find

$$v_{escape} = \sqrt{\frac{2GM}{R}}\ ,$$

where M is the mass of the massive body and R its radius. We know that the speed of light is c; thus we can find a condition on the radius of the object given its mass by reversing the above formula

$$R_s = \frac{2GM}{c^2}\ . \tag{1}$$

If the mass of the object is concentrated within a radius R_s, the light cannot emerge from regions within that radius. Objects like this were initially called dark stars. However this idea was soon abandoned since light was found to have zero mass. Consequently, for Newton's theory of gravity, light is not affected by the gravitational attraction of a massive body: light can therefore escape from any massive object and such a dark star cannot exist in the universe.

In 1915 Einstein developed the theory of General Relativity, showing that gravity influences light's motion. In this theory the spacetime is a four-dimensional differentiable manifold with a Lorentzian metric $g_{\mu\nu}$; the latter is related to the distribution of matter, described by the energy-momentum tensor, by the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\ , \tag{2}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature: they are both determined by the metric $g_{\mu\nu}$. A few months after, Schwarzschild found the homonym solution, noting that it became singular for a radius value equal to that in (1), which was thereafter called Schwarzschild radius. It took several years to discover that this singularity was not physical, but was only related to the chosen coordinate system. Only in 1958, thanks to David Finkelstein, the surface given by the Schwarzschild radius was identified as an event horizon: matter and light are trapped inside and cannot escape. Thus the event horizon can only be crossed from the outside to the inside. Since the Schwarzschild solution describes only the space outside the massive body, the question arises whether it is really possible to enclose the mass of the body within the Schwarzschild radius starting from the gravitational collapse of a star.

Meanwhile in astrophysics, in 1931 Chandrasekhar discovered that a non rotating body in equilibrium on the degenerate electron pressure, commonly called white dwarf, has no stable solutions for $M_{\rm wd} > 1,4 M_\odot$: the white dwarf will then collapse into a neutron star, a body in equilibrium on the degenerate neutron pressure. In 1939 Oppenheimer and others discovered that a star whose mass exceeds the TOV (Tolman-Oppenheimer-Volkoff) limit continues to collapse: gravity overcomes any kind of pressure and a neutron star is not formed as the degenerate neutron pressure is not sufficient to keep the star in equilibrium. This happens for bodies such that $M_{\text{nucleus}} \sim 3M_{\odot}$ (or equivalently for initial masses $M > 15M_{\odot}$), but this limit is still very uncertain. At this point in known physics there is no other phenomenon that can stop the collapse: we therefore refer to it as black hole, since the mass M of the collapsing body will be within the radius R_s.

Thus in the 1960s the golden age of General Relativity, began also thanks to the discovery of pulsars, that are rotating neutron stars. Until that time, neutron stars, like black holes, were regarded as theoretical curiosities: the discovery of pulsars showed their physical relevance and spurred a further interest in all types of compact objects that might be formed by gravitational collapse. Therefore black holes become mainstream subjects of research. In this period Kerr found the solution for a rotating black hole, while Newman found the solution for a rotating and electrically charged black hole. Through the work of Werner Israel, Brandon Carter and David Robinson the no-hair theorem emerged: a stationary, asymptotically flat black hole solution to general relativity coupled to electromagnetism is fully characterized by the parameters mass, electric charge and angular momentum. Thanks to Komar, a way to calculate such quantities has been found, leading to the formulation of black hole thermodynamics in the early 1970s through to the work of James Bardeen, Jacob Bekenstein, Carter, and Hawking.

In recent years the question of how the thermodynamics of a black hole is formulated in the presence of magnetic charge or NUT (Newman-Unti-Tamburino) charge has arisen, since they are part of the family of solutions representing stationary and axisymmetric black holes. We then try to better understand these solutions which present line singularities on the axis and we will also analyze the various methods that have emerged in recent literature in the case of the NUT charge.

Contents

Chapter 1

Introductory concepts

In this chapter some basic concepts will be introduced to deal with the calculation of the conserved charges of a black hole. First of all, we will recall two fundamental notions: the Einstein field equations, with its general stationary and axisymmetric solution and the notion of Killing vector. Secondly, we will deal with the concept of hypersurfaces: in particular we will focus on the notions of normal vector and volume element and we will state the Stokes' theorem in the context of differential geometry. Finally, we will show some important features of the event horizon.
In this work we will use the Planck units, where $c = G = \hbar = k_B = 4\pi\varepsilon_0 = 1$.

1.1 Solutions of Einstein equations

Let M be a four-dimensional differentiable manifold and g the metric on M, we call spacetime the couple (M, g). The Einstein field equations relates the metric of spacetime with the matter distribution, expressed by the energy-momentum tensor $T_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi\, T_{\mu\nu} \ , \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the scalar curvature: they are both determined by the metric $g_{\mu\nu}$. Exact solutions of such equations can be found in the electrovacuum case: it means that the energy-momentum tensor in (1.1) is only the electromagnetic one, given by

$$T^E{}_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) \ , \tag{1.2}$$

where $F_{\mu\nu}$ is the electromagnetic Faraday tensor, defined, as usual, from the electromagnetic four-potential A_μ, $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$. In this case the Einstein equations must be couple with the Maxwell equations for the four-potential, that in vacuum takes the form:

$$\partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) = 0 \ . \tag{1.3}$$

For a stationary and axisymmetric spacetime, that has two commuting vector fields $k = \partial_t$ and $m = \partial_\varphi$, associated respectively with time translations and rotations around the symmetry axis, the most general solution representing a single source and asymptotically flat black hole regular outside the horizon is the dyonic Kerr-Newman metric, that in Boyer-Lindquist coordinates is given by

$$\begin{aligned} ds^2 = &- \frac{\Delta - a^2 \sin^2\theta}{\Xi} dt^2 - 2a \sin^2\theta \frac{r^2 + a^2 - \Delta}{\Xi} dt d\varphi + \\ &+ \frac{\left(r^2 + a^2\right)^2 - a^2 \sin^2\theta \Delta}{\Xi} \sin^2\theta d\varphi^2 + \frac{\Xi}{\Delta} dr^2 + \Xi d\theta^2 \ , \end{aligned} \tag{1.4}$$

where

$$\Delta = r^2 - 2mr + a^2 + q^2 + p^2 \,, \qquad \Xi = r^2 + a^2 \cos^2\theta \,.$$

The parameters q, p, m and a are respectively the electric charge, the magnetic charge, the mass and the rotational parameter.
The electromagnetic four-potential for this solution is

$$A = \left[\frac{qr - pa\cos\theta}{\Xi}, \, 0, \, 0, \, \frac{p\cos\theta\left(r^2 + a^2\right) - aqr\sin^2\theta}{\Xi} \right] \,. \tag{1.5}$$

The metric is singular for the r-values such that $\Delta(r) = 0$. This equation has two solutions

$$r_{\pm} = m \pm \sigma \,, \tag{1.6}$$

where

$$\sigma = \sqrt{m^2 - a^2 - q^2 - p^2} \,. \tag{1.7}$$

The values $r = r_+$ and $r = r_-$ are coordinate singularities: they define two surfaces that are called event horizons, but only the first horizon is physically relevant. Note that σ can also be a complex number: in this case the function $\Delta(r)$ has no real zeroes and the solution describes a naked singularity, since there is only the curvature singularity $\Xi = 0$, without any event horizon. This is a problematic situation because it would open to the possibility of observing the curvature singularity, and the spacetime will fail to be a smooth manifold as requested by General Relativity. The presence of an event horizon is precisely what allows us not to interact with that singularity. Therefore, from now on, we will consider σ to be real, i.e. the parameters m, a, q and p must satisfy

$$m^2 \geq a^2 + q^2 + p^2 \,. \tag{1.8}$$

1.2 Killing vector

Consider the spacetime (M, g) and the diffeomorphism $\phi \colon M \to M$. If this diffeomorphism leaves the metric unchanged then it is called isometry. If there is an isometry, we can construct orbits along which the metric remains unchanged: the vector tangent to these orbits is called the Killing vector. So a Killing vector characterizes the symmetries of the metric and it is defined by the Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{1.9}$$

or similarly

$$\nabla_{(\mu}\xi_{\nu)} = 0 \,. \tag{1.10}$$

An important consequence of the Killing equation (1.9) is the following lemma

Lemma 1.2.1. *For a Killing vector field ξ*

$$\nabla_\rho \nabla_\mu \xi^\nu = R^\nu{}_{\mu\rho\sigma} \xi^\sigma \,. \tag{1.11}$$

Proof. From the definition of the Riemann tensor

$$[\nabla_\mu, \nabla_\nu]\xi_\rho = -R^\sigma{}_{\rho\mu\nu} \xi_\sigma \,.$$

Using the Killing equation (1.9) and making cyclic permutations of the indices $(\mu\nu\rho)$, we can write

$$\nabla_\mu \nabla_\nu \xi_\rho + \nabla_\nu \nabla_\rho \xi_\mu = -R^\sigma{}_{\rho\mu\nu} \xi_\sigma \,,$$

$$\nabla_\nu \nabla_\rho \xi_\mu + \nabla_\rho \nabla_\mu \xi_\nu = -R^\sigma{}_{\mu\nu\rho}\xi_\sigma \ ,$$
$$\nabla_\rho \nabla_\mu \xi_\nu + \nabla_\mu \nabla_\nu \xi_\rho = -R^\sigma{}_{\nu\rho\mu}\xi_\sigma \ .$$

Adding the first equation to the second and subtracting the third, we obtain

$$2\nabla_\nu \nabla_\rho \xi_\mu = (-R^\sigma{}_{\rho\mu\nu} - R^\sigma{}_{\mu\nu\rho} + R^\sigma{}_{\nu\rho\mu})\xi_\sigma \ .$$

Recalling the Bianchi identity

$$R^\sigma{}_{[\rho\mu\nu]} = 0 \ ,$$

it is obvious to find

$$\nabla_\nu \nabla_\rho \xi_\mu = R^\sigma{}_{\nu\rho\mu}\xi_\sigma \ .$$

By renaming the indices and raising the index ν

$$\nabla_\rho \nabla_\mu \xi^\nu = R^\sigma{}_{\nu\rho}{}^\mu \xi_\sigma = R^\nu{}_{\mu\rho\sigma}\xi^\sigma \ ,$$

where the properties of the Riemann tensor were used in the last equality. ■

We also report another lemma that will be used in the next chapter

Lemma 1.2.2. *For a Killing vector field* ξ

$$\nabla_\nu \nabla_\mu \nabla^\nu \xi^\mu = 0 \ . \tag{1.12}$$

Proof. Recall the definition of the Riemann tensor

$$[\nabla_\nu, \nabla_\mu]\, u^\rho = R^\rho{}_{\sigma\nu\mu} u^\sigma \ .$$

Setting $u^\rho = t^{\rho\lambda} v_\lambda$, the definition becomes

$$[\nabla_\nu, \nabla_\mu]\, t^{\rho\lambda} v_\lambda = R^\rho{}_{\sigma\nu\mu} t^{\sigma\lambda} v_\lambda \ .$$

Applying the Leibniz rule to the left member

$$v_\lambda\, [\nabla_\nu, \nabla_\mu]\, t^{\rho\lambda} + t^{\rho\lambda} \underbrace{[\nabla_\nu, \nabla_\mu]\, v_\lambda}_{-R^\sigma{}_{\lambda\nu\mu} v_\sigma} = R^\rho{}_{\sigma\nu\mu} t^{\sigma\lambda} v_\lambda \ .$$

This equation is true for any v, so

$$[\nabla_\nu, \nabla_\mu]\, t^{\rho\lambda} = R^\rho{}_{\sigma\nu\mu} t^{\sigma\lambda} + R^\lambda{}_{\sigma\nu\mu} t^{\rho\sigma} \ .$$

Contracting the index ρ with ν and the index λ with μ

$$[\nabla_\nu, \nabla_\mu]\, t^{\nu\mu} = R^\nu{}_{\sigma\nu\mu} t^{\sigma\mu} + R^\mu{}_{\sigma\nu\mu} t^{\nu\sigma} = R_{\sigma\mu} t^{\sigma\mu} - R_{\sigma\nu} t^{\nu\sigma} = 2R_{\sigma\mu} t^{[\sigma\mu]} = 0 \ ,$$

where in the last step we used the symmetry of the Ricci tensor.
Now choose $t^{\nu\mu} = \nabla^\nu \xi^\mu$, from the last equation we have

$$\nabla_\nu \nabla_\mu \nabla^\nu \xi^\mu = \nabla_\mu \nabla_\nu \nabla^\nu \xi^\mu \ ,$$

but, for the Killing equation (1.10), $\nabla^\nu \xi^\mu$ is completely antisymmetric, therefore

$$\nabla_\nu \nabla_\mu \nabla^\nu \xi^\mu = 0 \ .$$

■

1.3 Hypersurfaces

Let M be an four-dimensional manifold with metric g, a hypersurface is a three-dimensional submanifold Σ on M. A hypersurface can be defined by setting single function to a constant value:

$$\Phi\left(x^{\alpha}\right)=\Phi^{*} \ . \tag{1.13}$$

This constraint can be written in parametric equations of the form

$$x^{\alpha}=x^{\alpha}\left(y^{a}\right) \ , \tag{1.14}$$

where $\{y^a\}$ is a coordinate system on the hypersurface Σ. Therefore, equation (1.13) can be viewed as a restriction on the coordinates.
The vector field

$$\eta^{\mu}=g^{\mu\nu}\nabla_{\nu}\Phi \tag{1.15}$$

is normal to the hypersurface, since the value of Φ changes only in the direction orthogonal to Σ. If the vector η is timelike, then the hypersurface is said to be spacelike; if η is spacelike, the hypersurface is timelike; if η is null, the hypersurface is also null. Any vector field proportional to the normal vector field η,

$$\xi^{\mu}=\Psi\left(x^{\alpha}\right)\eta^{\mu} \tag{1.16}$$

for some function Ψ, will itself be a normal vector field.

If the hypersurface is not null, a unit normal vector n can be defined as

$$n^{\mu}=\varepsilon\,\frac{\eta^{\mu}}{|\eta^{\mu}\eta_{\mu}|^{1/2}} \ , \tag{1.17}$$

where ε is a number that can be either 1 or -1. Such a normal vector field is unique except for a global sign, which changes its orientation. We require that the normal vector field n points in the direction of increasing Φ, that is

$$n^{\mu}\nabla_{\mu}\Phi>0 \ . \tag{1.18}$$

From (1.17), the request (1.18) takes the form

$$|g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi|^{-1/2}\,\varepsilon\,g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi=|g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi|^{1/2}\,\varepsilon\,n^{\mu}n_{\mu}>0 \ ,$$

$$\Rightarrow \varepsilon\,n^{\mu}n_{\mu}>0 \ ,$$

but $n^{\mu}n_{\mu}=-1$ for spacelike hypersurfaces and $n^{\mu}n_{\mu}=1$ for timelike hypersurfaces, then

$$\varepsilon=\begin{cases} -1 & \text{if } \Sigma \text{ is spacelike,} \\ +1 & \text{if } \Sigma \text{ is timelike,} \end{cases} \tag{1.19}$$

therefore: $\varepsilon=n^{\mu}n_{\mu}$. Because of this choice, if Σ is the hypersurface defined by $t=const$, then the unit normal vector n is future-directed.

If the hypersurface Σ is null, the unit normal vector n is not defined because $g^{\mu\nu}\nabla_{\mu}\Phi\nabla_{\nu}\Phi=0$. In this particular case we consider the normal vector field η defined in (1.17): this vector is also tangent to Σ, since null vectors are orthogonal to themselves; in fact $\eta^{\mu}\eta_{\mu}=0$. It can be also shown that η is tangent to null geodesics contained in Σ: consider the integral curves $C^{\mu}(\alpha)$ of the vector field η, which satisfy

$$\frac{dC^{\mu}}{d\alpha}=\eta^{\mu} \ . \tag{1.20}$$

These curves will be contained in Σ, because η is tangent to the hypersurface. Then we calculate

$$\eta^\mu \nabla_\mu \eta_\nu = \eta^\mu \nabla_\mu \nabla_\nu \Phi = \eta^\mu \nabla_\nu \nabla_\mu \Phi = \eta^\mu \nabla_\nu \eta_\mu = \frac{1}{2} \nabla_\nu \left(\eta^\mu \eta_\mu\right) = \frac{1}{2} \nabla_\nu \left(\eta^2\right) ,$$

where in the second equality we used the torsion-free condition, so covariant derivatives acting on scalars commute. Since η^2 is constant on Σ, its derivative will be normal to the hypersurface. Therefore we must have

$$\nabla_\nu \left(\eta^2\right) = f\, \eta_\nu \ \Rightarrow\ \eta^\mu \nabla_\mu \eta_\nu = \frac{f}{2}\, \eta_\nu \tag{1.21}$$

for some scalar function $f(x^\alpha)$. The above equation is equivalent to the geodesic equation (in non-affine parameterization), so the integral curves $C^\mu(\alpha)$ are geodesics. At this point we are free to re-parameterize the curve $C^\mu(\alpha)$ with an affine parameter $\lambda(\alpha)$, or equivalently, since the normal vector field η is defined up to a scalar function $\Psi(x)$, we can scale that normal vector field such that $\xi^\mu \nabla_\mu \xi_\nu = 0$, with ξ given by (1.16). The null geodesics $C^\mu(\lambda)$, with λ affine parameter, whose union is the null hypersurface Σ, are called the generators of Σ. In other words, a null hypersurface is generated by null geodesics.

It is often convenient to use a coordinate system on a manifold such that it naturally adapts to some hypersurface Σ. One way is to use the so-called Gaussian normal coordinates: choose coordinates $\{y^a\}$, $a = 1, 2, 3$ on the hypersurface Σ; now, at each point $p \in \Sigma$, take the geodesic for which n^μ is the tangent vector at p. Let z be the affine parameter on each geodesic. For any point q in a neighborhood of Σ, we then assign the coordinates $\{z, y^1, y^2, y^3\}$, where $\{y^a\}$ are the coordinates of the point p connected to q by the geodesic constructed as illustrated above. The coordinates $\{z, y^1, y^2, y^3\}$ are known as Gaussian normal coordinates.

The induced metric on the hypersurface Σ is obtained by restricting the action of the metric $g_{\mu\nu}$ on M to the tangent vectors to Σ. By the equation (1.14), we can defined the vectors

$$e_i^\alpha = \frac{\partial x^\alpha}{\partial y^i} \tag{1.22}$$

that are tangent to curves contained in Σ. Then the induced metric on Σ can be calculated as

$$h_{ij} = g_{\mu\nu}\left(x^\alpha\left(y^a\right)\right) \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} = g_{\mu\nu}\left(x^\alpha\left(y^a\right)\right) e_i^\mu e_j^\nu\ . \tag{1.23}$$

In Gaussian normal coordinates the metric of the spacetime can be written as

$$ds^2 = \varepsilon dz^2 + h_{ij} dy^i dy^j\ , \tag{1.24}$$

where h_{ij} is the induced metric on Σ.

Along with an induced metric, hypersurfaces inherit an induced volume element from the manifold M. For the manifold the volume element is given by

$$dV = \sqrt{|g|}\, d^4x\ , \tag{1.25}$$

where $g = det\left(g_{\mu\nu}\right)$. Similarly, in the case of a non-null hypersurface, the volume element on Σ will be

$$d\Sigma = \sqrt{|h|}\, d^3y\ , \tag{1.26}$$

where $h = det\left(h_{ij}\right)$, with h_{ij} the same of (1.24). Since the hypersurface is embedded in the manifold M, we can defined an oriented volume element $d\Sigma_\mu$ by the combination of the unit normal vector n_μ and $d\Sigma$. Of course, as for n, the oriented volume element can have two different orientations; we choose the convention

$$n^\mu d\Sigma_\mu > 0\ , \tag{1.27}$$

which implies

$$d\Sigma_\mu = \varepsilon n_\mu d\Sigma \ . \tag{1.28}$$

In fact

$$n^\mu d\Sigma_\mu = \varepsilon\, n^\mu n_\mu\, d\Sigma = \varepsilon^2\, d\Sigma = d\Sigma > 0 \ .$$

Therefore it is necessary not to include the null case because if Σ is null $h = 0$ and n_μ does not exist.
Another way to write the oriented volume element is

$$d\Sigma_\mu = \epsilon_{\mu\alpha\beta\gamma} e_1^\alpha e_2^\beta e_3^\gamma d^3y \ , \tag{1.29}$$

where $\epsilon_{\mu\alpha\beta\gamma}$ are the components of the Levi-Civita tensor and are related to the components $\tilde{\epsilon}_{\mu\alpha\beta\gamma}$ of the Levi-Civita symbols by the following equations

$$\epsilon_{\mu\alpha\beta\gamma} = \sqrt{|g|}\, \tilde{\epsilon}_{\mu\alpha\beta\gamma} \ . \tag{1.30}$$

Because the Levi-Civita tensor is completely antisymmetric, as for (1.28), the sign of $d\Sigma_\mu$ as defined in (1.29) would depend on the ordering of the coordinates y^1, y^2 and y^3, which is arbitrary; note that from the convention in (1.27) follows that the scalar $f \equiv \epsilon_{\mu\alpha\beta\gamma}\, n^\mu e_1^\alpha e_2^\beta e_3^\gamma$ must be a positive quantity.: this sets the conventional ordering.
We now demonstrate that (1.28) and (1.29) are equivalent. First of all, note that

$$\varepsilon f n_\mu = \varepsilon \epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha e_2^\beta e_3^\gamma n^\mu n_\mu = \epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha e_2^\beta e_3^\gamma \ ,$$

As f is a scalar, it has the same value in every coordinate system: then, without lost of generality, we can consider the Gaussian normal coordinates. From (1.24) it is obvious to find that $\sqrt{|g|} = \sqrt{|h|}$, because $g = \varepsilon\, h$, where ε can be either 1 or -1. Therefore, since $e_1^\alpha = \delta_1^\alpha$, $e_2^\beta = \delta_2^\beta$, $e_3^\gamma = \delta_3^\gamma$ and $n = (1, 0, 0, 0)$ in this particular coordinate system, we find $f = \sqrt{|h|}$. Thus we get

$$\epsilon_{\mu\alpha\beta\gamma}\, e_1^\alpha e_2^\beta e_3^\gamma = \varepsilon \sqrt{|h|} n_\mu \ .$$

This equality concludes the proof.

Let's take an example: consider the Schwarzschild spacetime: the metric is given by (1.4) setting $a = p = q = 0$ (non-charged and non-rotating black hole). It can be rewrite as

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{1.31}$$

Let Σ be the hypersurface defined by $t = const$, or, using the above notation, $\Phi(t, r, \theta, \varphi) = t$, that is constant on Σ. Obviously, the tangent vectors to Σ are

$$\begin{aligned} e_r &= e_r^\alpha\, \partial_\alpha = \delta_r^\alpha\, \partial_\alpha = \partial_r \ , \\ e_\theta &= e_\theta^\alpha\, \partial_\alpha = \delta_\theta^\alpha\, \partial_\alpha = \partial_\theta \ , \\ e_\varphi &= e_\varphi^\alpha\, \partial_\alpha = \delta_\varphi^\alpha\, \partial_\alpha = \partial_\varphi \ . \end{aligned}$$

Then $\{r, \theta, \varphi\}$ is a good set of coordinates on Σ; the induced metric on Σ is

$$h_{ij} dy^i dy^j = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{1.32}$$

Now we calculate the unit normal vector from (1.17). For a function f, $\nabla_\nu f = \partial_\nu f$, so $\nabla_\nu \Phi$ is non-vanishing only for $\nu = t$. Therefore the normal vector field η, defined in (1.15), has only one non-null component

$$\eta^t = g^{tt} \nabla_t \Phi = g^{tt} = -\left(1 - \frac{2m}{r}\right)^{-1} \ .$$

The normal vector field is timelike, then the hypersurface is spacelike and the normal vector n, choosing $\varepsilon = -1$, becomes

$$n = n^{\mu}\,\partial_{\mu} = \left[\left(1 - \frac{2m}{r}\right)^{-1/2}, 0, 0, 0\right] . \tag{1.33}$$

In the null case the oriented surface element can be written as in (1.29), where $e_1^{\alpha} = \xi_{\alpha}$, where

$$\xi^{\alpha} = \frac{dC^{\alpha}}{d\lambda}$$

is the tangent vector to the generators of Σ.

Later we will consider two-dimensional surfaces S as the boundary of an hypersurface Σ on the manifold M. The surface S is a submanifold of Σ and is described by an equation $\psi\left(y^a\right) = \psi^*$ or similarly by the parametric equations $y^a = y^a\left(w^A\right)$, where $\{w^A\}$ is a coordinate system on S. Since Σ is defined by $x^{\alpha}\left(y^a\right)$, we can combine this relation with the one of the surface S to obtain the equations $x^{\alpha}\left(w^A\right)$, which describe S as embedded in M.

The surface element on S has two equivalent expressions

$$dS_{\mu\nu} = \epsilon_{\mu\nu\beta\gamma} e_2^{\beta} e_3^{\gamma}\, d^2 w , \tag{1.34}$$

$$dS_{\mu\nu} = -2 n_{[\mu}\sigma_{\nu]} \sqrt{|h^{(2)}|}\, d^2 w , \tag{1.35}$$

where $h^{(2)}$ is the determinant of the induced metric on S, $h_{ij}^{(2)}$, n is the unit normal vector to Σ and σ is the unit normal vector to S; σ is also normal to n. The equivalence of the above equations is guaranteed by the same arguments used for the equality between (1.28) and (1.29).

1.4 Gauss-Stokes Theorem

Let R be a region of the spacetime manifold M bounded by a closed three-dimensional hypersurface ∂R, then for any vector field V defined within R

$$\int_R \nabla_{\mu} V^{\mu} \sqrt{|g|}\, d^4 x = \int_{\partial R} V^{\mu}\, d\Sigma_{\mu} . \tag{1.36}$$

This result is known as Gauss' theorem [1]. The theorem is valid also for hypersurfaces that have segments that are timelike, spacelike or null.

Let now Σ be an hypersurface on the manifold M with boundary $\partial\Sigma$, a closed surface. Then for any antisymmetric tensor field $X^{\mu\nu}$

$$\int_{\Sigma} \nabla_{\nu} X^{\mu\nu} d\Sigma_{\mu} = \frac{1}{2} \oint_{\partial\Sigma} X^{\mu\nu} dS_{\mu\nu} . \tag{1.37}$$

This result is another version of Gauss' theorem, usually called Stokes' theorem. Instead of $dS_{\mu\nu}$, we will often use the following surface element

$$d\Sigma_{\mu\nu} \equiv \frac{1}{2}\, dS_{\mu\nu} . \tag{1.38}$$

Therefore the Stokes' theorem becomes

$$\int_{\Sigma} \nabla_{\nu} X^{\mu\nu} d\Sigma_{\mu} = \oint_{\partial\Sigma} X^{\mu\nu} d\Sigma_{\mu\nu} . \tag{1.39}$$

[1] For a proof of Gauss' theorem and Stokes' theorem see [2], section 3.3

1.5 Killing horizons and surface gravity

The horizon of a black hole has a particular feature: it is a null hypersurface, especially it is a Killing horizon. So let's define what a killing horizon is.

Definition 1.5.1. *A null hypersurface Σ is called a Killing horizon if there is a Killing vector field ξ normal to Σ.*

Therefore the notion of Killing horizon is independent from that of an event horizon: a Killing horizon need not necessarily to be an event horizon. However, there is an essential link between event horizons and Killing horizons. In particular, there are two independent results, usually referred to as rigidity theorems. The first, due to Carter, states that for a static black hole, the static Killing vector field $k = \partial_t$ must be normal to the event horizon, whereas for a stationary and axisymmetric black hole, with the property that the planes spanned by k and the rotational Killing vector field $m = \partial_\varphi$ are orthogonal to a family of two dimensional surface (this is the orthogonality property), there exists a Killing vector field $\xi = \partial_t + \Omega_H \partial_\varphi$ which is normal to the event horizon; Ω_H is a constant called the angular velocity of the horizon. This result is a purely geometric fact: it holds without invoking the Einstein field equations. The second result, due to Hawking, directly proves that in vacuum or electrovac spacetimes, the event horizon of any stationary black hole must be a Killing horizon. Hawking's theorem makes no assumptions of symmetries beyond stationarity, but it does rely on the properties of the Einstein field equations.
To every Killing horizon we can associate a quantity called surface gravity. Consider the Killing horizon Σ with normal vector η. As shown in section 1.3 it satisfies the geodesic equation $\nabla_\eta \eta = 0$ in affine parameterization. The Killing vector ξ normal to Σ will be proportional to η on Σ: $\xi = \Psi(x^\alpha)\,\eta$, for some function Ψ. Therefore it follows that ξ satisfies the geodesic equation in non affine parameterization

$$\nabla_\xi \xi = \kappa\,\xi\,, \quad \text{on } \Sigma\,, \tag{1.40}$$

where κ is a function called surface gravity.
This formula for the surface gravity is quite difficult to apply, but we can find a direct relation between κ and the Killing vector field ξ. In fact, from the Frobenius' theorem follows that the necessary and sufficient condition that ξ be hypersurface orthogonal is

$$\xi_{[\mu} \nabla_\nu \xi_{\rho]}\big|_\Sigma = 0\,. \tag{1.41}$$

We can rewrite this condition as

$$\xi_\mu \nabla_\nu \xi_\rho + \xi_\nu \nabla_\rho \xi_\mu + \xi_\rho \nabla_\mu \xi_\nu - \xi_\nu \nabla_\mu \xi_\rho - \xi_\rho \nabla_\nu \xi_\mu - \xi_\mu \nabla_\rho \xi_\nu = 0\,.$$

By the Killing equation (1.9) follows that

$$\xi_\mu \nabla_\nu \xi_\rho = -\left(\xi_\nu \nabla_\rho \xi_\mu + \xi_\rho \nabla_\mu \xi_\nu\right) = -\left(\xi_\nu \nabla_\rho \xi_\mu - \xi_\rho \nabla_\nu \xi_\mu\right)\,.$$

By contraction with $\nabla^\nu \xi^\rho$ we find

$$\begin{aligned}\xi_\mu \left(\nabla^\nu \xi^\rho\right)\left(\nabla_\nu \xi_\rho\right) &= -\left(\nabla^\nu \xi^\rho\right)\left(\xi_{[\nu} \nabla_{\rho]} \xi_\mu - \xi_{[\rho} \nabla_{\nu]} \xi_\mu\right) = -2\left(\nabla^\nu \xi^\rho\right) \xi_\nu \nabla_\rho \xi_\mu = \\ &= -2\kappa \xi^\rho \nabla_\rho \xi_\mu = -2\kappa^2 \xi_\mu\,,\end{aligned}$$

where in the first step we noticed that equation (1.10) implies the antisymmetrization in the indices ν, ρ in the second factor; then we used (1.9) renaming the indices if necessary; finally we used (1.40) twice.
Dividing both sides by ξ_μ we finally get

$$\kappa^2 = -\frac{1}{2}\left(\nabla_\mu \xi_\nu\right)\left(\nabla^\mu \xi^\nu\right)\,, \tag{1.42}$$

where the expression on the right-hand side is to be evaluated at the horizon.
Note that the surface gravity associated with a Killing horizon is in principle arbitrary. In fact, if Σ is a Killing horizon of ξ with surface gravity κ, then it is also a Killing horizon of $c\xi$, for any real constant c: the corresponding surface gravity will be $|c|\kappa$. There is no natural normalization of ξ on Σ since $\xi^2 = 0$ there, but in an asymptotically flat spacetime there is a natural normalization at spatial infinity. For example for the time translation Killing vector field k we choose

$$k_\mu k^\mu = -1, \quad \text{for } r \to \infty \, . \tag{1.43}$$

This in turn fixes the surface gravity of any associated Killing horizon.

Why κ is called surface gravity? The reason is clear when we consider a static and asymptotically flat spacetime. In that case the surface gravity is the acceleration of a static observer near the horizon as measured by a static observer at infinity. Consider for example the Schwarzschild spacetime, whose metric is given by (1.31). A static observer has four-velocity u proportional to the time translation Killing vector field k

$$k^\mu = V(x^\alpha)\, u^\mu \, . \tag{1.44}$$

Since the four-velocity is normalized, that is $u_\mu u^\mu = -1$, the function V is

$$V = \sqrt{-k_\mu k^\mu} \, . \tag{1.45}$$

In this case

$$V = \sqrt{1 - \frac{2m}{r}} \quad \Rightarrow \quad u = \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \partial_t \, .$$

Note that $V = 1$ for a static observer at infinity. The four-acceleration a is given by

$$a = \nabla_u u \, . \tag{1.46}$$

In components

$$a^\mu = u^\nu \nabla_\nu u^\mu = u^\nu \Gamma^\mu{}_{\nu\sigma} u^\sigma = u^t u^t \Gamma^\mu{}_{tt} = \left(1 - \frac{2m}{r}\right)^{-1} \Gamma^\mu{}_{tt} \, .$$

The only non null component is $a^r = m/r^2$, because the Christoffel symbols for the metric (1.31) are null if $\mu \neq r$. The acceleration norm is

$$|a(r)| = \sqrt{g_{rr} a^r a^r} = \frac{m}{r^2} \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} \, .$$

Consider now a particle that is moving along static trajectories: $|a(r)|$ is the acceleration required to hold the particle at constant r. The acceleration diverges for $r = 2m$: thus it will take an infinite acceleration to keep an object on a static trajectory at the horizon. Now suppose that the particle is held in place by an observer at infinity through an infinitely long and massless string; this situation is different since the particle and it can be shown that the force exerted by the observer at infinity to hold a unit test mass in place, that we will call F_∞, differs from the force exerted locally by the redshift factor V. In formula: $F_\infty = VF$. Therefore the acceleration of the particle measured by the observer at infinity a_∞ is the force applied by this observer per unit mass: it turns out to be $a_\infty(r) = V(r)a(r)$. For $r = 2m$ we find the acceleration measured at infinity of a particle on a static trajectory at the horizon: this is exactly what we usually refer to as surface gravity. Then we define the surface gravity $\kappa = a_\infty(r = 2m)$ for the Schwarzschild spacetime. Thus we find

$$\kappa = \frac{1}{4m} \, . \tag{1.47}$$

This is the same result we would find starting from (1.42).
In the case of a stationary, but not static, spacetime the surface gravity loses this interpretation because is no longer possible to consider static observers very close to the horizon of the black hole. In fact, if we consider observers that do not move with respect to hypersurfaces of constant t, they rotate around the black hole. This effect is known as frame-dragging.

Chapter 2

Energy, angular momentum and charge

In this chapter we will present the Komar integral, a way to calculate the conserved charges in General Relativity starting from a symmetry of the spacetime. We then apply this method to the simplest black holes, in order to understand the parameters present in the metric of these solutions. Finally, we will see a decomposition of the Komar integral, in order to distinguish the various contributions of the spacetime to the conserved charges.

2.1 Komar Integral

From classical mechanics we know that there is a constant of motion for each symmetry of the system. In GR the concept of symmetry of a spacetime is related to the Killing vector fields; then we look for a way to define a conserved quantity starting from these symmetries. In particular we are interested in mass, charge and angular momentum of spacetime, since, for the no-hair theorem, they are the parameters that fully characterize stationary black hole solutions in asymptotically flat GR. One way to define these conserved charges is the Komar integral.

To understand the Komar integral, it is instructive to start from the definition of the electric charge: this conserved charge is not linked to a Killing vector field, but rather to the symmetry group $U(1)$ of Maxwell equations. Consider a generic spacetime (M, g), not necessarily a black hole: Maxwell's equations relate the electromagnetic field strength tensor $F_{\mu\nu}$ to the electric current four-vector J_e^μ

$$\nabla_\nu F^{\mu\nu} = 4\pi J_e^\mu \ , \tag{2.1}$$

$$\nabla_{[\nu} F_{\mu\lambda]} = 0 \ . \tag{2.2}$$

A direct consequence of Maxwell equations is that J^μ is a conserved current, that is

$$\nabla_\mu J_e^\mu = 0 \ . \tag{2.3}$$

Let Σ be a spacelike hypersurface on M, generally defined as the hypersurface at constant t. We define the total electric charge on Σ to be

$$Q = \int_\Sigma J_e^\mu \, d\Sigma_\mu = \frac{1}{4\pi} \int_\Sigma \nabla_\nu F^{\mu\nu} d\Sigma_\mu \ , \tag{2.4}$$

where in second last step we used (2.1). This definition is nothing more than the definition of charge in classical electromagnetism readjusted to curved spacetimes in covariant form.

Since $F^{\mu\nu}$ is an antisymmetric tensor, for the Stokes' theorem (1.39) we can express the charge as an integral over the boundary $\partial\Sigma$ of Σ

$$Q = \frac{1}{4\pi} \oint_{\partial\Sigma} F^{\mu\nu}\, d\Sigma_{\mu\nu} \; . \tag{2.5}$$

By the antisymmetry of $F^{\mu\nu}$

$$F^{\mu\nu} d\Sigma_{\mu\nu} = -\frac{1}{2} F^{\mu\nu} \left(n_\mu \sigma_\nu - n_\nu \sigma_\mu\right) \sqrt{|h^{(2)}|}\, d^2 w = -F^{\mu\nu} n_\mu \sigma_\nu \sqrt{|h^{(2)}|}\, d^2 w \; ,$$

so we can write the charge in the following way

$$Q = -\frac{1}{4\pi} \oint_{\partial\Sigma} n_\mu \sigma_\nu F^{\mu\nu} \sqrt{|h^{(2)}|}\, d^2 w \; . \tag{2.6}$$

Previously we said that J_e^μ is a conserved current as it satisfies equation (2.3). For the Gauss' theorem this divergenceless current implies that Q is a conserved charge. In fact, let R be a region of the spacetime manifold M defined as the region between two spatial hypersurfaces, Σ_1 and Σ_2, extending all the way to infinity; its boundary ∂R is a closed hypersurface composed of Σ_1, Σ_2 and a hypersurface at infinity connecting them. The latter hypersurface can be ignored, since the four-vector J_e^μ vanishes at infinity. Therefore, using (2.3) and the Gauss' theorem (1.36), we can write

$$\begin{aligned} 0 &= \int_R \nabla_\mu J^\mu \sqrt{|g|} d^4 x = \int_{\partial R} J^\mu \, d\Sigma_\mu = \int_{\Sigma_1} J^\mu \, d\hat{\Sigma}_\mu + \int_{\Sigma_2} J^\mu \, d\hat{\Sigma}_\mu = \\ &= \int_{\Sigma_2} J^\mu \, d\Sigma_\mu - \int_{\Sigma_1} J^\mu \, d\Sigma_\mu = Q\left(\Sigma_2\right) - Q\left(\Sigma_1\right) \; , \end{aligned}$$

where on each hypersurface $d\hat{S}_\mu = -n_\alpha \sqrt{|h|}\, d^3 y$, where n_α is the outward normal vector to the closed hypersurface. To get the conventional volume element dS_μ used in the charge definition (2.4), let $n_{1\alpha}$ and $n_{2\alpha}$ be the conventional unit normal vectors respectively to Σ_1 and Σ_2, as shown in figure 2.1: they are both future directed. It is clear that for Σ_2

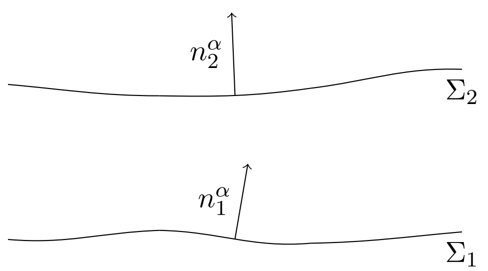

Figure 2.1: Normal vectors to spacelike hypersurfaces

the normal vector n_α inherited from R coincides with $n_{2\alpha}$, whereas for Σ_1 the inherited normal vector has the opposite direction of the conventional one: $n_\alpha = -n_{1\alpha}$. For this reason a minus sign appears in front of the conventional integral over Σ_1. Then we see that $Q\left(\Sigma_1\right) = Q\left(\Sigma_2\right)$: the charge is independent of the hypersurface Σ on which is evaluated.

Therefore it is necessary to find conserved currents J associated with a Killing vector field ξ in order to define a conserved charge. We will now show that such a current can be constructed from any Killing vector field. Consider the current

$$J^\mu = R^{\mu\nu} \xi_\nu = R^\mu{}_\nu \xi^\nu \; . \tag{2.7}$$

From Einstein's equations (1.1) it follows that

$$R^{\mu\nu} = 8\pi \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right) , \tag{2.8}$$

so we can write the current through the energy-momentum tensor as

$$J^{\mu} = 8\pi \xi_{\nu} \left(T^{\mu\nu} - \frac{1}{2} T g^{\mu\nu} \right) . \tag{2.9}$$

We now note that J can be written in term of the covariant derivative of an antisymmetric tensor thanks to the following lemma

Lemma 2.1.1. *For a Killing vector field* ξ

$$\nabla_{\nu} \nabla_{\mu} \xi^{\nu} = R_{\mu\nu} \xi^{\nu} . \tag{2.10}$$

Proof. By contracting (1.11)

$$\nabla_{\nu} \nabla_{\mu} \xi^{\nu} = R^{\nu}{}_{\mu\nu\sigma} \xi^{\sigma} = R_{\mu\sigma} \xi^{\sigma} = R_{\mu\nu} \xi^{\nu} .$$

■

Therefore (2.7) becomes

$$J^{\mu} = \nabla_{\nu} \nabla^{\mu} \xi^{\nu} . \tag{2.11}$$

Proposition 2.1.2. *For a Killing vector field* ξ *the current defined in* (2.7) *is a conserved current, that is:* $\nabla_{\mu} J^{\mu} = 0$.

Proof. It follows from Lemma 1.2.2

■

We can thus associate the following conserved charge to this current

$$\mathcal{Q}_{\xi} = -\frac{c_{\xi}}{8\pi} \int_{\Sigma} J^{\mu} \, d\Sigma_{\mu} = -\frac{c_{\xi}}{8\pi} \int_{\Sigma} \nabla_{\nu} \nabla^{\mu} \xi^{\nu} \, d\Sigma_{\mu} = \frac{c_{\xi}}{8\pi} \oint_{\partial\Sigma} \nabla^{\mu} \xi^{\nu} d\Sigma_{\nu\mu} . \tag{2.12}$$

where the minus sign has been introduced in order to have the same constant of [1] and [12] before the integral; c_{ξ} is a constant determined in the limit of "weak" gravity. The last member of the equation is the Komar integral associated with the Killing vector field ξ. We can now define the total energy and the total angular momentum of spacetime.

Energy is the conserved quantity associated with the invariance of a system by time translation; in General Relativity we can define the energy as the conserved charge associated with a timelike Killing vector field k

$$M = \frac{1}{4\pi} \oint_{\partial\Sigma} \nabla^{\mu} k^{\nu} \, d\Sigma_{\nu\mu} = \frac{1}{4\pi} \oint_{\partial\Sigma} \nabla^{\mu} k^{\nu} n_{\mu} \sigma_{\nu} \sqrt{|h^{(2)}|} \, d^2 w , \tag{2.13}$$

where we choose $c_k = 2$. To determine this constant we consider the weak field limit where $g_{\mu\nu} \sim \eta_{\mu\nu}$ and a distribution of dust with energy-momentum tensor $T_{\mu\nu} = \mathrm{diag}(\rho, 0, 0, 0)$. Starting from the first equality in (2.12) combined with (2.9), we obtain

$$E = -c_k \int_{\Sigma} \left(T^{\mu}{}_{\nu} - \frac{1}{2} \delta^{\mu}{}_{\nu} T \right) k^{\nu} \, d\Sigma_{\mu} .$$

Choose now Σ to be the spacelike hypersurface of constant t; in a coordinate system $\{t, x^1, x^2, x^3\}$ where x^i are coordinates on Σ, from (1.29) it is easy to find that $dS_{\mu} =$

$\delta^t_\mu \sqrt{|g|}\, d^3x$, where $\sqrt{|g|} \sim \sqrt{|\eta|} = 1$. The timelike Killing vector field has components $k^\nu = \delta^\nu_t$, so

$$E = -c_k \int_\Sigma \left(T^t{}_\nu - \frac{1}{2}\delta^t_\nu T \right) \delta^\nu{}_t \, d^3x = -c_k \int_\Sigma \left(-T_{tt} + \frac{1}{2} T \right) d^3x = \frac{c_k}{2} \int_\Sigma \rho \, d^3x \ ,$$

where in the second step we lowered the index t of the energy-momentum tensor through the Minkowski metric. Comparing this result with the energy of the energy density ρ in the Minkowski spacetime we find $c_k = 2$.

Angular momentum is the conserved charge associated with a rotational Killing vector field $m = \partial_\varphi$

$$J = -\frac{1}{8\pi} \oint_{\partial\Sigma} \nabla^\mu m^\nu \, d\Sigma_{\nu\mu} = -\frac{1}{8\pi} \oint_{\partial\Sigma} \nabla^\mu m^\nu n_\mu \sigma_\nu \sqrt{|h^{(2)}|}\, d^2w \ , \tag{2.14}$$

where we choose $c_m = -1$. To determine this constant consider again the weak field limit and let Σ be the same hypersurface as before, then

$$J = -c_m \int_\Sigma \left(T^\mu{}_\nu - \frac{1}{2} \delta^\mu{}_\nu T \right) m^\nu \, d\Sigma_\mu \ .$$

In the coordinate system $\{t, x^1, x^2, x^3\}$, where now x^i are the Cartesian coordinates, the rotational Killing vector field is $m = -x^2 \partial_{x_1} + x^1 \partial_{x_2}$, we obtain

$$J = -c_m \int_\Sigma x^1 T^t{}_2 - x^2 T^t{}_1 \, d^3x = -c_m \, \varepsilon_{3ij} \int_\Sigma x^i T^{jt} \, d^3x \ ,$$

which for $c_m = -1$ this is the result for the third component of angular momentum in Minkowski spacetime with energy-momentum tensor $T_{\mu\nu}$.

For a black hole solution, in order to consider the contribution of the horizon to the conserved charges, the latter are defined by an integral over a two-sphere at spatial infinity

$$M = \frac{1}{4\pi} \oint_{\Sigma_\infty} \nabla^\mu k^\nu d\Sigma_{\nu\mu} \ , \tag{2.15}$$

$$J = -\frac{1}{8\pi} \oint_{\Sigma_\infty} \nabla^\mu m^\nu d\Sigma_{\nu\mu} \ , \tag{2.16}$$

$$Q = \frac{1}{4\pi} \oint_{\Sigma_\infty} F^{\mu\nu} d\Sigma_{\mu\nu} \ . \tag{2.17}$$

Finally, note that a Killing vector is defined up to multiplication by a constant: if ξ satisfies the Killing equation, also $\zeta = c^* \xi$ satisfies that equation. This would make the energy and the angular momentum of spacetime change according to the value of c^*. Actually the constant in front of the Komar integral is determined for asymptotically flat spacetimes by imposing that $\xi^2 = -1$ asymptotically if ξ is timelike, while $\xi^2 = 1$ asymptotically if ξ is spacelike (already for asymptotically (A)dS spacetimes the normalization is not trivial). If ξ is not correctly normalized, then the constant in front of the Komar integral will change accordingly.

2.2 Energy of Schwarzschild black hole

The Schwarzschild metric is given by (1.31). We already calculated the unit normal vector n to the spacelike hypersurface Σ of constant t: the result is in equation (1.33). Given the symmetry of the solution, we choose the boundary of $\partial\Sigma$ as the surface of constant r: in particular it is a two-sphere at spatial infinity. The unit normal vector to $\partial\Sigma$, normalized as in (1.17), and orthogonal to n is

$$\sigma = \left(1 - \frac{2m}{r}\right)^{1/2} \partial_r \ .$$

Starting from (2.15) and rewriting the surface element using (1.38) and (1.35), we find the term

$$\begin{aligned} n_\mu \sigma_\nu \nabla^\mu m^\nu &= n^\mu \sigma^\nu g_{\nu\rho} \nabla_\mu m^\rho = n^t \sigma^r g_{rr} \nabla_t k^r = n^t \sigma^r g_{rr} {\Gamma^r}_{t\nu} k^\nu = n^t \sigma^r g_{rr} {\Gamma^r}_{tt} = \\ &= -\frac{1}{2} n^t \sigma^r g_{rr} g^{rr} \partial_r g_{tt} = \frac{1}{2}\frac{\partial}{\partial r}\left(1 - \frac{2m}{r}\right) = \frac{m}{r^2} \ . \end{aligned}$$

From the induced metric on the two-sphere at spatial infinity follows that $\sqrt{|h^{(2)}|} = r^2 \sin\theta$; therefore the energy of the Schwarzschild black hole is

$$M = \lim_{r\to\infty} \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta\, r^2 \sin\theta\, \frac{m}{r^2} = \frac{m}{2} \int_0^\pi d\theta \sin\theta = m \ .$$

Note that this integral is equal to m for each two-sphere of radius $r > 2m$: all the energy is enclosed within the event horizon and the parameter m is interpreted as the mass of the black hole.

2.3 Energy and charge of Reissner-Nordström black hole

The Reissner-Nordström black hole is an exact solution of Einstein-Maxwell equations representing a non-rotating and electrically charged black hole. The metric is given by (1.4) setting $a = p = 0$; thus we find

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \ , \qquad (2.18)$$

with the four-vector electromagnetic potential

$$A = \left[\frac{q}{r}, 0, 0, 0\right] \ . \qquad (2.19)$$

Let Σ be the usual spacelike hypersurface of constant t and $\partial\Sigma$ its boundary, that is a surface of constant r. In the same way of Schwarzschild black hole, the unit normal vectors to $\partial\Sigma$ are

$$n = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1/2} \partial_t \ , \qquad \sigma = \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{1/2} \partial_r \ .$$

Starting from (2.6), consider the following term

$$n_\mu \sigma_\nu F^{\mu\nu} = n^t \sigma^r F_{tr} = -n^t \sigma^r \partial_r A_t = \frac{q}{r^2} \ .$$

Therefore we find that the charge is

$$Q = -\frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta\, r^2 \sin\theta\, \frac{q}{r^2} = -\frac{q}{2} \int_0^\pi d\theta\, \sin\theta = -q \ .$$

The minus sign in front of q is due to the fact that we considered the vector potential of a negative charge of modulus q. Anyway, note that the metric remains unchanged by the transformation $q \to -q$ since it depends on q^2: there will be no difference in considering a

charge q or $-q$.
In order to find the energy, same steps to those seen in the previous section are made

$$n_\mu \sigma_\nu \nabla^\mu m^\nu = -\frac{1}{2} n^t \sigma^r g_{rr} g^{rr} \partial_r g_{tt} = \frac{1}{r^2}\left(m - \frac{q^2}{r}\right) ,$$

$$M = \lim_{r\to\infty} \frac{1}{8\pi} \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, r^2 \sin\theta \left(\frac{2m}{r^2} - \frac{2q^2}{r^3}\right) = \lim_{r\to\infty}\left(m - \frac{q^2}{r}\right) = m .$$

Note that the Komar integral is not equal for each sphere outside the horizon because of the term q/r. As we will clarify in section 2.5 the parameter m is the total energy of the spacetime, but it does not coincide with the mass of the black hole.

2.4 Energy and angular momentum of Kerr black hole

The Kerr solution describes a rotating, non-charged, black hole. In Boyer-Lindquist coordinates the metric is given by (1.4) setting $q = p = 0$. We do not rewrite the metric as it would remain identical to (1.4), except for the metric function Δ, that now is

$$\Delta = r^2 - 2mr + a^2 .$$

In the same coordinates the metric can be written in the so-called canonical form

$$ds^2 = -N^2 dt^2 + \gamma \left(d\varphi - \Omega dt\right)^2 + \frac{\Xi}{\Delta} dr^2 + \Xi d\theta^2 , \tag{2.20}$$

where

$$N^2 = \frac{\Delta \Xi}{R^2} , \qquad R^2 = \left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta ,$$
$$\Omega = \frac{2mar}{R^2} , \qquad \gamma = \frac{R^2 \sin^2\theta}{\Xi} .$$

Let's now turn our attention to the calculation of energy. The Kerr metric is not diagonal because of the presence of the term $g_{t\varphi}$, so the unit normal vector n to the spacelike hypersurface Σ of constant t will generally be a linear combination of the two basis vectors ∂_t and ∂_φ: $n = \alpha \partial_t + \beta \partial_\varphi$. By imposing the orthogonality with the basis tangent vectors $\{\partial_r, \partial_\theta, \partial_\varphi\}$ to Σ, we find the relation $\beta = \Omega\, \alpha$. Finally, asking for $n^2 = -1$ since it is timelike, the unit normal vector takes the form

$$n = N^{-1} \left(\partial_t + \Omega \partial_\varphi\right) .$$

The boundary $\partial\Sigma$ of Σ is a surface of constant r; since the induced metric on Σ is diagonal, the unit normal vector to $\partial\Sigma$ and tangent to Σ is

$$\sigma = \sqrt{\frac{\Xi}{\Delta}}\, \partial_r .$$

Starting from (2.15), noting that $\sqrt{|h^{(2)}|} = R \sin\theta$ and that

$$n_\mu \sigma_\nu \nabla^\mu m^\nu = n^\mu \sigma^\nu g_{\nu\rho} \nabla_\mu m^\rho = n^t \sigma^r g_{rr} \nabla_t m^r + n^\varphi \sigma^r g_{rr} \nabla_\varphi m^r ,$$

the energy is given by

$$M = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi d\theta \, \sin\theta \, \left[R \left(n^t \sigma^r g_{rr} \nabla_t k^r + n^\varphi \sigma^r g_{rr} \nabla_\varphi k^r\right)\right] , \tag{2.21}$$

where

$$\nabla_t k^r = \Gamma^r{}_{tt} = -\frac{1}{2} g^{rr} g_{tt,r} = \frac{m\Delta\left(r^2 - a^2 - \cos^2\theta\right)}{\Xi^3} ,$$

$$\nabla_\varphi k^r = \Gamma^r{}_{t\varphi} = -\frac{1}{2} g^{rr} g_{t\varphi,r} = \frac{am\Delta}{\Xi^3} \sin^2\theta \left(a^2\cos^2\theta - r^2\right) .$$

We have not written the limit $r \to \infty$ in (2.21) because the result would be the same on each sphere outside the horizon. However we can solve the integral only in the limit $r \to \infty$; in this case we can develop in series of $1/r$ the term in square brackets in (2.21). By the following asymptotic behaviors

$$\Delta \sim r^2 , \qquad \Xi \sim r^2 , \qquad N \sim 1 , \qquad \Omega \sim \frac{2am}{r^3} , \qquad R \sim r^2 ,$$

$$\Gamma^r{}_{tt} \sim \frac{m}{r^2} , \qquad \Gamma^r{}_{t\varphi} \sim -\frac{am\sin^2\theta}{r^2} .$$

the term in square brackets takes the form

$$m + \mathcal{O}\left(\frac{1}{r}\right) ,$$

so the energy is

$$E = \frac{m}{4\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta = m .$$

As in Schwarzschild, the parameter m coincides with the mass of the black hole.
Similarly for the angular momentum, starting from (2.16), we can write

$$J = -\frac{1}{8\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \left[R\left(n^t\sigma^r g_{rr}\nabla_t m^r + n^\varphi\sigma^r g_{rr}\nabla_\varphi m^r\right)\right] , \qquad (2.22)$$

where

$$\nabla_t m^r = \Gamma^r{}_{t\varphi} ,$$

$$\nabla_\varphi m^r = \Gamma^r{}_{\varphi\varphi} = -\frac{1}{2} g^{rr} g_{\varphi\varphi,r} = -\frac{\Delta\sin^2\theta}{\Xi}\left[r - \frac{ma^2\sin^2\theta}{\Xi^2}\left(r^2 - a^2\cos^2\theta\right)\right] .$$

Using the above asymptotic behaviors together with the following

$$\Gamma^r{}_{\varphi\varphi} \sim -r\sin^2\theta ,$$

the term in square brackets in (2.22) takes the form

$$-3ma\sin^2\theta + \mathcal{O}\left(\frac{1}{r}\right) ,$$

so the angular momentum is

$$J = \frac{3ma}{8\pi} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin^3\theta = ma .$$

Therefore the parameter a in (1.4) is the ratio between angular momentum and energy of the black hole.

2.5 Energy and angular momentum decomposition

Mass and angular momentum are defined respectively by (2.15) and (2.16) in a coordinate-independent manner. For a black hole solution, the hypersurface Σ of constant t as two boundaries: a two-sphere at spatial infinity and the black hole horizon. Therefore, by Stokes' theorem (1.39), for a timelike Killing vector field k

$$-\frac{1}{4\pi}\int_\Sigma R^\mu{}_\nu k^\nu \, d\Sigma_\mu = \frac{1}{4\pi}\oint_{\Sigma_\infty} \nabla^\mu k^\nu \, d\Sigma_{\nu\mu} - \frac{1}{4\pi}\oint_H \nabla^\mu k^\nu \, d\Sigma_{\nu\mu} \,. \tag{2.23}$$

Then we can write the Komar energy as the sum of two contributions

$$M = M_{GS} + \widehat{M}_H \,, \tag{2.24}$$

where

$$M_{GS} = -\frac{1}{4\pi}\int_\Sigma R^\mu{}_\nu k^\nu \, d\Sigma_\mu = -2\int_\Sigma \left(T^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T\right) k^\nu \, d\Sigma_\mu \tag{2.25}$$

is the energy contribution of spacetime outside the black hole due to external sources, while

$$\widehat{M}_H = \frac{1}{4\pi}\oint_H \nabla^\mu k^\nu \, d\Sigma_{\nu\mu} \tag{2.26}$$

is the horizon contribution. The energy-momentum tensor can be decomposed into a material part, indicated by a suffix M and an electromagnetic part, indicate by a suffix E, that is

$$T^\mu{}_\nu = T_M{}^\mu{}_\nu + T_E{}^\mu{}_\nu \,,$$

where the electromagnetic part is given by (1.2). Consequently M_{GS} is divided into two terms: an external contribution of material sources M_M

$$M_M = -2\int_\Sigma \left(T_M{}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T_M\right) k^\nu \, d\Sigma_\mu \tag{2.27}$$

and an external contribution M_E due to an electromagnetic field

$$M_E = -2\int_\Sigma T_E{}^\mu{}_\nu k^\nu \, d\Sigma_\mu \,, \tag{2.28}$$

where we used that the electromagnetic energy-momentum tensor has null trace. Similarly for the angular momentum

$$J = J_{GS} + \widehat{J}_H \,, \tag{2.29}$$

where

$$\begin{aligned} J_{GS} &= \frac{1}{8\pi}\int_\Sigma R^\mu{}_\nu m^\nu \, d\Sigma_\mu = \int_\Sigma \left(T^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T\right) m^\nu \, d\Sigma_\mu = \\ &= \underbrace{\int_\Sigma \left(T_M{}^\mu{}_\nu - \frac{1}{2}\delta^\mu{}_\nu T_M\right) m^\nu \, d\Sigma_\mu}_{J_M} + \underbrace{\int_\Sigma T_E{}^\mu{}_\nu m^\nu \, d\Sigma_\mu}_{J_E} \,, \end{aligned} \tag{2.30}$$

and

$$\widehat{J}_H = -\frac{1}{8\pi}\oint_H \nabla^\mu m^\nu \, d\Sigma_{\nu\mu} \,. \tag{2.31}$$

We will deal with solutions of the vacuum Einstein-Maxwell equations, where $T_M{}^{\mu\nu} = 0$; so the external material contributions M_M and J_M vanish and the Komar integral's decomposition becomes

$$M = \widehat{M}_H + M_E \,, \tag{2.32}$$

$$J = \widehat{J}_H + J_E \ . \tag{2.33}$$

We now use this decomposition to calculate the energy contribution of the black hole and that of the external spacetime for the Reissner-Nordström solution. The horizon contribution $\widehat{M}_H$ is identical to the Komar integral, except that the integration surface is not at spatial infinity, but at $r = r_H$. Therefore, performing the same steps as those in section 2.3, we find

$$\oint_H \nabla^\mu k^\nu d\Sigma_{\nu\mu} = \frac{1}{8\pi}\left(2m - \frac{2q^2}{r_H}\right)\int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\varphi = m - \frac{q^2}{r_H} \ .$$

Let's now calculate the electromagnetic contribution M_E: since Σ is the spacelike hypersurface of constant t, the index μ of the volume element in (2.28) must be equal to t (this is clear from formula (1.29)). Moreover, the timelike Killing vector fields has components $k^\nu = \delta^\nu_t$, so the only component of the energy-momentum tensor that contributes in the integral is

$$T^t_{\ t} = \frac{1}{4\pi}\left[F^{tr}F_{tr} - \frac{1}{4}\left(F^{tr}F_{tr} + F^{rt}F_{rt}\right)\right] = \frac{1}{8\pi}F^{tr}F_{tr} = \frac{1}{8\pi}g^{tt}g^{rr}\left(\partial_r A_t\right)^2 \ .$$

Consequently the electromagnetic contribution becomes

$$M_E = \frac{q^2}{4\pi}\int_0^\pi d\theta\, \sin\theta \int_0^{2\pi} d\varphi \int_{r_H}^\infty \frac{dr}{r^2} = \lim_{r\to\infty} -\frac{q^2}{r}\bigg|_{r_H}^{x} = \frac{q^2}{r_H} \ .$$

The two contributions add up to $M = m$.
In conclusion the total energy of the Reissner-Nordström black hole is the sum of two contributions: the energy of the black hole and the energy due to the presence of a non-null electromagnetic energy-momentum tensor. So there is a contribution of the outer space outside the black hole. For this reason the Komar integral is not the same on every two-sphere outside the horizon, as happens in the Schwarzschild case, and the parameter m is not the mass of the black hole.

2.6 Dual Komar integrals: magnetic and NUT charges

Komar charges can be written in a more compact form using forms. Let's consider the energy as an example: we can define a 1-form $\hat{k}$ associated to the timelike Killing vector field k of components $\hat{k}_\mu = g_{\mu\nu}k^\nu$. We can then construct a 2-form by taking its exterior derivative. The Komar mass (2.15) is then given by the surface integral over the Hodge dual of $d\hat{k}$

$$M = -\frac{1}{8\pi}\oint_{\Sigma_\infty} *d\hat{k} \ . \tag{2.34}$$

In fact, since the 2-form $d\hat{k}$ has components $(d\hat{k})_{\mu\nu} = 2\nabla_\mu k_\nu$, its Hodge dual reduces to expression (2.15) when restricted to the surface.
Similarly for the electric charge we can write

$$Q = \frac{1}{4\pi}\oint_{\Sigma_\infty} *F \ , \tag{2.35}$$

where F is the 2-form constructed from the Faraday tensor.
We can now define the magnetic charge through the dual of the electric charge

$$P = \frac{1}{4\pi}\oint_{\Sigma_\infty} F = \frac{1}{8\pi}\oint_{\Sigma_\infty} F_{\mu\nu}\, dx^\mu \wedge dx^\nu = -\frac{1}{16\pi}\oint_{\Sigma_\infty} \epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}dS_{\mu\nu} \ . \tag{2.36}$$

These three expressions are equivalent to each other. In particular the second one shows that the magnetic charge can be calculated without introducing the metric, which gives local information about spacetime: the magnetic charge describes actually a topological characteristics of the solution, in fact it is associated with Dirac strings[1]. Quite the opposite, the computation of the electric charge depends on the metric, in fact it is not a topological characteristic of the solution.
In section 5.1 we will define the NUT charge: such a charge is interpreted as a sort of magnetic mass. Therefore, similarly to the magnetic charge, we can define the NUT charge as the dual of the Komar mass

$$\tilde{M} = -\frac{1}{8\pi}\oint_{\Sigma_\infty} d\hat{k} = -\frac{1}{8\pi}\oint_{\Sigma_\infty} \nabla_\mu k_\nu \, dx^\mu \wedge dx^\nu = \frac{1}{16\pi}\oint_{\Sigma_\infty} \epsilon^{\mu\nu\alpha\beta}\nabla_\alpha k_\beta dS_{\mu\nu} \, . \tag{2.37}$$

Note that these charges are also conserved; however they are not linked to any symmetry. We computed the values of P and $\tilde{M}$ for the Reissner-Nordström-NUT spacetime, whose metric can be found from (1.4) by setting $a = 0$ and adding the NUT charge, which will be defined in section 5.1. The computation of (2.36) returns the parameter p up to a sign, which depends on the choice of the vector potential but does not affect the metric. The computation of (2.37) returns the NUT parameter n.
The dual Komar integrals P and $\tilde{M}$ are standard formulas employed for the computation of the magnetic charge and the NUT charge respectively. In section 6.2 we will present two alternative approaches for the computation of the Smarr formula which make use of these definitions.

[1] we will talk about Dirac strings in section 5.1

Chapter 3

Rod Structure

We already said that the coupled vacuum Maxwell-Einstein equations simplify considerably for stationary and axisymmetric solutions. We now focus our attention on this particular class of solutions, analyzing their general structure. This chapter, together with the next, will be very useful in finding an elegant way to calculate the conserved charges in presence of line singularities [1].

3.1 Stationary and axisymmetric solutions

Consider a four-dimensional manifold M with two commuting and linearly independent Killing vector fields $\xi_{(i)}$, $i = 1, 2$, where one of the two Killing vector fields is timelike for the stationary condition, while the other one is spacelike and it is related to the axisymmetry of the spacetime. The following results can be extended to the case of a d-dimensional manifold with $d-2$ commuting and linearly independent Killing vector fields, where $d-3$ spacelike Killing vector fields gives the so-called axisymmetry of the spacetime.
We can find coordinates x^i, $i = 1, 2$ such that

$$\xi_{(i)} = \frac{\partial}{\partial x^i} \ .$$

Under certain conditions that we assume to be verified [2], the two planes orthogonal to the two Killing vector fields are integrable: this means that for any given point of the spacetime there is a two-dimensional submanifold that includes this point and have the property that its two-dimensional tangent space is orthogonal to all of the Killing vector fields. By choosing coordinates on one of these two-dimensional submanifolds and dragging them along the integral curves of the two Killing vector fields, we obtain two coordinates y^1 and y^2 such that the vector $\frac{\partial}{\partial y^j}$ are orthogonal everywhere to all the Killing vector fields for all $j = 1, 2$. Basically the metric can be divided into two blocks.
Through changes of the coordinates y^1, y^2, the metric can be written in the form

$$ds^2 = G_{ab} dx^a dx^b + e^{2\gamma} \left(d\rho^2 + dz^2 \right) \ , \tag{3.1}$$

where the coordinate ρ is related to the matrix G of components G_{ab} via

$$\rho = \sqrt{|\mathrm{det} G|} \ . \tag{3.2}$$

The latter relations is a consequence of the vacuum Maxwell-Einstein equations. Clearly all the metric functions γ and g_{ab}, with $a, b = 1, 2$, are functions of (ρ, z).

[1] the references for this chapter are [7] and [8].
[2] for more detail see Theorem 2.1 in [7].

Since the coordinate ρ satisfies (3.5), the G matrix is always invertible when $\rho \neq 0$, while for $\rho = 0$ we see that $\det G(0,z) = 0$. Therefore the eigenvalues of $G(0,z)$ include the eigenvalue zero for a given z and

$$\dim\left[\ker(G(0,z))\right] \geq 1$$

In order to avoid curvature singularities on the axis a necessary condition is that

$$\dim\left[\ker(G(0,z))\right] = 1$$

except for isolated values of z. In fact having two zero eigenvalues in a an interval $[z_1, z_2]$ leads to the divergence of the curvature invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ for a given z in that interval: therefore we get a curvature singularity.
Naming the isolated z-values $z_1, ..., z_N$, the z-axis splits up into $N+1$ intervals $[-\infty, z_1]$, $[z_1, z_2]$, ..., $[z_{N-1}, z_N]$, $[z_N, \infty]$ called rods.
Consider now a specific rod $[z_n, z_{n+1}]$: the vector $v_n \in \mathbb{R}^2$ such that

$$G_{ab}(0,z)v_n^a = 0 \qquad \text{for } z \in [z_n, z_{n+1}] \tag{3.3}$$

is called direction of the rod $[z_n, z_{n+1}]$.
The rod structure of a solution is the specification of the rod intervals $[z_n, z_{n+1}]$ with the corresponding direction v_n.
Note that the direction of a rod is a linear combination of the two Killing vector fields ξ_i and therefore it defines a Killing vector field of the spacetime. In particular the Killing vector field v_n vanishes along the associated rod: the rod will be a Killing horizon for the corresponding direction.
In order to characterize the rods, consider the limit where ρ goes to zero. Since we can always make a constant coordinate transformation of the coordinates x^i such that $G_{1i} = 0$ for $i = 1, 2$ and $z \in [z_n, z_{n+1}]$ [3], to leading order, we can write

$$\mathrm{G} = \begin{bmatrix} \pm a(z)\rho^2 & 0 \\ 0 & \pm\frac{1}{a(z)} \end{bmatrix} \quad \text{for } \rho \to 0$$

where the signs in front of the two matrix elements are not correlated, while the exponent of ρ is determined by the condition (3.5). In this new coordinates the direction of the rod $[z_n, z_{n+1}]$ becomes the two-dimensional vector $v_n = (1, 0)$, and its norm goes to zero as ρ^2, in fact

$$v_n^2 = G_{ab}v_n^a v_n^b \sim \pm a(z)\rho^2 \quad \text{for } \rho \to 0$$

Therefore, since from Einstein equations it follows that $e^{2\gamma} \sim c^2 a(z)$ for $\rho \to 0$ where c is a positive number, the quantity $\rho^{-2}e^{-2\gamma}v_n^2$ has a finite limit on the polar axis and will be constant along the corresponding rod. If this quantity is negative, positive or zero for $\rho \to 0$ we say the rod is respectively timelike, spacelike or null: we will consider only timelike and spacelike rods. Finite timelike rods correspond to horizons, while the semi-infinite ones correspond to acceleration horizons; finite spacelike rods potentially correspond to conical singularities.
For the horizon rod we can calculate the associated surface gravity, since a rod is a Killing horizon. In particular, starting from (1.42) and using the constrain (3.2), we find

$$\kappa_H = \lim_{\rho \to 0}\left(-\rho^{-2}e^{-2\gamma}G_{ab}v_H^a v_H^b\right)^{1/2} \tag{3.4}$$

where v_H is the horizon rod direction.
We can define a surface gravity for each rod through a similar formula, since they are Killing horizons; however, surface gravity assumes a physically relevant role only for the black hole horizon.

[3] for more details see theorem 3.1 in [7]

3.2 Rod structure of the Kerr black hole

Let us now give an example of how to determine the rod structure of a solution starting from a metric, not necessarily in Weyl coordinates. Consider the Kerr metric that in Boyer-Lindquist coordinates is given by (1.4) setting $q = p = 0$. To find the matrix G it is not necessary to be in Weyl coordinates: it is simply a 2×2 matrix of components $G_{11} = g_{tt}$, $G_{12} = G_{21} = g_{t\varphi}$ and $G_{22} = g_{\varphi\varphi}$. From $\det G = -\Delta \sin^2\theta$ we find the coordinate

$$\rho = \sqrt{\Delta}\,\sin\theta \tag{3.5}$$

while the z coordinate, which can be determined in such a way that the metric fits into the ansatz (3.1), is given by

$$z = (r - m)\cos\theta \tag{3.6}$$

Using this, we can in principle write the Kerr metric in the canonical form (3.1); however, it is useful to write the Kerr metric in the prolate spherical coordinates by using (A.5). In this coordinates the components of the matrix G becomes

$$\begin{aligned}
G_{11} &= -\frac{(\sigma x + m)^2 - 2m\,(\sigma x + m) + a^2 y^2}{(\sigma x + m)^2 + a^2 y^2} \\
G_{12} &= -\frac{2am\left(1 - y^2\right)(\sigma x + m)}{(\sigma x + m)^2 + a^2 y^2} = G_{21} \\
G_{22} &= \frac{G_{12}^2 - \rho^2}{G_{11}}
\end{aligned}$$

To find the rod structure of such a metric, we search the isolated z-values for which the kernel of G has dimension greater than one are easily find. In fact, since G is a 2 matrix, we must have $\dim(\ker(G(0,z))) = 2$ for this points: G must therefore be the null matrix. This happens only when $x = 1$ and $y = 1$ or $x = 1$ and $y = -1$, that from (A.5) corresponds to $z = \pm\sigma$ and $\rho = 0$.

Consequently there are two semi-infinite rods $[-\infty, -\sigma]$ and $[\sigma, \infty]$ and one finite rod

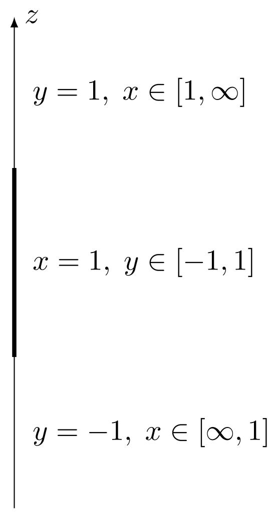

Figure 3.1: Rod structure for the Kerr metric

$[-\sigma, \sigma]$. For $z \in [-\infty, -\sigma]$ and $\rho = 0$, from (A.6) we see that $y = -1$ and $x = -z/\sigma$, while for $z \in [\sigma, \infty]$ and $\rho = 0$ we have that $y = 1$ and $x = z/\sigma$. For $z \in [-\sigma, \sigma]$ and $\rho = 0$ we have that $x = 1$ and $y = z/\sigma$. The z-axis is then divided into three zones in (x, y) coordinates, as shown in figure 3.1.

For both the intervals of the semi-infinite rods we see that $G_{12} = G_{22} = 0$ while $G_{11} \neq 0$: this means that the two rods have direction $v = (0, 1)$, so they are in the ∂_φ direction and therefore spacelike.
For the finite rod: we search for the direction of components $v = (v^1, v^2)$ such that $\sum_{j=1,2} G_{ij} v^j = 0$, that gives two equivalent equations for $i = 1, 2$. Without lost of generality we can set $v^1 = 1$: it follows that $v^2 = -G_{11}/G_{12}$ evaluated in $x = 1$. It is easy to find from the above expressions that

$$v = (1, \Omega_H) \tag{3.7}$$

where

$$\Omega_H = \frac{a}{2m(\sigma + m)} = \frac{a}{r_H^2 + a^2}, \quad r_H = \sigma + m$$

is the angular velocity of the event horizon. Note that the direction v precisely is the null Killing vector for the event horizon: in fact it is null along that rod. In other words, for a Killing horizon the null Killing vector is the same as the direction of the timelike rod, as we discussed before.

Chapter 4

Ernst Potentials

In this chapter [1] we will define the Ernst potentials of a stationary and axisymmetric solution to the coupled Einstein-Maxwell equations in vacuum. They are two complex potentials for which the Einstein-Maxwell equations assume a very compact form in terms of three dimensional differential operators.
As mentioned in the previous chapter, a stationary and axisymmetric spacetime has two commuting Killing vectors $k = \partial_t$ and $m = \partial_\varphi$ associated with time translations and rotations around the symmetry axis. The most generic solution for such a spacetime in Weyl coordinates can be written, in the Lewis-Weyl-Papapetrou form, as

$$ds^2 = -f\left(dt - \omega d\varphi\right)^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right] \ , \tag{4.1}$$

where the three metric functions f, ω and γ depends only on the non-Killing coordinates (ρ, z).
We consider the following electromagnetic potential compatible with the spacetime symmetries

$$A = A_t dt + A_\varphi d\varphi \ , \tag{4.2}$$

where the components A_t and A_φ depends only on the coordinates (ρ, z).
In term of the metric (4.1) the gravitational field equations (1.1) becomes

$$\vec{\nabla} \cdot \left[\rho^{-2} f^2 \vec{\nabla}\omega + 4\rho^{-2} f A_t \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t\right)\right] = 0 \ , \tag{4.3}$$

$$f \nabla^2 f = \left(\vec{\nabla} f\right)^2 - \rho^{-2} f^4 \left(\vec{\nabla}\omega\right)^2 + 2f\left[\left(\vec{\nabla} A_t\right)^2 + \rho^{-2} f^2 \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t\right)^2\right] \ , \tag{4.4}$$

while the Maxwell field equations (1.3) becomes

$$\vec{\nabla} \cdot \left[\rho^{-2} f \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t\right)\right] = 0 \ , \tag{4.5}$$

$$\vec{\nabla} \cdot \left[f^{-1} \vec{\nabla} A_t - \rho^{-2} f \omega \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t\right)\right] = 0 \ , \tag{4.6}$$

where the differential operators $\vec{\nabla}$ and ∇^2 are respectively the standard flat gradient and Laplacian in cylindrical coordinates (ρ, z, φ), given by (A.2) and (A.3).
Equation (4.5) suggests the definition of a magnetic scalar potential $\tilde{A_\varphi}$ as

$$\rho^{-1} f \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t\right) = \hat{\varphi} \times \vec{\nabla} \tilde{A}_\varphi \ , \tag{4.7}$$

[1] the references for this chapter are [5].

such that the equation will be automatically satisfied for the integrability condition: in fact equation (4.5) takes the form

$$\vec{\nabla} \cdot \left(\rho^{-1} \hat{\varphi} \times \vec{\nabla} \tilde{A_\varphi} \right) = 0 \ ,$$

which using (A.4) together with the fact that the functions do not depend on the coordinate φ is automatically verified because derivatives along different directions commute.
From the following vector calculus identity

$$\hat{\varphi} \times (\hat{\varphi} \times \vec{\nabla} \tilde{A_\varphi}) = \hat{\varphi} \, (\hat{\varphi} \cdot \vec{\nabla} \tilde{A_\varphi}) - \vec{\nabla} \tilde{A_\varphi} \, (\hat{\varphi} \cdot \hat{\varphi}) = -\vec{\nabla} \tilde{A_\varphi} \ ,$$

equation (4.7) can be written as

$$\vec{\nabla} \tilde{A_\varphi} = -\rho^{-1} f \, \hat{\varphi} \times \left(\vec{\nabla} A_\varphi + \omega \vec{\nabla} A_t \right) \ , \tag{4.8}$$

which is the standard definition for the scalar potential $\tilde{A_\varphi}$.
Now it is advantageous to define the electromagnetic complex Ernst potential

$$\mathbf{\Phi} = A_t + i \tilde{A_\varphi} \ . \tag{4.9}$$

Noting that $\vec{\nabla} \cdot (\rho^{-1} \, \hat{\varphi} \times A_t \vec{\nabla} \tilde{A_\varphi}) = -\vec{\nabla} \cdot (\rho^{-1} \, \hat{\varphi} \times \tilde{A_\varphi} \vec{\nabla} A_t)$, equation (4.3) can be written in the form

$$\vec{\nabla} \cdot \left(\rho^{-2} + f^2 \vec{\nabla} \omega + 2\rho^{-1} \mathrm{Im} \left(\bar{\mathbf{\Phi}} \vec{\nabla} \mathbf{\Phi} \right) \right) = 0 \ , \tag{4.10}$$

which suggest the definition of a new potential χ as

$$-\, \rho^{-1} f^2 \vec{\nabla} \omega - 2 \, \hat{\varphi} \times \mathrm{Im} \left(\bar{\mathbf{\Phi}} \vec{\nabla} \mathbf{\Phi} \right) = \hat{\varphi} \times \vec{\nabla} \tilde{A_\varphi} \ , \tag{4.11}$$

such that equation (5.15) is automatically satisfied for the integrability condition.
Again, from the above vector calculus identity follows that

$$\vec{\nabla} \chi = \rho^{-1} f^2 \, \hat{\varphi} \times \vec{\nabla} \omega - 2 \mathrm{Im} \left(\bar{\mathbf{\Phi}} \vec{\nabla} \mathbf{\Phi} \right) \ , \tag{4.12}$$

which is the standard definition for the function χ.
If we define the gravitational complex Ernst potential as

$$\mathcal{E} = f - \bar{\mathbf{\Phi}} \mathbf{\Phi} + i \chi \ , \tag{4.13}$$

therefore it can be shown that equations (4.4) and (4.6) becomes respectively

$$\left(\mathrm{Re} \mathcal{E} + |\mathbf{\Phi}|^2 \right) \nabla^2 \mathcal{E} = \vec{\nabla} \mathcal{E} \cdot \left(\vec{\nabla} \mathcal{E} + 2 \bar{\mathbf{\Phi}} \vec{\nabla} \mathbf{\Phi} \right) \ , \tag{4.14}$$

$$\left(\mathrm{Re} \mathcal{E} + |\mathbf{\Phi}|^2 \right) \nabla^2 \mathbf{\Phi} = \vec{\nabla} \mathbf{\Phi} \cdot \left(\vec{\nabla} \mathcal{E} + 2 \bar{\mathbf{\Phi}} \vec{\nabla} \mathbf{\Phi} \right) \ . \tag{4.15}$$

Note that in all the equations written up to now the metric function $\gamma(\rho, z)$ does not appear: it is completely decoupled from the other functions, as seen in the previous chapter. Once we know the Ernst potentials, $\gamma(\rho, z)$ fully determined by an integral.
A minus sign may appear in the definition of the two functions $\tilde{A_\varphi}$ and χ depending on the ordering of the three spatial coordinates. In our discussion we will use the ordering (ρ, z, φ). Once this ordering has been set, the equations (4.8) and (4.12) can be rewritten in a more explicit form by noting that for a generic function $h(\rho, z)$

$$\left(\hat{\varphi} \times \vec{\nabla} h(\rho, z) \right)_\rho = \tilde{\epsilon}_{\varphi z \rho} \partial_z h(\rho, z) = -\partial_z h(\rho, z) \ ,$$

$$\left(\hat{\varphi} \times \vec{\nabla} h(\rho, z) \right)_z = \tilde{\epsilon}_{\varphi \rho z} \partial_\rho h(\rho, z) = \partial_\rho h(\rho, z) \ .$$

Then introducing the two-dimensional Levi-Civita symbol, we obtain

$$\partial_i \tilde{A}_\varphi = \rho^{-1} f \tilde{\epsilon}_{ij} \left(\partial_j A_\varphi + \omega \partial_j A_t \right) \ , \tag{4.16}$$

$$\partial_i \chi = -\rho^{-1} f^2 \tilde{\epsilon}_{ij} \partial_j \omega + 2 \left(\tilde{A}_\varphi \partial_i A_t - A_t \partial_i \tilde{A}_\varphi \right) \ , \tag{4.17}$$

where i, j label the coordinates ρ, z, with $x^1 = \rho$ and $x^2 = z$.

Given a particular metric, the Ernst potentials $\mathcal{E}$ and $\mathbf{\Phi}$ can be calculated not necessarily in Weyl coordinates. In fact, the definitions of the functions $\tilde{A}_\varphi$ and χ, respectively (4.8) and (4.12), depend on the flat gradient operator in cylindrical coordinates, so, if the metric is in a different coordinate system from that of Weyl, we simply modify the differential operators for the three dimensional flat space in the new coordinate system. Let's make an example: consider the metric (1.4) of the dyonic Kerr-Newman black hole. The metric is expressed in Boyer-Lindquist coordinates $\{t, r, \theta, \varphi\}$, but the metric functions f and ω can be found anyway: this is the first step necessary to calculate the Ernst potentials. First of all, we note that the transformation from the Boyer-Lindquist coordinates to the Weyl coordinates does not involve the coordinates associated with the Killing vector fields, therefore the metric block associated with the t, φ coordinates remains unchanged. Thus the function f is simply given by $f = -g_{tt}$. Then we find

$$f(r,\theta) = \frac{\Delta(r) - a^2 \sin^2\theta}{\Xi(r,\theta)} = 1 + \frac{q^2 + p^2 - 2mr}{r^2 + a^2\cos^2\theta} \ . \tag{4.18}$$

The function ω can be found by requiring that $f\omega = g_{t\varphi}$. Then we get

$$\omega(r,\theta) = \frac{a \sin^2\theta (q^2 + p^2 - 2mr)}{r^2 + a^2\cos^2\theta + q^2 + p^2 - 2mr} \ . \tag{4.19}$$

Starting from the component $g_{\varphi\varphi}$ of the metric we can find the transformation for the Weyl coordinate ρ by requiring that $\rho^2 = f g_{\varphi\varphi} + f^2\omega^2$. From this condition we get

$$\rho = \sqrt{\Delta(r)} \sin\theta \ ,$$

which is the same relation find out in (3.5) from the condition (3.2). From this relation together with (3.6) we can find the metric in Weyl coordinates. However, it is easier to find the Ernst potentials using the coordinate r and $y := \cos\theta$. What we need to know now is the divergence operator in the flat space of coordinates $\{r, y, \varphi\}$. In order to do this, it is convenient to write the metric (1.4) in the following way

$$ds^2 = -f\left(dt - \omega d\varphi\right)^2 + f^{-1}\left\{ e^{2\gamma}\left[(r-m)^2 - \sigma^2 y^2\right]\left(\tfrac{dr^2}{\Delta} + \tfrac{dy^2}{1-y^2}\right) + (1-y^2)\Delta d\varphi^2 \right\} \ , \tag{4.20}$$

where the metric function γ can be determined by requiring that

$$e^{2\gamma} f^{-1}\left[(r-m)^2 - \sigma^2 y^2\right] = \Xi$$

from this we get

$$e^{2\gamma(r,y)} = \frac{r^2 - 2mr + q^2 + p^2 + a^2 y^2}{(r-m)^2 - \sigma^2 y^2} \ ,$$

Using (A.12) and (A.5), it is easy to find that the above metric fit the ansatz (4.1). Therefore the gradient operator in the flat space (r, y, φ) becomes

$$\vec{\nabla} h(r,y) = \frac{1}{\sqrt{(r-m)^2 - \sigma^2 y^2}} \left(\hat{r}\, \sqrt{\Delta(r)}\, \frac{\partial h}{\partial r} + \hat{y}\, \sqrt{1-y^2}\, \frac{\partial h}{\partial y} \right) \ . \tag{4.21}$$

where we omitted the φ component since the metric functions do not depend on that coordinate.
We can now calculate the Ernst potentials: starting from (4.16) with (4.21), we find the following equations for the three components

$$\begin{cases} \partial_r \tilde{A}_\varphi = \frac{f}{\Delta(r)} \left(\partial_y A_\varphi + \omega \partial_y A_t\right) \ , \\ \partial_y \tilde{A}_\varphi = -\frac{f}{1-y^2} \left(\partial_r A_\varphi + \omega \partial_r A_t\right) \ , \\ \partial_\varphi \tilde{A}_\varphi = 0 \ . \end{cases} \tag{4.22}$$

From the first equation we arrive at

$$\partial_r \tilde{A}_\varphi = \frac{pr^2 - a^2py^2 + 2aqry}{\left(r^2 + a^2y^2\right)^2} \quad \Rightarrow \quad \tilde{A}_\varphi = -\frac{aqy + pr}{r^2 + a^2y^2} \ . \tag{4.23}$$

Therefore the electromagnetic Ernst potential, using (1.5), becomes

$$\mathbf{\Phi} = \frac{q - ip}{r + iay} \ . \tag{4.24}$$

Note that, using only the first equation in (4.22), the Ernst potential could also have a second contribution that depends only on y: we see that this contribution is zero by integrating the second equation in (4.22). Eventually integration constants can be absorbed by rescaling the coordinates.
Similar steps lead to the gravitational Ernst potential: starting from (4.17), we find the following equations

$$\begin{cases} \partial_r \chi = -\frac{4^2}{\Delta(r)} \partial_y \omega + 2\left(\tilde{A}_\varphi \partial_r A_t - A_t \partial_r \tilde{A}_\varphi\right) \ , \\ \partial_y \chi = \frac{4^2}{1-y^2} \partial_r \omega + 2\left(\tilde{A}_\varphi \partial_y A_t - A_t \partial_y \tilde{A}_\varphi\right) \ , \\ \partial_{\tilde{A}\varphi} \chi = 0 \ . \end{cases} \tag{4.25}$$

From the first equation we arrive at

$$\partial_r \chi = -\frac{4mrax}{\left(r^2 + a^2y^2\right)^2} \quad \Rightarrow \quad \chi = \frac{2amy}{r^2 + a^2y^2} \ . \tag{4.26}$$

Knowing the electromagnetic Ernst potential and the function f, we can write

$$\mathcal{E} = 1 - \frac{2m}{r + iay} \ . \tag{4.27}$$

The same previous observations on any additional contributions can be made.
The Ernst potentials for other metrics obtained by canceling some parameter in (1.4) can be calculated in a similar manner to that shown now, or by canceling the same parameters in (4.24) and (4.27).

Chapter 5

Komar charges in presence of line singularities

In this chapter we show a method for calculating the conserved charges in the presence of line singularities. The latter can be described as string-like defects on the symmetry axis in the case of stationary and axisymmetric spacetimes. In Weyl coordinates, both black holes and defects can be described as rods located on the axis, so that conserved charges can be expressed only by integral on a cylinder surrounding that rod. Then, introducing the Ernst potentials, we find and expression involving only the potentials evaluated in the so-called isolated z-values plus a term whose presence will be discussed.
This method is subsequently applied first to the same cases considered in chapter 2 and then to black holes with Dirac and Misner strings, analyzing the contribution of the latter to the conserved charges.

5.1 Line singularities

In the following we will deal with two different types of line singularities: the Dirac string and the Misner string, respectively related to the magnetic charge and the NUT (Newman-Unti-Tamburino) charge. Another line singularity is the cosmic string, related to angular defects. We now show how Dirac and Misner strings emerge from the corresponding charge.

The magnetic charge can be thought as the end of a line of dipoles as shown in figure 5.1. This fictitious line of dipoles that stretches off to infinity is the so-called Dirac string. The Dirac string is the only way to incorporate magnetic monopoles into Maxwell's equations. In fact the vector potential of the magnetic monopole has a curl that is directed radially

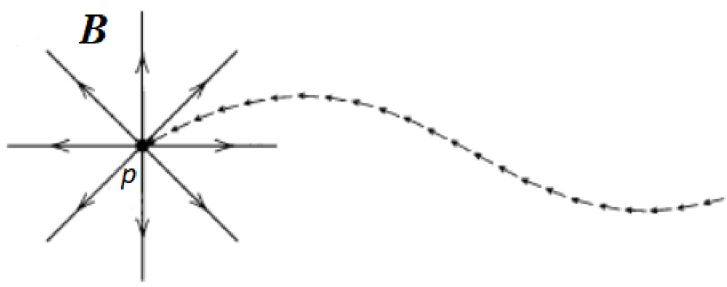

Figure 5.1: Magnetic monopole p and the Dirac string

outward for all points except on the string, generating a non null flux p. On the string itself the vector potential is singular: this singular behavior is equivalent to an intense magnetic

field inside the string that brings a return contribution to the flux (-p) to cancel the pole's outward flow. Another important feature is that the string is not a physical observable, it is just a theoretical artifact; in fact its location is arbitrary: a choice of different string positions is equivalent to different choices of gauge for the vector potential. Consequently, the string can be moved to other locations but can never be removed.
Therefore the Dirac string emerges when there is a discontinuity in the vector potential. Consider for example the Reissner-Nordström black hole with magnetic charge p: the electromagnetic four-potential is given by (1.5), setting $a = 0$. Thus it becomes

$$A = \left[\frac{q}{r}, 0, 0, p\cos\theta\right] \ .$$

For the vector potential to be well defined the boundary conditions must be

$$\begin{aligned} A_\varphi\big|_{\theta=0} &= 0 \ , \\ A_\varphi\big|_{\theta=\pi} &= 0 \ . \end{aligned} \tag{5.1}$$

In our case the φ component of the vector potential does not satisfy these conditions, since $A_\varphi\big|_{\theta=0} = p$ and $A_\varphi\big|_{\theta=0} = -p$: thus there would be two symmetrically distributed Dirac strings on the z-axis. But taking advantage of the gauge invariance of the vector potential, we can consider the component $A_\varphi = p\cos\theta + b_0$, where b_0 is a constant gauge parameter; then the boundary conditions become

$$\begin{aligned} A_\varphi\big|_{\theta=0} &= p + b_0 \ , \\ A_\varphi\big|_{\theta=\pi} &= -p + b_0 \ . \end{aligned}$$

Therefore we can choose a value for b_0 in order to eliminate one Dirac string. In presence of the magnetic charge, however, it is never possible to eliminate the Dirac string.

Consider now the NUT charge. If a metric of the type (4.1) is such that asymptotically

$$\omega \sim -2n\cos\theta \ ,$$

then the solution describes an object with NUT charge n. We discuss the meaning of this new charge, for which there is no Newtonian analog, by considering the simplest vacuum solution with this charge: the Taub-NUT solution, whose metric is

$$ds^2 = -f(r)\left(dt + 2n\cos\theta d\varphi\right)^2 + f^{-1}(r)dr^2 + \left(r^2+n^2\right)\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \ , \tag{5.2}$$

where

$$f(r) = \frac{\Delta}{\Xi} = \frac{r^2 - 2mr - n^2}{r^2 + n^2} \ . \tag{5.3}$$

This spacetime has some peculiar properties. The first concerns the interpretation of the parameter n: in the Newtonian limit we know that the diagonal components of the metric are related to the gravitostatic Newtonian potential; however, the Taub–NUT solution has non diagonal non-vanishing components in this limit; these off-diagonal components g_{ti} can be related to a gravitomagnetic potential $\mathcal{A}$, whose only non vanishing component is

$$\mathcal{A}_\varphi = 2n\cos\theta \ .$$

This is essentially the electromagnetic field of a magnetic monopole of charge proportional to n. Thus the NUT parameter n is interpreted as a sort of magnetic mass. Note that this is a possible interpretation of the NUT charge related to the introduction of a fictitious potential in the Newtonian limit.
The metric is not asymptotically flat because the off-diagonal components $g_{t\varphi}$ does not vanish at infinity. Moreover, the metric is perfectly regular at $r = 0$: it does not have

curvature singularities; however, for $\theta = 0$ and $\theta = \pi$ the metric is no more invertible, since $g = \det(g_{\mu\nu}) = 0$. These points are not coordinate singularity as for the Schwarzschild black hole: it is precisely the metric function ω that has a singular behavior, since it takes different values on the symmetry axis passing from the south pole axis to the north pole axis. This behavior is the same that happens for the electromagnetic potential in the case of the Dirac string. The solution has then line singularities at $\theta = 0$ and $\theta = \pi$, called Misner strings. Similarly to the magnetic potential, we can introduce a constant s such that asymptotically

$$g_{t\varphi} \sim 2n(\cos\theta + s) \; .$$

This parameter allow us to eliminate one of the two Misner strings: if $s = n$ there is only one Misner string n the north pole axis ($\cos\theta = 1$), while the south pole axis ($\cos\theta = -1$) is perfectly regular; the choice $s = -n$ corresponds to the opposite situation. Therefore the parameter s regulates the distribution of both Misner strings and, as we will later see, regulates also their strength: thus changing the value of s corresponds to a physical transformation of the system.

Misner proposed a way to remove these singularities and make the metric regular everywhere by introducing a time periodical identification condition; however, the metric becomes physically problematic due to the presence of closed timelike curves and this led him to declare the NUT parameter as nonphysical. In the last few years, it has been noted that such a spacetime is less physically problematic if the Misner strings are left. In fact, the Misner strings are transparent to geodesics, making the spacetime geodesically complete [1] : this feature arises from purely geometric considerations. Moreover, the condition $|s/n| \leq 1$ guarantees the absence of closed timelike and null geodesics, preserving the principle of causality. This raised a possibility that the NUT charge may actually be relevant for astrophysics: already in 1997 astrophysicists probed the possibility of detecting the NUT parameter by microlensing.

Therefore we will study solutions with NUT charge without removing the line singularities. The NUT parameter introduces also angular defects. In fact, consider a small circle around the symmetry axis, assuming fixed t and r: if the ratio between the circumference and the radius of the circle in the limit $\theta \to 0$ or $\theta \to \pi$ is different from 2π then the axis has a conical singularity. The string associated to this line singularity is called cosmic string. In order to avoid such a singularity we can introduce a parameter C in the range of the angular coordinate φ, such that $\varphi \in [0, 2\pi C)$ and set it so that the ratio is equal to 2π. The invariant length of the circumference of the circle is

$$\mathcal{C} = \int_0^{2\pi C} \sqrt{|g_{\varphi\varphi}|}\, d\varphi \; , \tag{5.4}$$

while its radius is

$$\mathcal{R} = \int_0^{\theta} \sqrt{|g_{\theta\theta}|}\, d\theta \; . \tag{5.5}$$

Therefore we have to calculate

$$\lim_{\theta\to 0,\pi} \frac{\text{circumference}}{\text{radius}} = \lim_{\theta\to 0,\pi} \frac{2\pi C \sqrt{|g_{\varphi\varphi}|}}{\theta \sqrt{|g_{\theta\theta}|}} \; . \tag{5.6}$$

From the metric (5.2) we find

$$\lim_{\theta\to 0,\pi} \frac{2\pi C}{\theta} \sqrt{\left| \sin^2\theta - \frac{4n^2\Delta}{\Xi^2} \cos^2\theta \right|} \; .$$

[1] a spacetime is said to be geodesically complete if every geodesic is complete, that is, if the geodesic has affine parameter λ, the definition domain of that parameter can be extended to $-\infty < \lambda < \infty$

This limit is divergent in both cases since the term $\cos\theta$ in the circumference does not approach zero for $\theta \to 0, \pi$. Consequently both north and south pole axes has a divergent angular defect for any value of the constant C, that we will keep equal to one so that the angular coordinate φ has the "standard" range $[0, 2\pi)$.
In order to check this result, we can make the same calculations in Weyl coordinates, where the above ratio (5.6) is given by

$$\lim_{\rho\to 0} \frac{\text{circumference}}{\text{radius}} = \lim_{\rho\to 0} \frac{2\pi C \sqrt{|g_{\varphi\varphi}|}}{\rho \sqrt{|g_{\rho\rho}|}} . \tag{5.7}$$

In this coordinate system we have to distinguish the two cases:

- North pole axis: $z > \sigma$
  Starting from (A.16) and (A.17) for $\rho \to 0$ to leading order we find

  $$g_{\varphi\varphi} = -\frac{4n^2(\sigma^2 - z^2)}{z^2 + \sigma^2 + 2mz} + \mathcal{O}\left(\rho^2\right) , \qquad g_{\rho\rho} = \frac{z^2 + \sigma^2 + 2mz}{z^2 - \sigma^2} + \mathcal{O}\left(\rho^2\right) .$$

  Therefore the quantity (5.7) is divergent because of the presence of ρ in the denominator.

- South pole axis: $z < -\sigma$
  Starting from (A.16) and (A.17) for $\rho \to 0$ to leading order we find

  $$g_{\varphi\varphi} = -\frac{4n^2(\sigma^2 - z^2)}{z^2 + \sigma^2 - 2mz} + \mathcal{O}\left(\rho^2\right) , \qquad g_{\rho\rho} = \frac{z^2 + \sigma^2 - 2mz}{z^2 - \sigma^2} + \mathcal{O}\left(\rho^2\right) .$$

  Also in this case the quantity (5.7) is divergent.

Finally the NUT charge carries both Misner and cosmic strings. In section 5.5.3 we will see that the NUT charge causes the presence of a Dirac string also in the case $p = 0$.

When line singularities are present, apart from the standard boundary on the event horizon and the two-sphere at spatial infinity, there are two small cylinders surrounding the strings, as shown in figure 5.2. In both cases the line singularities are located on the

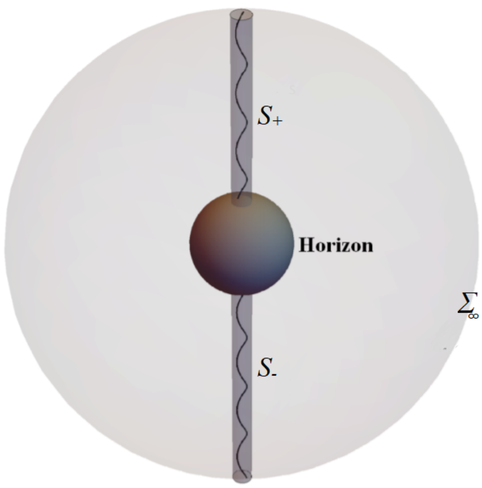

Figure 5.2: Boundaries in presence of line singularities

north or south pole, that is, on the z-axis in Weyl coordinates. In particular they are in correspondence of the spacelike semi-infinite rods, for a rod structure like the one shown in section 3.2. So the semi-infinite spacelike rods account for defects.

5.2 Komar charges decomposition for the rod structure

Consider a stationary and axisymmetric spacetime, where the two Killing vector fields $k = \partial_t$ and $m = \partial_\varphi$. The total energy, angular momentum and charge of the spacetime are given by an integral over a two-sphere at spatial infinity respectively by equations (2.15), (2.16) and (2.17).
In presence of line singularities, the spacetime is bounded by the two-sphere at spatial infinity Σ_∞ and by cylinders Σ_n around each rod in Weyl coordinates. Therefore using the decomposition presented in section 2.5, the Komar energy can be written as a sum of an electromagnetic contribution and a rod contributions

$$M = \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} \nabla^\nu k^\mu d\Sigma_{\mu\nu} + M_E \ , \tag{5.8}$$

where M_E is given by (2.28). Note that the integral over the cylinders surrounding the rods reduces to (2.26) in absence of line singularities because the only rod that contributes to the conserved charges is the horizon rod; the two semi-infinite spacelike rod becomes important only in presence of line singularities. This decomposition can also be done in coordinates other than those of Weyl, by suitably varying the integration surfaces around the horizon and the defects; the Weyl coordinates are suitable since all event horizons and defects are represented by rods on the axis $\rho = 0$.
Let Σ be the spacelike hypersurface of constant t, then by (1.29) it is clear that necessarily $\mu = t$ in (2.28). Therefore, denoting the electromagnetic energy-momentum tensor T_E with T, the electromagnetic contribution to the energy is

$$M_E = -2 \int_\Sigma \sqrt{|g|}\, T^t{}_\nu\, \delta^\nu_t d^3y = -2 \int_\Sigma \sqrt{|g|}\, T^t{}_t \ d^3y \ . \tag{5.9}$$

Because of the ansatz (4.2) the only non vanishing components of the field strength tensor $F_{\mu\nu}$ are F_{it}, $F_{i\varphi}$, where i labels other coordinates different from the ones associated to the two Killing vector fields, which are t, φ. Therefore from (1.2) follows that

$$T^t{}_t = \frac{1}{8\pi} \left(F_{it} F^{it} - F_{i\varphi} F^{i\varphi} \right) \ .$$

Replacing it in the previous equation we have

$$\begin{aligned} M_E &= -\frac{1}{4\pi} \int_\Sigma \sqrt{|g|} \left(F_{it} F^{it} - F_{i\varphi} F^{i\varphi} \right) d^3y = \\ &= -\frac{1}{4\pi} \int_\Sigma \sqrt{|g|} \left[(\partial_i A_t) F^{it} - (\partial_i A_\varphi) F^{i\varphi} \right] d^3y \ . \end{aligned}$$

The Maxwell equations are given by (1.3): for $\nu = t$ and $\nu = \varphi$ we can write respectively

$$\begin{aligned} \partial_i \left(\sqrt{|g|} F^{it} \right) &= 0 \ , \\ \partial_i \left(\sqrt{|g|} F^{i\varphi} \right) &= 0 \ , \end{aligned}$$

so that the electromagnetic contribution becomes

$$M_E = -\frac{1}{4\pi} \int_\Sigma \partial_i \left[\sqrt{|g|} \left(A_t F^{it} - A_\varphi F^{i\varphi} \right) \right] d^3y \ . \tag{5.10}$$

We can now think of the quantity that multiplies the square root of the determinant of the metric as the ti component of a completely antisymmetric tensor field $X^{\nu\mu}$; for the latter holds:

$$\nabla_\mu X^{\nu\mu} = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} X^{\nu\mu} \right) \ . \tag{5.11}$$

Consequently (5.10) can be written as

$$M_E = -\frac{1}{4\pi}\int_\Sigma \sqrt{|g|}\,\nabla_i\left(\sqrt{|g|}X^{ti}\right)d^3y = -\frac{1}{4\pi}\int_\Sigma \nabla_i\left(X^{ti}\right)dS_i\ .$$

We can now use the Stokes' theorem (1.37): pay attention that the indices present in the two sides of the equation, as saturated indices, are not correlated. The boundary $\partial\Sigma$ of Σ is the union of the two-sphere at spatial infinity and the small cylinders around the rods; its surface element, since Σ is the hypersurface of constant t, has only non vanishing components $dS_i \equiv dS_{ti} = -dS_{it}$. Taking into account that the conventional orientation of the surface element is different from the one that $\partial\Sigma$ inherits from Σ, we find

$$\int_\Sigma \nabla_i X^{ti} dS_t = \oint_{\partial\Sigma} X^{ti} dS_i = \oint_{\Sigma_\infty} X^{ti} dS_i - \sum_n \oint_{\Sigma_n} X^{ti} dS_i\ .$$

Replacing the tensor X, we can write the magnetic contribution as a sum over rod contributions

$$M_E = \sum_n \frac{1}{4\pi}\oint_{\Sigma_n}\left(A_t F^{it} - A_\varphi F^{i\varphi}\right)dS_i\ , \tag{5.12}$$

where we have implicitly assumed that

$$-\frac{1}{4\pi}\oint_{\Sigma_\infty}\left(A_t F^{it} - A_\varphi F^{i\varphi}\right)dS_i = 0\ . \tag{5.13}$$

However, this condition is not always satisfied. On one hand, it depends on the properties of the electric and magnetic fields, which fall off at infinity as $1/r^2$ for the solutions considered in this work. On the other hand, it depends on the Gauge choice of the four-vector potential because of the presence of both A_t and A_φ. We will discuss in more detail the contribution of such a term in section 5.5.1. Therefore, when condition (5.13) is not satisfied, the result (5.12) should also include a surface integral at infinity.
Returning to (5.8), the first term can be rewritten only as a function of the metric components. In fact

$$\nabla^\nu k^\mu = g^{\nu\lambda}\nabla_\lambda k^\mu = g^{\nu\lambda}\Gamma^\mu{}_{\lambda t}\ ,$$

because the Killing vector k has constant components, so that their partial derivative is equal to zero, and the only one non vanishing is the t component. The contraction with the surface element gives

$$\begin{aligned} g^{\nu\lambda}\Gamma^\mu{}_{\lambda t}\,d\Sigma_{\mu\nu} &= g^{ij}\Gamma^t{}_{jt}d\Sigma_{ti} + g^{ta}\Gamma^i{}_{at}\,d\Sigma_{it} = \left(\frac{1}{2}g^{ij}g^{ta}\partial_j g_{ta} + \frac{1}{2}g^{ta}g^{ij}\partial_j g_{ta}\right)d\Sigma_{ti} = \\ &= g^{ij}g^{ta}\partial_j g_{ta}\,d\Sigma_i = \frac{1}{2}g^{ij}g^{ta}\partial_j g_{ta}\,dS_i\ , \end{aligned}$$

where in the last step we used the relation (1.38); the index a labels only the coordinates t, φ.
Note that we are not necessarily in Weyl coordinates: the results obtained so far are valid for any solution whose metric is a block matrix, that is $g_{ai} = 0$.
Therefore, if the assumption (5.13) is valid, the total energy of the spacetime can be decomposed as a sum on rod contributions

$$M = \sum_n \frac{1}{8\pi}\oint_{\Sigma_n}\left[g^{ij}g^{ta}\partial_j g_{ta} + 2\left(A_t F^{it} - A_\varphi F^{i\varphi}\right)\right]dS_i\ , \tag{5.14}$$

where the first term may be viewed as the gravitational contribution to the source mass, and the second term as the electromagnetic contribution. it is remarkable that even the bulk contribution, that is M_E (the contribution of all the spacetime outside the rods), can be expressed in terms of the data on the axis.

Similarly, the total angular momentum can be written as a sum of a rod contributions and an electromagnetic contribution

$$J = \sum_n -\frac{1}{8\pi} \oint_{\Sigma_n} \nabla^\mu m^\nu d\Sigma_{\nu\mu} + J_E \ , \tag{5.15}$$

where J_E is reported in (2.30). Following similar steps to those made for M_E, the electromagnetic contribution J_E can be transformed into a sum of rod integrals. In fact in (2.30), for the same previous reasons, necessarily $\mu = t$, then, since the Killing vector field m as components $m^\nu = \delta^\nu_\varphi$, the only term that contributes to the bulk integral is

$$T^t{}_\varphi = \frac{1}{4\pi} F_{i\varphi} F^{it} \ .$$

Consequently the electromagnetic contribution becomes

$$J_E = \frac{1}{4\pi} \int_\Sigma \partial_i A_\varphi F^{it} d^3y = \frac{1}{4\pi} \int_\Sigma \partial_i \left(\sqrt{|g|}\, A_\varphi F^{it} \right) d^3y = \sum_n -\frac{1}{4\pi} \oint_{\Sigma_n} A_\varphi F^{it} dS_i \ ,$$

where we made the same steps illustrated for the energy contribution M_E. The last step is valid under the assumption

$$\frac{1}{4\pi} \oint_{\Sigma_\infty} A_\varphi F^{it}\, dS_i = 0 \ . \tag{5.16}$$

The same observations previously exposed for (5.13) also apply to the latter assumption. Returning to (5.15), we can write the first term in function of the metric components in the same way as seen for energy, with the only difference due to the different Killing vector. From the contraction of the covariant derivative with the surface element we obtain

$$g^{\nu\lambda} \Gamma^\mu{}_{\lambda\varphi} d\Sigma_{\mu\nu} = g^{ij} g^{ta} \partial_j g_{\varphi a} d\Sigma_i = \frac{1}{2} g^{ij} g^{ta} \partial_j g_{\varphi a} dS_i \ .$$

Therefore, the angular momentum is decomposed into a sum of rod integrals

$$J = \sum_n -\frac{1}{16\pi} \oint_{\Sigma_n} \left(g^{ij} g^{ta} \partial_j g_{ta} + 4 A_\varphi F^{it} \right) dS_i \ . \tag{5.17}$$

Even the electric charge can be decomposed into a sum of the fluxes through the cylinders Σ_n. In fact, by Stokes theorem (1.37)

$$\frac{1}{4\pi} \oint_{\Sigma_\infty} F^{\mu\nu} d\Sigma_{\mu\nu} = \frac{1}{4\pi} \int_\Sigma \nabla_\nu F^{\mu\nu} dS_\mu + \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} F^{\mu\nu} d\Sigma_{\mu\nu} \ , \tag{5.18}$$

but $\nabla_\nu F^{\mu\nu} = 0$ for the Maxwell equations in vacuum, so the first integral in the second member is equal to zero; from the fact that the only two non-null components of the surface element are $d\Sigma_{ti} = -d\Sigma_{it} = dS_i/2$ and using the antisymmetry of the Faraday tensor, the charge becomes

$$Q = \sum_n \frac{1}{4\pi} \oint_{\Sigma_n} F^{ti} dS_i \ . \tag{5.19}$$

5.3 Komar charges in terms of Ernst potentials

Consider now the Lewis-Weyl-Papapetrou metric in (4.1): in matrix form, the metric and its inverse, obtained through the cofactors matrix, are given by

$$g_{\mu\nu} = \begin{bmatrix} -f & f\omega & 0 & 0 \\ f\omega & f^{-1}\rho^2 - f\omega^2 & 0 & 0 \\ 0 & 0 & f^{-1}e^{2\gamma} & 0 \\ 0 & 0 & 0 & f^{-1}e^{2\gamma} \end{bmatrix} , \tag{5.20}$$

$$g^{\mu\nu} = \begin{bmatrix} -f^{-1} + \rho^{-2} f\omega^2 & \rho^{-2} f\omega & 0 & 0 \\ \rho^{-2} f\omega & \rho^{-2} f & 0 & 0 \\ 0 & 0 & fe^{-2\gamma} & 0 \\ 0 & 0 & 0 & fe^{-2\gamma} \end{bmatrix} . \tag{5.21}$$

Recall the definition of the complex Ernst potentials defined by (4.16) and (4.17) in Weyl coordinates. Since Σ_n is a cylindrical surface characterized by constant t and $\rho = 0$, the index i in the energy formula (5.14) must be equal to ρ (this is clear from the formula (1.34)). Therefore the rod gravitational contribution to the energy is

$$M_n^G = \frac{1}{8\pi} \int_0^{2\pi} d\varphi \int_{z_n}^{z_{n+1}} dz \sqrt{|g|}\, g^{\rho\rho} \left(g^{tt} \partial_\rho g_{tt} + g^{t\varphi} \partial_\rho g_{t\varphi} \right) ,$$

where from the above matrix is obvious to find

$$\sqrt{|g|} = \frac{\rho e^{2\gamma}}{f} \quad \Rightarrow \quad \sqrt{|g|}\, g^{\rho\rho} = \rho .$$

Integrating over the cyclic coordinate φ, since no function depends on this coordinate, we get

$$\begin{aligned} M_n^G &= \frac{1}{4} \int_{z_n}^{z_{n+1}} dz\, \rho \left[\left(\rho^{-2} f\omega^2 - f^{-1} \right) \partial_\rho (-f) + \rho^{-2} f\omega^2 \partial_\rho f + \rho^{-2} f^2 \omega \partial_\rho \omega \right] = \\ &= \frac{1}{4} \int_{z_n}^{z_{n+1}} dz\, \left(\rho f^{-1} \partial_\rho f + \rho^{-1} f^2 \omega \partial_\rho \omega \right) = \\ &= \frac{1}{4} \int_{z_n}^{z_{n+1}} dz\, \omega \left[\partial_z \chi + 2 \left(A_t \partial_z \tilde{A}_\varphi - \tilde{A}_\varphi \partial_z A_t \right) \right] + \frac{1}{4} \int_{z_n}^{z_{n+1}} dz\, \rho f^{-1} \partial_\rho f , \end{aligned}$$

where in the last step we used the definition (4.17) of $\chi = \mathrm{Im}\mathcal{E}$. We will see that the term in the second integral vanishes in the limit $\rho \to 0$ when the black hole is rotating. So we will keep that term in order to include all the possible cases.
The rod electromagnetic contribution to the energy is

$$\begin{aligned} M_n^E &= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{z_n}^{z_{n+1}} dz \sqrt{|g|} \left(A_t F^{\rho t} - A_\varphi F^{\rho\varphi} \right) = \\ &= \frac{1}{2} \int_{z_n}^{z_{n+1}} dz \sqrt{|g|}\, g^{\rho\rho} \left(g^{tt} A_t \partial_\rho A_t + g^{t\varphi} A_t \partial_\rho A_\varphi + g^{\varphi\varphi} A_\varphi \partial_\rho A_\varphi + g^{\varphi t} A_\varphi \partial_\rho A_t \right) . \end{aligned}$$

By the inverse of the matrix metric (5.21) we obtain

$$\begin{aligned} M_n^E &= \frac{1}{2} \int_{z_n}^{z_{n+1}} dz\, \left[\rho^{-1} f \left(\partial_\rho A_\varphi + \omega \partial_\rho A_t \right) \left(\omega A_t - A_\varphi \right) - \rho^{-1} f A_t \partial_\rho A_t \right] = \\ &= -\frac{1}{2} \int_{z_n}^{z_{n+1}} dz\, \left(\omega A_t - A_\varphi \right) \partial_z \tilde{A}_\varphi - \frac{1}{2} \int_{z_n}^{z_{n+1}} dz\, \rho f^{-1} A_t \partial_\rho A_t , \end{aligned}$$

where in the last step we used the definition (4.16) of $\tilde{A}_\varphi = \mathrm{Im}\boldsymbol{\Phi}$. Also in this case the second integral vanishes in the limit $\rho \to 0$ only when the black hole is rotating. Now, it is important to note that the metric function ω takes a constant value ω_n along each rod (this is a consequence of the constancy of the rod directional vectors on each rod), and the same for the angular velocity along a rod $\Omega_n = 1/\omega_n$ and for the corotating electric potential

$$\Phi_n = -A_\mu \xi_n^\mu\big|_n = -A_t - \Omega_n A_\varphi \ . \tag{5.22}$$

Equally constant on each rod is the quantity

$$-\,\omega_n \Phi_n = A_\varphi + \omega A_t \ . \tag{5.23}$$

We can therefore add to the above integral the null quantity $\tilde{A}_\varphi \partial_z(-A_\varphi - \omega A_t)$ in order to get

$$M_n^E = -\frac{1}{2}\int_{z_n}^{z_{n+1}} dz \left[\omega(A_t\partial_z\tilde{A}_\varphi - \tilde{A}_\varphi\partial_z A_t) - \partial_z(\tilde{A}_\varphi A_\varphi)\right] - \frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \rho f^{-1} A_t \partial_\rho A_t \ .$$

Adding the two gravitational and electromagnetic contributions we find

$$M_n = M_n^G + M_n^E = \frac{1}{4}\int_{z_n}^{z_{n+1}} dz \left[\omega\partial_z \mathrm{Im}\mathcal{E} + 2\partial_z(A_\varphi \mathrm{Im}\boldsymbol{\Phi})\right] + M^* \ ,$$

where

$$M^* = \frac{1}{4}\int_{z_n}^{z_{n+1}} dz\, \rho f^{-1}\left(\partial_\rho f - 2A_t\partial_\rho A_t\right) \ . \tag{5.24}$$

Using again the constancy of ω along each rod, the rod contribution to the total energy becomes

$$M_n = \frac{\omega_n}{4}\mathrm{Im}\mathcal{E}\big|_{z_n}^{z_{n+1}} + \frac{1}{2}(A_\varphi \mathrm{Im}\boldsymbol{\Phi})\big|_{z_n}^{z_{n+1}} + M^* \ . \tag{5.25}$$

The term M^* is necessary to include the non-rotating case, in which $\omega = 0$.
Similarly for the angular momentum, starting from (5.17), the rod gravitational contribution, after the integration over φ, is

$$\begin{aligned} J_n^G &= -\frac{1}{8}\int_{z_n}^{z_{n+1}} dz\, \sqrt{|g|}\, g^{\rho\rho}\left(g^{tt}\partial_\rho g_{\varphi t} + g^{t\varphi}\partial_\rho g_{\varphi\varphi}\right) = \\ &= -\frac{1}{8}\int_{z_n}^{z_{n+1}} dz \left(2\omega - \rho\partial_\rho\omega - 2\rho f^{-1}\omega\partial_\rho f - \rho^{-1} f^2\omega^2\partial_\rho\omega\right) = \\ &= -\frac{1}{8}\int_{z_n}^{z_{n+1}} dz \left\{2\omega\left(1 - \rho f^{-1}\partial_\rho f\right) - \omega^2\left[\partial_z\chi + 2\left(A_t\partial_z\tilde{A}_\varphi - \tilde{A}_\varphi\partial_z A_t\right)\right]\right\} \ , \end{aligned}$$

where the second addend of the integral vanishes in the limit $\rho \to 0$ for rotating black holes. The rod electromagnetic contribution is

$$\begin{aligned} J_n^E &= -\frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \sqrt{|g|}\, A_\varphi\left(g^{\rho\rho} g^{ta}\partial_\rho A_a\right) \\ &= -\frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, A_\varphi\omega\rho^{-1} f\left(\partial_\rho A_\varphi + \omega\partial_\rho A_t\right) + \frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \rho f^{-1} A_\varphi\partial_\rho A_t = \\ &= \frac{1}{4}\int_{z_n}^{z_{n+1}} dz\, \omega\left[\left(A_\varphi + \omega A_t\right)\partial_z\tilde{A}_\varphi - \omega\left(A_t\partial_z\tilde{A}_\varphi - \tilde{A}_\varphi\partial_z A_t\right) + \partial_z\left(A_\varphi\tilde{A}_\varphi\right)\right] + \\ &\quad + \frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \rho f^{-1} A_\varphi\partial_\rho A_t \ , \end{aligned}$$

where in the last step we added the null term $A_\varphi \partial_z (A_\varphi + \omega A_t)$ because of the constancy on each rod of (5.23). The second integral term is null for $\rho \to 0$ in the case of rotating black holes.
The total rod contribution to the spacetime angular momentum becomes

$$J_n = \frac{1}{8}\int_{z_n}^{z_{n+1}} dz\, \omega \left[-2 + \omega \partial_z \mathrm{Im}\mathcal{E} + 2\partial_z \left(A_\varphi \mathrm{Im}\mathbf{\Phi}\right) + 2\left(A_\varphi + \omega A_t\right)\partial_z \mathrm{Im}\mathbf{\Phi}\right] + J^* \ ,$$

where

$$J^* = \frac{1}{4}\int_{z_n}^{z_{n+1}} dz \rho f^{-1}\left(\omega \partial_\rho f + 2A_\varphi \partial_\rho A_t\right) \ . \tag{5.26}$$

Using the constancy of ω and (5.23) we find

$$J_n = \frac{\omega_n}{4}\left\{-(z_{n+1} - z_n) + \left[\omega_n \left(\frac{\mathrm{Im}\mathcal{E}}{2} - \Phi_n \mathrm{Im}\mathbf{\Phi}\right) + A_\varphi \mathrm{Im}\mathbf{\Phi}\right]\Bigg|_{z_n}^{z_{n+1}}\right\} + J^* \ . \tag{5.27}$$

Note that for semi-infinite rods, if $\omega_n \neq 0$, the length term in the above formula will give an infinite rod angular momentum, but we will see that this contribution cancels out with that of the opposite rod for particular choices.
We can also express the charge in terms of the complex Ernst potentials: starting from (5.19), we obtain

$$\begin{aligned}
Q_n &= -\frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \sqrt{|g|}\, F^{\rho t} = \\
&= -\frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \omega \rho^{-1} f \left(\partial_\rho A_\varphi + \omega \partial_\rho A_t\right) = \\
&= \frac{1}{2}\int_{z_n}^{z_{n+1}} dz\, \omega\, \partial_z \tilde{A}_\varphi \ ,
\end{aligned}$$

where in the last step we used the definition (4.16). By the constancy of ω on each rod we get

$$Q_n = \frac{\omega_n}{2}\mathrm{Im}\mathbf{\Phi}\Big|_{z_n}^{z_{n+1}} \ . \tag{5.28}$$

5.4 Schwarzschild, Reissner-Nordström and Kerr black holes

Now let's quickly analyze the three simplest cases for which, in chapter 2, we have already calculated the conserved charges by directly evaluating the Komar integral on a two-sphere at spatial infinity. This section is important to better understand the role of the quantities M^* and J^* introduced in the previous section.
Let's start from the Schwarzschild black hole: the rod structure of this solution is the same as that of the Kerr black hole seen in section 3.2. Since we have no line singularities, the only rod contribution is that of the horizon. First of all, we change the coordinate system, passing from the Schwarzschild coordinates to the prolate spherical coordinates through (A.12), where from (1.7) follows that $\sigma = m$ since the parameters a, q and p are set to zero. In this coordinate system the metric takes the form [2]

$$ds^2 = -\frac{x-1}{x+1}dt^2 + \frac{m^2(x+1)}{x-1}dx^2 + \frac{m^2(x+1)^2}{1-y^2}dy^2 + m^2(x+1)^2(1-y^2)d\varphi^2 \ . \tag{5.29}$$

[2]Note that m is an overall conformal factor if we define a new time coordinate $t' = t/m$, and can therefore be removed from the metric

The horizon rod corresponds to $x = 1$ and $y \in [-1, 1]$, therefore, starting from (5.14) and integrating over the cyclic coordinate φ, we find

$$M = \frac{1}{4}\int_{-1}^{1} dy\, \sqrt{|g|}\, g^{xx} g^{tt} \partial_x g_{tt} \quad \text{where } \sqrt{|g|} = m^3 (x+1)^2 \;\Rightarrow\; M = \frac{m}{2}\int_{-1}^{1} dy = m\,.$$

We can also calculate the contributions of the two semi-infinite spacelike rods (where y is constant) and see that they are null since $\partial_y g_{tt} = 0$. Actually we have done nothing but use the decomposition described in the section 2.5 in order to rewrite the boundary integral over the two-sphere at spatial infinity as a boundary integral over the event horizon; we just used a different coordinate system. However this simple case is useful for understanding the expression (5.25). In fact, the Ernst potential for the Schwarzschild solution is given by (4.27) setting $a = q = p = 0$; thus we find

$$\mathcal{E} = 1 - \frac{2m}{r}\,. \tag{5.30}$$

Consequently the first term in (5.25) depending explicitly on Ernst potential is null since the complex gravitational potential is real. The contribution to the energy of the spacetime is therefore enclosed in M^*; in particular it is due precisely to the term $\rho f^{-1}\partial_\rho f$, that is nothing but the integrand in the above calculation in the prolate spherical coordinates. If we want to see it explicitly in Weyl coordinates, starting from (A.6), we can write the metric in the following way

$$ds^2 = -f dt^2 + f^{-1}\left[e^{2\gamma}\left(d\rho^2 + dz^2\right) + \rho^2 d\varphi^2\right]\,, \tag{5.31}$$

where

$$f(\rho, z) = \frac{\sqrt{(m+z)^2 + \rho^2} - m - z}{\sqrt{(m-z)^2 + \rho^2} + m - z}\,.$$

This metric fits the ansatz (4.1) with $\omega = 0$. Then it is quite easy to find that

$$\lim_{\rho\to 0} \frac{\rho}{f}\,\partial_\rho f = 2 \neq 0\,.$$

Then M^* just gives the parameter m as a result.

Consider now the Reissner-Nordström black hole. The gravitational Ernst potential is the same as the Schwarzschild black hole, while the electromagnetic one is given by (4.24) setting $a = p = 0$; thus we find

$$\boldsymbol{\Phi} = \frac{q}{r} = A_t\,. \tag{5.32}$$

Also in this case the contribution to energy is due only to the horizon and is totally contained in M^*, since the Ernst potential are real and the metric function ω is zero. The term M^* has now two contributions since the electromagnetic four-potential is non null. We will not illustrate the steps, but, writing the metric (2.18) in prolate spherical coordinates through (A.12), we find $M = M_H = m$: the subscript H means that it is the contribution of the integral around the horizon rod. Note that it does not coincide with the black hole's contribution to energy, as there is also a volume contribution (5.9), due to the electromagnetic energy-momentum tensor, which has been transformed as an integral around the horizon rod. Therefore M_H contains both the contribution of the black hole, due to the first term in (5.24) and that of outer spacetime, due to the second term in (5.24).

Finally, we consider the Kerr black hole: the gravitational Ernst potential is given by (4.27), setting $q = p = 0$; thus we find

$$\mathcal{E} = 1 - \frac{2m}{r + iay} \quad\Rightarrow\quad \mathrm{Im}\mathcal{E} = \frac{2amy}{r^2 + a^2y^2}\,. \tag{5.33}$$

In this case the total energy of the spacetime is only given by the first term in (5.25). The metric function ω for the Kerr metric is given by (4.19), setting to zero the electric and magnetic charges; we simply rewrite it in prolate spherical coordinates through (A.12) and evaluate it on the horizon, that is $x = 1$. Then we get

$$\omega_H = \omega|_{x=1} = \frac{2m(\sigma + m)}{a} \; .$$

The Ernst potential dependent term remains unchanged in prolate spherical coordinates: we just have to evaluate it in the y interval corresponding to $z \in [-\sigma, \sigma]$. Therefore, using that $r_H^2 + a^2 = 2mr_H$, the total energy is

$$M = M_H = \frac{\omega_H}{4} \mathrm{Im}\mathcal{E}\big|_{y=-1}^{y=1} = \omega_H \, \frac{ma}{2m(\sigma + m)} = m \; .$$

Why the contribution M^* is null for the Kerr solution? Let's write that contribution in (x, y) coordinates for the horizon rod; the integrand becomes

$$\underbrace{\sqrt{|g|}\, g^{xx}}_{\sigma(x^2-1)} \; f^{-1}\partial_x f \; .$$

From here we see that the first term goes to zero for $x \to 1$, while, from (4.18) (always setting $q = p = 0$), the metric function f and its derivative have no singular behaviors for $x = 1$. Therefore this term is null when $x = 1$. This happens because for rotating black holes the horizons occur when $\Delta(r) = 0$, while the function f is non null on the horizons; for non-rotating black holes, the zeroes of Δ coincide with the same values for which $f = 0$ (in fact they are the same function), so on the horizon the term contained in M^* does not go to zero. The same remarks apply to the term J^* in (5.27).

5.5 Rotating black holes

We now turn our attention on rotating black holes. As noted in the previous section, formulas (5.25) and (5.27) respectively takes the more elegant form

$$M_n = \frac{\omega_n}{4} \mathrm{Im}\mathcal{E}\big|_{z_n}^{z_{n+1}} + \frac{1}{2}(A_\varphi \mathrm{Im}\mathbf{\Phi})\big|_{z_n}^{z_{n+1}} \tag{5.34}$$

and

$$J_n = \frac{\omega_n}{4} \left\{ -(z_{n+1} - z_n) + \left[\omega_n \left(\frac{\mathrm{Im}\mathcal{E}}{2} - \Phi_n \mathrm{Im}\Phi \right) + A_\varphi \mathrm{Im}\Phi \right] \Bigg|_{z_n}^{z_{n+1}} \right\} . \tag{5.35}$$

Using (5.28), we can rewrite the rod contribution to the angular momentum as

$$J_n = \frac{\omega_n}{2} \left(-\frac{z_{n+1} - z_n}{2} + M_n - \Phi_n Q_n \right) \; . \tag{5.36}$$

This formula is applicable both to timelike rods and spacelike ones.
Consider now a finite timelike rod, which describes a horizon, and suppose it matches the interval $[z_1, z_2]$. We need now to know the entropy and the temperature of a black hole, in order to write the Smarr formula, that is a formula which relates the conserved charges with other quantities evaluated at the horizon. The black hole entropy is usually known as the Bekenstein-Hawking entropy and it is given by $S_H = \mathcal{A}_H/4$, where $\mathcal{A}_H$ is the horizon area

$$\mathcal{A}_H = \int_H \sqrt{|h^{(2)}|}\, d^2 w \; . \tag{5.37}$$

In Weyl coordinates the horizon is reduced to a rod of length $L_H = z_2 - z_1$ on the z-axis. However, the horizon still has an area, that is given by

$$\mathcal{A}_H = \int_0^{2\pi} d\varphi \int_{z_1}^{z_2} dz \sqrt{|g_{zz} g_{\varphi\varphi}|} = 2\pi \, e^{\gamma} |\omega_H| L_H \ .$$

The black hole temperature is given by the Hawking temperature $T_H = \kappa_H / 2\pi$, where κ_H is the surface gravity of the horizon defined in (1.42). In section 3.1 we see that in Weyl coordinates for the ansatz (3.1) the surface gravity reduces to (3.4). We can therefore calculate the surface gravity starting from the latter equation for the Lewis-Weyl-Papapetrou metric. Note that the ansatz (4.1) is a little different from (3.1): to find the correct results we have to made the transformation $e^{-2\gamma} \to e^{-2\gamma} f$ in (3.4). Therefore we get

$$\kappa_H = \frac{e^{-\gamma}}{|\omega_H|} \ .$$

Consequently we have

$$T_H S_H = \frac{\kappa_H \mathcal{A}_H}{8\pi} = \frac{L_H}{4} \ . \tag{5.38}$$

Then, from the latter equation, using (5.36) and recalling that $\Omega_H = 1/\omega_H$, it follows that for the horizon rod

$$M_H = 2 T_H S_H + 2 \Omega_H J_H + \Phi_H Q_H \ , \tag{5.39}$$

that is the usual Smarr formula: it does not contain information on the Dirac and Misner strings, associated with the magnetic charge and the NUT charge.
The global Smarr formula will in principle have additional terms due to the strings. We now discuss some examples of spacetimes with line singularities.

5.5.1 Dyonic Kerr-Newman black hole

Consider the dyonic Kerr-Newman black hole: the metric in Boyer-Lindquist coordinates is given by (1.4). The electromagnetic fields for this solution are given by (1.5). In particular, observe that the φ component of the four-potential can be written as

$$A_\varphi = p \cos\theta - a \sin^2\theta A_t + b_0 \ , \tag{5.40}$$

where b_0 is a constant related to the distribution of the two Dirac strings.
First of all, note that the two assumptions (5.13) and (5.16) are satisfied only for $b_0 = 0$. In fact, since the integral is over a two-sphere at spatial infinity, the index i in both equations is equal to r. By the following asymptotic behaviors

$$\begin{aligned}
&A_t = \frac{q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), && A_\varphi = p\cos\theta + b_0 + \mathcal{O}\left(\frac{1}{r}\right), && F_{rt} = -\frac{q}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \\
&g^{rr} = 1 + \mathcal{O}\left(\frac{1}{r}\right), && g^{tt} = -1 + \mathcal{O}\left(\frac{1}{r}\right), && F_{r\varphi} = \frac{qa\sin^2\theta}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \ , \\
&g^{t\varphi} = \frac{c}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), && g^{\varphi\varphi} = \frac{1}{r^2\sin^2\theta} + \mathcal{O}\left(\frac{1}{r^3}\right), && \sqrt{|g|} = r^2\sin\theta + \mathcal{O}\left(r\right),
\end{aligned}$$

we find that the two integral terms in (5.13) are asymptotically

$$A_t F^{rt} = g^{tt} g^{rr} A_t F_{rt} + g^{t\varphi} g^{rr} A_t F_{r\varphi} = \frac{q^2}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \ ,$$

$$A_\varphi F^{r\varphi} = \frac{qa(p\cos\theta + b_0)}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right) \ .$$

Therefore they fall off at infinity more quickly than $1/r^2$, so the condition for the mass is satisfied. The situation is different for the angular momentum: in this case, the term in (5.16) behaves asymptotically as

$$A_\varphi F^{rt} = \frac{q(p\cos\theta + b_0)}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) .$$

Therefore we find

$$\frac{1}{4\pi}\oint_{\Sigma_\infty} A_\varphi F^{rt}\, d\Sigma_r = \frac{q}{2}\int_0^\pi d\theta\, \sin\theta(p\cos\theta + b_0) = q b_0 .$$

Then condition (5.16) is valid for dyons provided the constant b_0 is set to zero. Then from now on we will consider $b_0 = 0$.
In order to calculated the conserved charge through the rod contributions, it is easier to work in prolate spherical coordinates. Through (A.12), the metric takes the form

$$ds^2 = -f\,(dt - \omega d\varphi)^2 + f^{-1}\left[e^{2\gamma}\sigma^2\left(x^2 - y^2\right)\left(\tfrac{dx^2}{x^2-1} + \tfrac{dy^2}{1-y^2}\right) + \rho^2 d\varphi^2\right] , \tag{5.41}$$

where

$$\begin{aligned}
f &= \frac{f^*}{\Xi} ,\\
f^* &= \sigma^2\left(x^2 - 1\right) - a^2\left(1 - y^2\right) ,\\
\Xi &= (\sigma x + m)^2 + a^2 y^2 ,\\
e^{2\gamma} &= \frac{f^*}{\sigma^2\left(x^2 - y^2\right)} ,\\
\omega &= -\frac{a(1-y^2)(2m(\sigma x + m) - q^2 - p^2}{f^*} .
\end{aligned} \tag{5.42}$$

The Ernst potentials of this metric are given by (4.27) and (4.24). The z axis is divided into three rods: the semi-infinite rods + and − represent the Dirac strings, which in principle may contribute to the mass, the angular momentum and the charge of the spacetime, while the finite rod represents the event horizon.
First of all, consider the horizon rod, where $x = 1$. On this rod we find

$$\omega_H = \omega|_{x=1} = \frac{r_H^2 + a^2}{a} ,$$

where $r_H = m + \sigma$. From (5.34), using (4.26) and (4.23), the total horizon mass is

$$M_H = \frac{\omega_H}{4}\chi\big|_{y=-1}^{y=1} + \frac{1}{2}(A_\varphi\tilde{A}_\varphi)\big|_{y=-1}^{y=1} = m - \frac{p^2 r_H}{r_H^2 + a^2} , \tag{5.43}$$

where the first term is the same calculation to the one of the Kerr black hole without electric and magnetic charges; the second term, since on the horizon $A_\varphi = py$, the only non null contribution in the product with $\tilde{A}_\varphi$ comes from the even part in y of the latter, that is $pr_H/(r_H^2 + a^2y^2)$. Note that the horizon energy M_H is different from the parameter m, because of the presence of a term related to the magnetic charge: we will soon see the reason for this difference.
From (5.28), it follows that the horizon charge is

$$Q_H = -q . \tag{5.44}$$

For the horizon angular momentum, from (5.35) we find

$$J_H = \frac{\omega_H}{2}\left(-\frac{L_H}{2} + M_H - \Phi_H Q_H\right) , \tag{5.45}$$

where

$$\Phi_H = -A_t - \Omega_H A_\varphi = -\frac{q r_H}{r_H^2 + a^2} . \tag{5.46}$$

Note that the horizon angular momentum satisfy $J_H = ma$. In fact, from the previous results, recalling that $L_H = \sigma$ is the length of the horizon rod, we find

$$J_H = \frac{(r_H^2 + a^2)(m - \sigma) - (p^2 + q^2) r_H}{2a} = am .$$

The rotational parameter a is therefore the angular momentum per unit mass.
Consider now the semi-infinite spacelike rods S_+ and S_-, where $y = 1$ and $y = -1$ respectively. Since $\omega_\pm = 0$, there is no contribution of the strings to the angular momentum and to the electric charge: $J_\pm = 0$ and $Q_\pm = 0$. The only non null string contribution is to the energy of the spacetime. In fact, from (5.34) using that $\omega_\pm = 0$, we find

$$M_+ = -\frac{1}{2}\left(A_\varphi \tilde{A}_\varphi\right)|_{x=1,y=1} ,$$

$$M_- = \frac{1}{2}\left(A_\varphi \tilde{A}_\varphi\right)|_{x=1,y=-1} .$$

Using (1.5) and (4.23), we get

$$M_\pm = \frac{p(p r_H \pm aq)}{2(r_H^2 + a^2)} . \tag{5.47}$$

We thus discovered that the Dirac strings are heavy: they give a non-zero contribution to the total mass of the dyonic Kerr-Newman black hole. The sum of the two string masses is

$$M_+ + M_- = \frac{p^2 r_H}{r_H^2 + a^2} .$$

Consequently the total energy becomes

$$M = M_H + M_+ + M_- = m . \tag{5.48}$$

Starting from the horizon Smarr formula (5.39) and using (5.43), where m is equal to the total energy, we find the following global Smarr formula

$$M = 2T_H S_H + 2\Omega_H J + \Phi_H Q + \frac{p^2 r_H}{r_H^2 + a^2} , \tag{5.49}$$

The last additional term can be interpreted as the product of the magnetic charge $P = -p$ computed through (2.36) with a horizon magnetic potential $\tilde{\Phi}_H$ defined by replacing q with p in the electric potential (5.46)

$$\tilde{\Phi}_H := -\frac{p r_H}{r_H^2 + a^2} . \tag{5.50}$$

Therefore the global Smarr formula takes the form

$$M = 2T_H S_H + 2\Omega_H J + \Phi_H Q + \tilde{\Phi}_H P , \tag{5.51}$$

This final result show us that, taking the string contributions into account, the Smarr relation for the total mass includes a magnetic term.

Although both assumptions (5.13) and (5.16) are Gauge dependent, the global Smarr formula is not affected by the addition of a constant term in A_t or A_φ. In fact if we keep a general constant b_0 in A_φ, then it is no longer true that $J_H = J$ because of the contribution from (5.16), indeed we get $J_H = J - qb_0$, where $J = am$. The co-rotating electric potential in the Smarr formula is evaluated at the horizon but relative to infinity, that is

$$\Phi_H = -\left(A_\mu \xi^\mu \big|_{r=r_H} - A_\mu \xi^\mu \big|_{r=\infty} \right) . \tag{5.52}$$

Similar results can be found for the Gauge transformation $A_t \to A_t + c_0$, where c_0 is a constant. In this case the condition (5.13) is no more satisfied. Moreover the imaginary part of the gravitational Ernst potential $\mathcal{E}$ will be modified by an additional term $-2c_0\tilde{A}_\varphi$: this is clear starting from (4.17). Anyhow the global Smarr formula (5.51) is restored.

5.5.2 Kerr-NUT

The Kerr-NUT spacetime is nothing but Kerr solution in presence of a NUT charge n. The metric in Boyer-Lindquist coordinates is given by

$$ds^2 = -\frac{\Delta}{\Xi}\left(dt + P_\theta d\varphi\right)^2 + \Xi\left(\frac{dr^2}{\Delta} + d\theta^2\right) + \frac{\sin^2\theta}{\Xi}\left(a dt - P_r d\varphi\right)^2 , \tag{5.53}$$

where

$$\begin{aligned}
P_\theta &= 2n\cos\theta + 2s - a\sin^2\theta , \\
P_r &= r^2 + a^2 + n^2 - 2as , \\
\Xi &= P_r + aP_\theta = r^2 + (n + a\cos\theta)^2 , \\
\Delta &= r^2 - 2mr + a^2 - n^2 ,
\end{aligned}$$

where s is the same parameter introduced in section 5.1.
First of all, we note that unless $s = 0$ the total angular momentum is divergent. In fact, starting from the Komar integral (2.16), we can rewrite it as a function of the metric components and their derivatives, through steps similar to those seen in section 5.2. Then we find

$$J = -\frac{1}{16\pi}\oint_{\Sigma_\infty} g^{ij} g^{ta} \partial_j g_{\varphi a} d\Sigma_{ti} . \tag{5.54}$$

Remember that the index a labels the coordinates t and φ, while the remaining coordinates, that in this case are r and θ, are labelled by i, j. Since the integral is over a two-sphere at spatial infinity, the index i in the above equation must necessarily be equal to r, so the total angular momentum becomes

$$J = -\frac{1}{16\pi}\oint_{\Sigma_\infty} \sqrt{|g|}\, g^{rr}\left(g^{tt}\partial_r g_{t\varphi} + g^{t\varphi}\partial_r g_{\varphi\varphi}\right) d\theta d\varphi . \tag{5.55}$$

By writing the metric (5.53) in matrix form, the inverse metric can be found by taking the inverse of the matrix metric through the cofactor matrix. By the following asymptotic behaviors

$$\sqrt{|g|} = r^2\sin\theta + \mathcal{O}(r) , \qquad g^{rr} = 1 + \mathcal{O}\left(\frac{1}{r}\right) ,$$

$$g^{t\varphi} = -\frac{2(n\cos\theta + s)}{r^2\sin^2\theta} + \mathcal{O}\left(\frac{1}{r^3}\right) , \qquad \partial_r g_{\varphi\varphi} = 2r\sin^2\theta + \mathcal{O}(1) ,$$

the second term in the integral (5.55) becomes

$$\lim_{r\to\infty}\frac{r}{4}\int_0^\pi d\theta\, \sin\theta\, 2(n\cos\theta + s) = \lim_{r\to\infty} sr .$$

Therefore this contribution to the total angular momentum is divergent unless $s = 0$. The physical angular momentum J can be then finite only for this choice of the parameter s, which corresponds to a symmetrical Misner string configuration.
Henceforward we assume $s = 0$. It is useful to work in prolate spherical coordinates (x, y) instead of the Boyer-Lindquist coordinates. Through (A.12), the metric (5.53) can be written in the form (5.41), where the metric functions have the same form as those in (5.42), except for

$$\Xi = (\sigma x + m)^2 + (ay + n)^2 \ ,$$
$$\omega = -\frac{2ny\sigma^2\left(x^2-1\right) + 2a(1-y^2)(m\sigma x + m^2 + n^2)}{f^*} \ .$$

Of course also the constant σ is different from that in (1.7). In this case its value is

$$\sigma = \sqrt{m^2 + n^2 - a^2} \ .$$

Since the Kerr-NUT is a solution of the vacuum Einstein equations, we consider only the gravitational Ernst potential, which in the presence of a NUT charge is given by

$$\varepsilon = \frac{\sigma x - m + i\,(ay - n)}{\sigma x + m + i\,(ay + n)} \ . \tag{5.56}$$

It can be calculated in a similar way as shown in chapter 4, or it can be directly obtained from the gravitational Ernst potential (4.27) for the dyonic Kerr-Newman metric through the transformation $m \to m + in$. From the above equation it is easy to find the imaginary part of the complex potential

$$\chi = \mathrm{Im}\varepsilon = \frac{2(may - n\sigma x)}{(\sigma x + m)^2 + (ay + n)^2} \ . \tag{5.57}$$

Even in the presence of a NUT charge, the rod structure is the same as that presented in section 3.2 for the Kerr metric.
Consider the horizon rod, which corresponds to $x = 1$ and $y \in [-1, 1]$. From (5.34) it follows that

$$M_H = \frac{\omega_H}{4}\chi\Big|_{y=-1}^{y=+1} = \frac{\omega_H}{4}\left(\chi_+ - \chi_-\right) \ , \tag{5.58}$$

where $\chi_+ = \chi\big|_{x=1,\,y=1}$ and $\chi_- = \chi\big|_{x=1,\,y=-1}$. From (5.57) we get [3]

$$M_H = \frac{a\omega_H}{2r_H} = \frac{m(\sigma + m) + n^2}{r_H} = m + \frac{n^2}{r_H} = \sigma \ , \tag{5.59}$$

where in the last step we used

$$\frac{mr_H + n^2}{r_H} = \frac{\sigma(\sigma + m)}{r_H} = \sigma \ .$$

For the horizon angular momentum, from (5.35) it is immediate to find

$$J_H = \frac{\omega_H}{2}\left(\frac{-L_H}{2} + M_H\right) = \frac{\omega_H}{2}\left[\frac{m(\sigma+m) + n^2 - \sigma(\sigma+m)}{r_H}\right] = \frac{a^2\omega_H}{2r_H} = aM_H \ . \tag{5.60}$$

In a similar way we can calculate the contributions of the two semi-infinite spacelike rods S_+ and S_-, where $y = 1$ and $y = -1$ respectively. Using (5.34) and together with (5.57), we obtain the following results

$$M_+ = \frac{\omega_+}{4}\lim_{x\to\infty}\chi\Big|_1^x = -\frac{\omega_+}{4}\chi_+ = \frac{n(a-n)}{r_H} \ , \tag{5.61}$$

[3] we remind the reader to appendix B, section B.1, for more explicit calculations

$$M_- = \frac{\omega_-}{4} \lim_{x\to\infty} \chi\big|_x^1 = \frac{\omega_-}{4}\chi_- = -\frac{n(a+n)}{r_H} \ , \tag{5.62}$$

where $\omega_\pm = \mp 2n$, while in the last steps we have used

$$\left[(\sigma+m)^2 + (n\pm a)^2\right]\left(\pm an - n^2\right) = 2n\left(\pm ma - n\sigma\right)(\sigma+m) \ .$$

For the string angular momentum, starting from (5.35), we find

$$J_\pm = \frac{\omega_\pm}{2}\left(\frac{-L_\pm}{2} + M_\pm\right) \ . \tag{5.63}$$

The total string contribution to energy can be written as

$$M_+ + M_- = \frac{\omega_-}{4}\left(\chi_+ + \chi_-\right) = -\frac{n^2}{r_H} = m - \sigma \ , \tag{5.64}$$

where in the last step we used: $n^2 = r_H(2\sigma - r_H)$.
Note that the parameter m, as usual, is the total energy of the spacetime; in fact [3]

$$M = M_H + M_+ + M_- = m \ . \tag{5.65}$$

Therefore m does not coincide with the mass of the black hole: the energy of spacetime does not come only from its horizon, but there is a contribution from the Misner strings. Like Dirac strings, Misner strings are heavy and in addition they also contribute to the angular momentum through (5.63). Note that the contribution of one string to the angular momentum is divergent, since $L_\pm = R - \sigma$ with $R \to \infty$, that is the length of the rod is infinite. However, for our symmetrical choice $s = 0$ the sum of the string angular momentum is finite, because $L_+ = L_-$ e $\omega_+ = -\omega_-$, in fact

$$J_+ + J_- = -n(M_+ - M_-) = -\frac{an^2}{r_H} = a(M_+ + M_-) \ .$$

Therefore the total angular momentum is the sum over the three rod contributions

$$J = J_H + J_+ + J_- = aM_H + a(M_+ + M_-) = aM \ . \tag{5.66}$$

The parameter a, as usual, is the ratio between the total angular momentum and the total energy of spacetime. Since the sum of string angular momenta is finite, it is convenient to define a reduced string angular momentum by

$$\tilde{J}_\pm = J_\pm + \frac{\omega_\pm L_\pm}{4} \ . \tag{5.67}$$

This angular momentum can be considered as the finite part of the Misner string angular momentum. Then we get

$$\tilde{J}_+ + \tilde{J}_- = J_+ + J_- = -n(M_+ - M_-) = -\frac{an^2}{r_H} = a(M_+ + M_-) \ .$$

Starting from (5.63) and using the above definition, the global Smarr relation for the Kerr-NUT solution can be written as

$$M = 2T_H S_H + 2\Omega_H J_H + 2\Omega_+ \tilde{J}_+ + 2\Omega_- \tilde{J}_- \ . \tag{5.68}$$

Therefore appear two contribution related to the angular momentum contributions of the two strings $\tilde{J}_+$ and $\tilde{J}_-$, while $\Omega - +$ and Ω_- are the interpreted as the string angular velocities.

5.5.3 Dyonic Kerr-Newman-NUT

We now consider the previous metric with both electric and magnetic charge. The metric in prolate spherical coordinates is the same as that in (5.41), where the metric functions have the same form as those in (5.42), except for

$$\begin{aligned}
\Xi &= (\sigma x + m)^2 + (ay + n)^2 \ , \\
\omega &= -\frac{2ny\sigma^2(x^2-1) + 2a(1-y^2)\left(m\sigma x + m^2 + n^2 - e^2/2\right)}{f^*} \ .
\end{aligned}$$

Also the constant σ is different from that in (1.7). In this case its value is

$$\sigma = \sqrt{m^2 + n^2 - a^2 - q^2 - p^2} \ .$$

The electromagnetic four-vector potential for this solution fits the ansatz (4.2), where the non-null components are given by

$$\begin{aligned}
A_t &= \frac{q(\sigma x + m) - p(ay + n)}{(\sigma x + m)^2 + (ay + n)^2} \ , \\
A_\varphi &= py + [2ny - a(1-y^2)]A_t \ .
\end{aligned}$$

Note that we have set $s = 0$ and $b_0 = 0$, so that both Misner and Dirac strings are symmetrically distributed.
The gravitational Ernst potential is the same as that in the previous section, given by (5.56), while its imaginary part is reported in (5.57). The electromagnetic Ernst potential is given by

$$\mathbf{\Phi} = \frac{q - ip}{\sigma x + m + i\,(ay + n)} \ . \tag{5.69}$$

It can be directly obtained from the complex potential (4.24) for the dyonic Kerr-Newman black hole through the transformation $m \to m + in$; its imaginary part is

$$\tilde{A}_\varphi = \mathrm{Im}\Phi = -\frac{q(ay + n) + p(\sigma x + m)}{(\sigma x + m)^2 + (ay + n)^2} \ . \tag{5.70}$$

Consider the horizon rod [4]. Starting from (5.34), (5.35) and (5.28), we obtain the following results for energy, charge and angular momentum

$$\begin{aligned}
M_H = &\frac{\nu^2[2(m^2+n^2)(\sigma+m) - me^2]}{\nu^4 - 4a^2n^2} + \\
&+ \left(\frac{pe^2}{2} - \mu r_H\right)\left[\frac{q(a+n) + pr_H}{(\nu^2+2an)^2} + \frac{q(n-a) + pr_H}{(\nu^2-2an)^2}\right] \ ,
\end{aligned} \tag{5.71}$$

$$Q_H = -\frac{2\nu^2[2r_H(mq - np) - qe^2]}{\nu^4 - 4a^2n^2} \ , \tag{5.72}$$

$$J_H = \frac{\omega_H}{2}\left(-\sigma + M_H - \Phi_H Q_H\right) \ , \tag{5.73}$$

where $\nu^2 = r_H^2 + n^2 + a^2 = 2\left(m^2 + n^2 + \sigma m - e^2/2\right)$ and $\mu = pm + qn$. The metric function ω evaluated on the horizon and the electric potential of the horizon are respectively

$$\omega_H = \omega(x=1) = \frac{2(m\sigma + m^2 + n^2 - e^2/2)}{a} = \frac{\nu^2}{a} \ , \tag{5.74}$$

[4] In order to avoid being too heavy, all the calculations of this section are reported in appendix B, section B.2

$$\Phi_H = -\frac{pn - qr_H}{\nu^2} \ . \tag{5.75}$$

Consider now the string rods. We obtain the following results

$$M_\pm = \pm\frac{n(ma \mp n\sigma)}{\nu^2 \pm 2an} - \left(\frac{pe^2}{2} - \mu r_H\right)\left[\frac{q(n \pm a) + pr_H}{(\nu^2 \pm 2an)^2}\right] \ , \tag{5.76}$$

$$Q_\pm = -\frac{n[pr_H + q(n \pm a)]}{\nu^2 \pm 2an} \ , \tag{5.77}$$

$$J_\pm = \frac{\omega_\pm}{2}\left(-\frac{L_\pm}{2} + M_\pm - \Phi_\pm Q_\pm\right) \ , \tag{5.78}$$

where

$$\omega_\pm = \mp 2n \ , \tag{5.79}$$

$$\Phi_\pm = \frac{p}{2n} \ . \tag{5.80}$$

As noted in the previous section, the string contribution to the angular momentum is divergent: the same remarks as before con be made. Defining the reduced angular momentum as in (5.67), we see that all these contributions satisfy the following global Smarr relation

$$M = 2T_H S_H + 2\Omega_H J_H + 2\Omega_+ \tilde{J}_+ + 2\Omega_- \tilde{J}_- + \Phi_H Q_H + \Phi_+ Q_+ + \Phi_- Q_- \ . \tag{5.81}$$

Note that in the presence of both Dirac and Misner strings, in addition to a mass and angular momentum contribution, there is also a charge contribution.

Chapter 6

Recent results following the Clément-Gal'tsov approach

In the previous chapter we have seen how to calculate the conserved charges in presence of line singularities: that method was proposed by Clément and Gal'tsov in [12] and, a few months later, was heavily criticized by García-Compeán, Manko and Ruiz in [13]. In this chapter we will comment the relevant aspects of criticism, showing how they can be overcome. Finally, we will analyze the recent literature concerning the Smarr formula of spacetimes with non-null NUT charge: in particular, we will quickly present two different methods, looking for points of connection with Clément-Gal'tsov approach.

6.1 Criticisms of Clément-Gal'tsov approach

In order to better understand the extent of this criticism, we report below some extracts of the paper [13] [1]. First of all, the abstract reads

> *We comment on physical inconsistences of the Clément-Gal'tsov approach to Smarr's mass formula in the presence of magnetic charge. We also point out that the results of Clément and Gal'tsov involving the NUT parameter are essentially based on the known study (dating back to 2006) of the Demiánski-Newman solutions which was not cited by them.*

We immediately see that the criticism concerns only solutions with Dirac strings.
We now report the beginning of the article, where we can see the authors are strongly convinced of the unphysical features of the Clément-Gal'tsov approach

> *In the paper [11], Clément and Gal'tsov considered the mass and angular momentum distributions in the dyonic Kerr-Newman (KN) black-hole spacetime to get the results different from those earlier obtained for this spacetime in [19]. The preprint [19] was later published under a slightly different title [20] better reflecting the topic of the special issue of Classical and Quantum Gravity on black holes and electromagnetic fields, and the paper [11] was not mentioned there because the physical inconsistences in the formulas (4.8) and (4.14) of [11] were so glaring, that we hoped Clément and Gal'tsov would be able to detect these themselves. However, it appears that the aforementioned authors were pretty sure about the correctness of their results because in the recent paper [12] they have extended their approach further to the solutions with the NUT parameter, hinting in passing that the title change of the preprint [19] might have had something*

[1] in the following quotes, references and notation have been slightly modified to fit the notation of our work.

> *to do with the critical tone of their previous work [11]. Therefore, we now feel ourselves obliged to respond the Clément and Gal'tsov's critique, and in what follows we will comment on the physical inconsistences of the papers [11], [12].*

Let us now focus on the criticized physical aspects. The first problem presented is the following

> *the model proposed and advocated by Clément and Gal'tsov as alternative to the usual interpretation of M (the mass fully confined inside the central body) has several frankly unphysical features. First, the semi-infinite strings introduced in [11] have different masses $M_\pm$, which apparently contradicts the equatorial symmetry of the dyonic KN solution [...] requiring $M_+ = M_-$.*

First of all, note that M is not the mass of the central body as the articles states, but it is the total energy of the spacetime, given by the Komar integral (2.15). In section 2.5 we see that, even in a well-known solution such as the Reissner-Nordström black hole, M is the black hole mass added to the external spacetime contribution, related to the electromagnetic energy-momentum tensor. It is true that both contributions can be written as integrals over the event horizon, such as in (5.14) where the only contribution comes from the horizon rod, however it coincides with the black hole mass only in the absence of both electric and magnetic charge.
With regard to the equatorial symmetry, we say that a solution is equatorially symmetric when all the metric fields and the Faraday tensor are even function under the transformation

$$\begin{aligned} z &\to -z \;, \\ \rho &\to \rho \;. \end{aligned} \tag{6.1}$$

In other words $h(\rho, z) = h(\rho, -z)$ for any function that characterizes the solution.
In spherical coordinates the above transformation is equivalent to

$$\begin{aligned} \theta &\to \pi - \theta \;, \\ r &\to r \;. \end{aligned} \tag{6.2}$$

Consider now the dyonic Kerr-Newman black hole: the metric (1.4) is equatorially symmetric; on the other hand the Faraday tensor constructed from (1.5) breaks this symmetry because of the presence of the magnetic charge: for example

$$F_{\theta\varphi}(r, \theta) \neq F_{\theta\varphi}(r, \pi - \theta) \;.$$

Therefore the whole solutions, which includes also the electromagnetic fields, has no equatorial symmetry. Consequently the difference between M_+ and M_- does not contradict any symmetry.
Then the article continues in this way

> *Moreover, it is easy to see that for small values of the magnetic charge p the masses M_+ and M_- of the two strings can even take opposite signs, which introduces undesirable negative masses into a well-behaved solution.*

This observation is correct: in fact, without lost of generality, we can assume the parameters a, q and p to be positive, therefore the rod contribution of the south pole axis to the energy M_- in (5.47) is negative if

$$p(\sigma + m) < aq \;. \tag{6.3}$$

In particular, always exist values of the parameters m, a, q and p such that both (1.8) and (6.3) are satisfied. The presence of a negative contribution to the energy can be attributed

to the fact that the string interact with the black hole by pushing it away (in this case the string is called strut, that can be viewed as an anti-gravitational object). This is more evident for the NUTty spacetimes, where the total string contribution given by the sum of M_+ and M_- is negative, while for the dyonic Kerr-Newman black hole the total string contribution is positive. In any case, it is important that all rod contributions give a total positive energy and this is always verified.
We also note that we are not dealing with a well-behaved solution due to the presence of Dirac strings.
Proceeding in the article we find

> *Mention also that the parameter a in the Clément-Gal'tsov treatment does not represent the total angular momentum per unit mass calculated on the horizon because the parts $S_\pm$ of the symmetry axis have zero angular momenta and nonzero masses, thus contradicting Carter's interpretation [14] of the dyonic KN solution.*

However, exactly on the first page of [14] the rotational parameter a is defined as

$$a = \frac{J}{M} \, , \tag{6.4}$$

where M and J, according to what Carter writes, are respectively the asymptotically defined mass and angular momentum, that is they are respectively defined by the Komar integrals (2.15) and (2.16). In section 5.5.1 we show that the relation (6.4) holds for the dyonic Kerr-Newman black hole since $J_H = J$ and $m = M$; therefore we are therefore consistent with the Carter's definition of the rotational parameter. Note that the above ratio reduces to

$$a = \frac{J_H}{M_H} \tag{6.5}$$

only in the absence of line singularities, where J_H and M_H have also contributions from the elctromagnetic energy-momentum tensor.
The last criticized aspect is the following

> *At the first try, the appearance of the additional term in the mass integral (3.11) of [11] leading to the above (1) [2] could be attributed to the clearly erroneous equations (3.2) of defining the magnetic scalar potential $\tilde{A}_\varphi$. At the same time, even if the calculations of Clément and Gal'tsov are somehow correct, the presence of the term involving the product $A_\varphi \tilde{A}_\varphi$, A_φ being the magnetic component of the electromagnetic 4-potential, must not really produce any effect on the usual physical interpretation of the dyonic KN solution because there are arguments in favor of vanishing of such a term. Indeed, [...] in the case of a magnetic monopole the potential A_φ can be made equal to zero on $S_\pm$ if one treats the Dirac string as a "gauge artifact", which allows for choosing an appropriate value of the integration constant b_0 in the expression of A on each part of the symmetry axis. Then the potential A_φ of the dyonic KN solution [...] will take zero value on S_+ ($y = 1$) after choosing $b_0 = p$, while on the lower part of the symmetry axis S_- ($y = -1$) the potential A_φ vanishes at $b_0 = p$. Consequently, in this case both M_+ and M_- also become zeros, which is consistent with the regularity of the metric on $S_\pm$. Obviously, this approach is equivalent to calculating $M_{S\pm}$ (and M_H too) by means of the usual Tomimatsu's mass integral.*

Here the real difference between the Clément-Gal'tsov approach and the Tomimatsu approach, that is the one used by Manko, comes to light. The Clément-Gal'tsov approach is

[2] the equation (1) of [13] summarize the rod energies (5.43) and (5.47)

nothing but the reformulation of the Tomimatsu approach in terms of the rod structure. In [11] Clément and Gal'tsov, in order to resolve the problem when the Tomimatsu formulas were applied to multi-dyons (it was observed that the resulting values for the black hole parameters failed to obey the standard Smarr relation, but obeyed a generalized Smarr relation with both electric and magnetic contributions), showed a new derivation for Tomimatsu formulas, since Tomimatsu gave very little details of his calculations. This derivation has been presented in terms of the rod structure in the previous chapter: comparing our results with the Tomimatsu formulas in [21], we find out that Tomimatsu have the second term in (5.34) missing.
In the above text García-Compeán, Manko and Ruiz states that the missing term vanishes if we choose two different values for the constant b_0 on each part of the symmetry axis. However, this solution is absurd since we cannot simultaneously choose two different integration constant for the same component of the four-vector electromagnetic potential.

6.2 Alternative approaches to thermodynamics of NUTty spacetimes

6.2.1 Wu-Wu method

In [15] Shuang-Qing Wu and Di Wu propose a systematic way to find the global Smarr formula of spacetimes that contain a nonzero NUT charge, starting from a Christodoulou-Ruffini-type squared-mass formula. The latter formula is a relation between the square of the energy of spacetime, the entropy of the black hole and the other conserved charges, that are charge and angular momentum. Such a formula is very convenient since the first law of black hole thermodynamic can be simply deduced via differentiating that formula with respect to all of its thermodynamic variables, and then the Smarr formula can be easily verified. For example for the Kerr-Newman ($p = 0$) black hole the Christodoulou-Ruffini-type squared-mass formula was found to be

$$M^2 = \frac{\pi}{4S}\left(\frac{S}{\pi} + Q^2\right)^2 + \frac{\pi J^2}{S} \ . \tag{6.6}$$

However, it is necessary to know the conserved charges for spacetimes with a non-null NUT charge in order to write such a formula. Instead of the usual conserved charges related to the symmetries of the specific spacetime, they consider three conserved charges for the Taub-NUT spacetime:

- Komar energy M defined by the standard integral at infinity (2.15): the authors report from other papers the result $M = m$, in accordance with (5.65) [3]. Wu and Wu reports also the result for the horizon mass: $M_H = r_H - m$, in accordance with (5.59).
- Gravitomagnetic mass N or dual mass $\tilde{M}$: the gravitomagnetic mass is calculated from a NUT-potential introduced so that the Einstein equations may be obtained from a three-dimensional Lagrangian density, while the dual mass (2.37) takes the Hodge dual of the Komar mass definition. These two quantities give the same result for spacetime with a zero cosmological constant; in particular they are both identical to the NUT charge n. However, neither of these two charges is associated with a symmetry of spacetime, which was our starting point for defining conserved charges.
- Angular momentum $J_n = mn$. This is a conjecture of the two authors: they suppose that, similarly to the angular $J = ma$ of the Kerr spacetime, an analogous angular

[3] Although reference is made to the results of the Kerr-NUT spacetime, the same results hold for the Taub-NUT spacetime for $a = 0$

momentum con be defined by substituting the rotational parameter a with the NUT charge n. Even though they states that there are a lot of reasons to support such an idea, it remains a pure hypothesis.

Starting from these conserved charges they arrive to the following square-mass formula

$$M^2 = \frac{\left(\mathscr{A}_H - 2N^2\right)^2}{4\mathscr{A}_H} + \frac{J_n^2}{\mathscr{A}_H} , \tag{6.7}$$

where $\mathscr{A}_H$ is the reduced horizon area: $\mathscr{A}_H = A_H/(4\pi)$. The hypothetical angular momentum J_n appears in the Christodoulou-Ruffini-type squared-mass formula for NUTty spacetime in an analogous way to how the standard angular momentum J appears in the corresponding formula for the Kerr-Newman black hole.
We can view (6.7) as an implicit function $M = M(\mathscr{A}_H, J_n, N)$ and then write the first law

$$dM = \frac{\kappa}{2} d\mathscr{A}_H + \tilde{\omega}_H dJ_n + \psi_H dN ,$$

where

$$\kappa = 2\frac{\partial M}{\partial \mathscr{A}_H} , \qquad \tilde{\omega}_H = \frac{\partial M}{\partial J_n} , \qquad \psi_H = \frac{\partial M}{\partial N} .$$

It's then possible to find the Smarr formula

$$M = \kappa \mathscr{A}_H + 2\tilde{\omega}_H J_n + \psi_H N .$$

The same approach is applied to the Kerr-Newman-NUT spacetime: the conserved charges, for this spacetime in addition to the previous ones, are the angular momentum $J = am$ and the electric charge $Q = q$. The squared-mass formula takes the form

$$M^2 = \frac{\left(\mathscr{A}_H - 2N^2 - Q^2\right)^2}{4\mathscr{A}_H} + \frac{J_n^2 + J^2}{\mathscr{A}_H} . \tag{6.8}$$

As in the previous case, by defining

$$\kappa = 2\frac{\partial M}{\partial \mathscr{A}_H} , \qquad \Omega_H = \frac{\partial M}{\partial J} , \qquad \tilde{\omega}_H = \frac{\partial M}{\partial J_n} , \qquad \psi_H = \frac{\partial M}{\partial N} , \qquad \Phi_H = \frac{\partial M}{\partial Q} ,$$

it is immediate to write the first law

$$dM = \frac{\kappa}{2} d\mathscr{A}_H + \Omega_H dJ + \tilde{\omega}_H dJ_n + \psi_H dN + \Phi_H dQ ,$$

and subsequently deduce the Smarr formula

$$M = \kappa \mathscr{A}_H + 2\Omega_H J + 2\tilde{\omega}_H J_n + \psi_H N + \Phi_H Q .$$

We finally note that the NUT charge shows both rotation-like and electromagnetic charge-like characteristics; similar results were found in the previous chapter, although charge characteristic emerges only in presence of an electric or magnetic charge.

6.2.2 Bordo-Gray-Kubizňák method

In [16] Bordo, Gray and Kubizňák propose two different first laws for rotating spacetimes with nonzero NUT charge. We now briefly summarize their method.
As we have seen in section 3.1, each rod of a general stationary and axisymmetric solution is a Killing horizon, associated to the direction of the specific rod. Consider now the dyonic Kerr-Newman-NUT spacetime; it admits three horizons: one is the event horizon of the black

hole, generated by the Killing vector $\xi_H = \partial_t + \Omega_H \partial_\varphi$, while the others are located along the spacelike rods (which account for strings), generated by the Killing vectors $\xi_\pm = \partial_t + \Omega_\pm \partial_\varphi$. Since we are dealing with Killing horizon, we can define a surface gravity of the strings: by using the standard definition (1.42), we find

$$\kappa_+ = \kappa_- = \frac{1}{2n} \,. \tag{6.9}$$

In principle, these surface gravities are a purely formal definition and do not necessarily have the same physical interpretation of temperature, as happens for the horizon surface gravity. Note that, up to a constant factor, they coincide with the angular velocities $\Omega_\pm$ of the strings. Therefore they define what they call Misner potential in the following way

$$\psi = \frac{\kappa_\pm}{4\pi} = \frac{1}{8\pi n} \,. \tag{6.10}$$

This potential is interpreted as the string temperature: however this interpretation is not argued and seems not very clear at the moment. Furthermore there would be a significant physical problem, since different parts of spacetime will have different temperatures; we also expect the strings to be in contact with the event horizon, thus leading to a non-equilibrium state. The authors generically call it Misner potential, since they know it could be also interpreted as an angular velocity.
The first law, or equivalently the Smarr formula, can be written through the Euclidean action. In order to do this, one can note that we can pass from a Lorentzian solution to a Euclidean one through a Wick rotation. The latter consists in the introduction of an imaginary time coordinate $\tau = -it$, where we have to identify τ with period $2\pi/\kappa_H$ to make the metric regular on the horizon (one has also to Wick rotate the NUT parameter, the rotational parameter, the electric charge and the magnetic charge so that both the metric and the electromagnetic potential vector remain real). Such Euclidean metric can be found from an Euclidean action. Starting from the Euclidean action, one can write the corresponding free energy $G = I\beta = I/T$, where I is the action and T is the Hawking temperature (previously denoted by T_H). Therefore they find two different first laws: at first, the free energy is viewed as an implicit function of T_H, ψ, Ω_H, Φ_e (it is the co-rotating electric potential, identical to our Φ_H) and the horizon magnetic charge Q_m^H; the latter has been computed by evaluating (2.36) at the event horizon. Then the thermodynamic quantities are found by taking the derivative of the free energy with respect to the corresponding variable. This first possibility leads to the Smarr formula

$$M = 2T_H S_H + 2\Omega_H J + 2\psi N + \phi_e Q_e + \phi_m Q_m^H \,. \tag{6.11}$$

The second possibility is to view to substitute the horizon magnetic charge with the asymptotic magnetic charge Q_m, that is given by the dual of the electric charge expression (2.4), in the variables of the free energy.

$$M = 2T_H S_H + 2\Omega_H \tilde{J} + 2\psi \tilde{N} + \phi_e Q_e^H + \tilde{\phi}_m Q_m \,, \tag{6.12}$$

where $\tilde{J}$, $\tilde{N}$ and $\tilde{\phi}_m$ are different from J, N and ϕ_m respectively.
In both cases the choice of variables seems to be postulated in such a way that the Smarr formula will take a specific form. Besides, is not clear why we should take into account also the horizon magnetic charge, or the asymptotic magnetic charge. Moreover, it seems that this method requires to already know the conserved charges and the associated "potentials": in fact the authors presented two different methods, suggesting an ambiguity in the choice of the relevant quantities.
We also point out that there is an ambiguity in the Euclidean action for manifolds with

boundary: one can add any boundary term without modifying the equations of motion. However, such boundary terms will modify the conserved charges, since they modify the expression of the free energy. The authors write the Euclidean action for a spacetime with nonzero cosmological constant Λ and then, once the free energy G is written, they take the limit $\Lambda \to 0$. Furthermore they states that NUTty spacetimes have no conical singularity for $\Lambda = 0$; however, in section 5.1 we show that the NUT charge introduces angular defects in the solution: the presence of divergent angular defect make us not sure if such an Euclidean action may be written down.

Below there are two tables summarizing the resulting Smarr formula obtained through the discussed method for the Taub-NUT spacetime and the Kerr-NUT spacetime.
Note that the product between κ and $\mathscr{A}_H$ gives the same result of $2T_H S_H$. Let's briefly analyze the Taub-NUT case: the Smarr formulas obtained one with the Clément-Gal'tsov method and the other with the Wu-Wu method are very different from each other. The comparison between them is quite complicated since the latter method start from the hypothesis of validity of the Christodoulou-Ruffini-type squared-mass formula, from which the first thermodynamic law is derived with a cascade effect, once the conserved charges have been chosen. A point of greater comparison would be possible if we could write a squared-mass formula from the Clément-Gal'tsov Smarr formula. With regarding to the Bordo-Gray-Kubizňák method, it gives similar results to ours: this similarity in the Smarr formulas was quite expected, since the only other thermodynamic quantity that appears in the first law for this spacetime, besides mass and entropy, is the Misner potential, which is proportional to the angular velocities of the strings.
Moreover, we point out that the Clément-Gal'tsov method is based on calculating the conserved charges (mass, angular momentum and electric charge) separately: once the various contributions have been calculated, starting from the horizon Smarr formula, a global Smarr formula is then found. On the other hand, by using the other exhibited methods, once the conserved charge have been chosen a Smarr formula will be automatically satisfied. This makes the latter methods difficult to apply to spacetime for which no Smarr formula is already known.
With regarding to the Kerr-NUT spacetime, first of all note that all the results can be written in a slightly different form by using that

$$r_H^2 + a^2 + n^2 = 2(mr_H + n^2) \ .$$

For the Wu-Wu results the same previous observations can be made. The result of Bordo, Gray and Kubizňák is similar to our also for this spacetime; in fact, it is immediate to see that $J_H = J$. More differences between the two methods come to light when the solution has both electric and magnetic charge (or even when only one of them is present): for example both J in (6.11) and $\tilde{J}$ in (6.12), besides being different from each other, are different from J_H in (5.73). Note that in the cases presented in the tables both $\tilde{N} = N$ and $\tilde{J} = J$, thus the Smarr formula (6.11) and (6.12) are equivalent.

CG	$M = 2T_H S_H + 2\Omega_+(\tilde{J}_+ - \tilde{J}_-)$		
	$M = m$	$T_H = 1/(4\pi r_H)$	$S_H = 2\pi\sigma r_H$
		$\Omega_+ = -\Omega_- = -1/(2n)$	$\tilde{J}_+ = -\tilde{J}_- = n^3/(2r_H)$
WW	$M = \kappa\mathscr{A}_H + 2\tilde{\omega}_H J_n + \psi_H N$		
	$M = m$	$\kappa = 1/(2r_H)$	$\mathscr{A}_H = 2\sigma r_H$
		$\psi_H = -n/\sigma$	$N = n$
		$J_n = mn$	$\tilde{\omega}_H = n/(2\sigma r_H)$
BGK	$M = 2T_H S_H + 2\psi N$		
	$M = m$	$T_H = 1/(4\pi r_H)$	$S_H = 2\pi\sigma r_H$
		$\psi = 1/(8\pi n)$	$N = -4\pi n^3/r_H$

Smarr formulas for Taub-NUT spacetime

CG	$M = 2T_H S_H + 2\Omega_H J_H + 2\Omega_+(\tilde{J}_+ - \tilde{J}_-)$		
	$M = m$	$T_H = \sigma/[2\pi(r_H^2 + a^2 + n^2)]$	$S_H = 2\pi(r_H^2 + a^2 + n^2)$
		$\Omega_H = a/(r_H^2 + a^2 + n^2)$	$J_H = a(r_H^2 + a^2 + n^2)/2r_H$
		$\Omega_+ = -1/(2n)$	$\tilde{J}_+ = n^2(n-a)/(2r_H)$
		$\Omega_- = 1/(2n)$	$\tilde{J}_- = -n^2(n+a)/(2r_H)$
WW	$M = \kappa\mathscr{A}_H + 2\Omega_H J + 2\tilde{\omega}_H J_n + \psi_H N$		
	$M = m$	$\kappa = \sigma/(r_H^2 + a^2 + n^2)$	$\mathscr{A}_H = r_H^2 + a^2 + n^2$
		$\Omega_H = a/\mathscr{A}_H$	$J = ma$
		$\tilde{\omega}_H = n/\mathscr{A}_H$	$J_n = mn$
		$\psi_H = -2nr_H/\mathscr{A}_H$	$N = n$
BGK	$M = 2T_H S_H + 2\Omega_H J + 2\psi N$		
	$M = m$	$T_H = \sigma/[2\pi(r_H^2 + a^2 + n^2)]$	$S_H = 2\pi(r_H^2 + a^2 + n^2)$
		$\Omega_H = a/(r_H^2 + a^2 + n^2)$	$J = a(r_H^2 + a^2 + n^2)/2r_H$
		$\psi = 1/(8\pi n)$	$N = -4\pi n^3/r_H$

Smarr formulas for Kerr-NUT spacetime

Conclusion

In this thesis we presented a way to calculate the conserved charges in GR in presence of line singularities: this happens when studying stationary and axisymmetric spacetime with NUT charge or magnetic charge. After a brief introduction necessary to recall some key concepts and to define the notion of three-dimensional hypersurfaces and surfaces in GR, we defined conserved charges starting from spacetime symmetries through the so-called Komar integral.
Then we introduced two essential formalisms. First of all, we gave the key definitions for understanding the rod structure for stationary and axisymmetric spacetimes with two commuting Killing vector fields; here we found that each rod has the property to be a Killing horizon, associated to the direction of the rod. Secondly we defined the Ernst potentials.
In chapter five we focused on line singularities: all the known types of such singularities are located on semi-infinite spacelike rods. Therefore in Weyl coordinates both black holes and defects can be universally described as rods located on the axis. Then combining the Komar charges with the rod structure formalism and the Ernst potentials, we decomposed the Komar integral into a sum of rod contributions, thus finding an elegant way to calculate conserved charges in the case of rotating black holes, since it involves only functions evaluated in the turning points. The decomposition has a more complicated expression for non-rotating black holes, because is not possible to eliminate some terms that usually go to zero on the axis.
Having applied this method to the dyonic Kerr-Newman spacetime, we found that the Dirac strings are heavy, that is their contribution to the energy of the spacetime is non-zero, leading to a magnetic term in the global Smarr formula. This result has been found for the particular choice $b_0 = 0$ (the additive constant in A_φ) which leads to a symmetrical string configuration. Anyhow, even for $b_0 \neq 0$ the global Smarr formula is restored.
Then we turned our attention to the solution with non-zero NUT charge. In presence of the latter both Misner and cosmic strings are present; if the solution is even only electrically charged, Dirac strings emerge. In the case of the Kerr-NUT spacetime we selected a symmetric configuration of Misner strings in order to guarantee a finite total angular momentum. The same configuration has been adopted for the dyonic Kerr-Newman-NUT spacetime: this more general case shows that the NUT charge contributes to all the conserved charges, that are energy, angular momentum and electric charge.
The flaw of the method employed in this work lies in the starting definition for the conserved charges: the Komar integral returns the correct charges up to a multiplicative constant, which is fundamental. This implies that not all observers measure the same energy and angular momentum: therefore there are privileged observers. In fact such a constant is correctly chosen only for well normalized Killing vector fields: this normalization is not obvious for not asymptotically flat spacetimes. Then a possible outlook is to try applying this method to spacetimes which are not asymptotically Minkowski: we have already seen that the method works for spacetimes which are asymptotically Taub-NUT. Therefore it would be natural to analyze the insertion of an acceleration, which will introduce infinite timelike rods corresponding to acceleration horizons in the solution; such solutions are asymptoti-

cally Rindler. In fact the acceleration would be the only parameter not considered for black holes of the class studied in this thesis, that is the Petrov type D class.
In addition, the Clément-Gal'tsov approach suggests that the different strings we have introduced (Dirac, Misner and cosmic strings) appear to be different characterisation of the same physical object. Indeed, in section 5.5.3 we do not distinguish between the Dirac strings and the Misner strings: we simply treated both south pole axis and north pole axis as a general line singularity.

Finally, by comparison with other methods applied to NUTty spacetimes in the recent

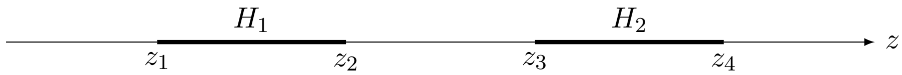

Figure 6.1: rods for a two black hole configuration

literature, we found that the method in this thesis is the only one that can be extended to new spacetimes for which no Smarr formula has been already found. In particular it represents a convenient framework which is useful for the analysis of solutions containing several horizons and defects. An example may be a system of two black holes on the same axis: the equilibrium is ensured by the rod $[z_2, z_3]$ shown in figure 6.1.

Appendix A

A.1 Differential operators

By writing the metric in Weyl coordinates with the Lewis-Weyl-Papapetrou ansatz (4.1), in section 4 we have seen that the Einstein field equations, which usually are written by curved differential operators, can be written in terms of flat differential operators.
For this reason we are interested in flat three-dimensional spacetime in cylindrical coordinates (ρ, z, φ), whose metric is given by

$$ds^2 = d\rho^2 + dz^2 + \rho^2 d\varphi^2 \; . \tag{A.1}$$

For any scalar function $h(\rho, z, \varphi)$ or vector function $\vec{X}(\rho, z, \varphi)$, the gradient, the laplacian and the divergence are respectively

$$\vec{\nabla} h(\rho, z, \varphi) = \left(\frac{\partial h}{\partial \rho}, \frac{\partial h}{\partial z}, \frac{1}{\rho}\frac{\partial h}{\partial \varphi} \right) \; , \tag{A.2}$$

$$\nabla^2 h(\rho, z, \varphi) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial h}{\partial \rho} \right) + \frac{1}{\rho^2}\frac{\partial h}{\partial \varphi} + \frac{\partial^2 h}{\partial z^2} \; , \tag{A.3}$$

$$\vec{\nabla} \cdot \vec{X}(\rho, z, \varphi) = \frac{1}{\rho}\frac{\partial \left(\rho X_\rho\right)}{\partial \rho} + \frac{\partial A_z}{\partial z} + \frac{1}{\rho}\frac{\partial X_\varphi}{\partial \varphi} \; . \tag{A.4}$$

Note that for the spacetimes considered these flat differential operators simplify further since all the scalar and vector functions do not depend on the φ coordinate, associated to the rotational Killing vector field.

A.2 Prolate spherical coordinates

Here we define the prolate spherical coordinates (x, y), which are convenient to use for stationary and axisymmetric solutions. They are related to the Weyl coordinates (ρ, z) by the following transformation

$$\begin{cases} \rho(x, y) = \sigma\sqrt{\left(x^2 - 1\right)\left(1 - y^2\right)} \\ \\ z(x, y) = \sigma x y \end{cases} \tag{A.5}$$

where σ is a positive constant. The ranges

$$\rho \geq 0 \; , \qquad -\infty < z < \infty \; ,$$

of the Weyl coordinates make x and y have ranges

$$x \geq 1 \; , \qquad -1 \leq y \leq 1 \; .$$

The inverse transformation of (A.5) is given by

$$\begin{cases} x(\rho,z) = \dfrac{R_+ + R_-}{2\sigma} \ , \\ y(\rho,z) = \dfrac{R_+ - R_-}{2\sigma} \ , \end{cases} \tag{A.6}$$

where

$$\begin{cases} R_+ = \sqrt{\rho^2 + (z+\sigma)^2} \ , \\ R_- = \sqrt{\rho^2 + (z-\sigma)^2} \ . \end{cases} \tag{A.7}$$

We are interested in how the metric changes under this coordinate transformation. In general the transformation law of basis dual vectors under a change of coordinates is

$$dy^\mu = \frac{\partial y^\mu}{\partial x^\alpha} dx^\alpha \quad \Rightarrow \quad dy^\mu dy^\nu = \frac{\partial y^\mu}{\partial x^\alpha}\frac{\partial y^\nu}{\partial x^\beta} dx^\alpha dx^\beta \ . \tag{A.8}$$

In our case, passing from the prolate spheroidal coordinates (x,y) to the Weyl coordinates (ρ,z), we find

$$d\rho^2 + dz^2 = dx^2 \left[\left(\frac{\partial\rho}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right] + dy^2 \left[\left(\frac{\partial\rho}{\partial y}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] + dxdy \left(\frac{\partial\rho}{\partial x}\frac{\partial\rho}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y}\right) \ .$$

From (A.5) follows that

$$\begin{aligned} \frac{\partial\rho}{\partial x} &= \frac{\sigma x\sqrt{1-y^2}}{\sqrt{x^2-1}} \ , \qquad & \frac{\partial z}{\partial x} &= \sigma y \ , \\ \frac{\partial\rho}{\partial y} &= -\frac{\sigma y\sqrt{x^2-1}}{\sqrt{1-y^2}} \ , \qquad & \frac{\partial z}{\partial y} &= \sigma x \ . \end{aligned} \tag{A.9}$$

Therefore we find

$$d\rho^2 + dz^2 = \sigma^2 \left(x^2 - y^2\right)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) \ . \tag{A.10}$$

Note that the term $dxdy$ does not appear. This transformation is useful since in (4.1) the block in (ρ,z) is diagonal and $g_{\rho\rho} = g_{zz}$, so there is the factor $d\rho^2 + dz^2$.
We can see how the metric changes also starting from the transformation of the metric components. The general transformation law for the metric is

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu} g_{\alpha\beta} \ , \tag{A.11}$$

where the symbol $'$ indicates the components of the coordinate system $\{y^\mu\}$. By (A.9) it is easy to find that $g_{xy} = 0$ for the ansatz (4.1). Thus the (ρ,z)-block remains diagonal also in prolate spherical coordinates. Actually this is true only in this particular case where $g_{\rho\rho} = g_{zz}$; in general we will find $g_{xy} \neq 0$.

The prolate spherical coordinates can also be related to the coordinates (r,θ) through the following transformation

$$\begin{cases} x(r) = \dfrac{r-m}{\sigma} \ , \\ y(\theta) = \cos\theta \ . \end{cases} \tag{A.12}$$

In this case if the original metric is diagonal in (r,θ), it remains diagonal also in (x,y) since the transformation matrix is diagonal.

A.3 Taub-NUT and solitons

Here we show how to write the components $g_{\varphi\varphi}$ and $g_{\rho\rho}$ for the Taub-NUT metric (5.2) in terms of the solitons defined by

$$\begin{cases} \mu_1 = \sigma - z + R_- = (y-1)(m-r-\sigma) \ , \\ \\ \mu_2 = -\sigma - z - R_+ = (y+1)(m-r-\sigma) \ , \end{cases} \tag{A.13}$$

where R_+ and R_- are defined by (A.7).
Inverting these equations we can write the spherical coordinates (r, y) in terms of the solitons as

$$r = \frac{\mu_1 - \mu_2}{2} - \sigma + m \ , \tag{A.14}$$

$$y = \frac{\mu_1 + \mu_2}{\mu_2 - \mu_1} \ . \tag{A.15}$$

Applying the latter equations to the Taub-NUT metric it is immediate to find the $g_{\varphi\varphi}$ component in terms of the solitons

$$\begin{aligned} g_{\varphi\varphi} = & \frac{4n^2(\mu_1+\mu_2)(4\sigma^2 - (2\sigma - \mu_1 + \mu_2)^2)}{(\mu_1 - \mu_2)\left[4\sigma^2 - 4m(2\sigma - \mu_1 + \mu_2) + (2\sigma - \mu_1 + \mu_2)^2\right]} \\ & - \frac{4\mu_1\mu_2\left[n^2 + (m - \sigma + \mu_1/2 - \mu_2/2)^2\right]}{(\mu_1 - \mu_2)^2} \ . \end{aligned} \tag{A.16}$$

With regard to the $g_{\rho\rho}$ component, the procedure is more complicated since we have to apply the transformation (A.11). However, from (4.1) we know that

$$g_{\rho\rho} = f^{-1} e^{2\gamma} \ ,$$

where for the Taub-NUT metric f is given by (5.3), while $e^{2\gamma}$ is given by the third equation in (5.42) setting $a = 0$. We can therefore simply substitute (A.14) and (A.15) in the above expression for $g_{\rho\rho}$ to get

$$g_{\rho\rho} = -\frac{8\mu_1\mu_2\left[-2\sigma^2\mu_1\mu_2(\rho^2+\mu_1^2)(\rho^2+\mu_2^2)+\sigma^2(\rho^2+\mu_1\mu_2)^2(\mu_1^2+\mu_2^2)+m\sigma(\mu_1-\mu_2)^2(\mu_1^2\mu_2^2-\rho^4)\right]}{(\rho^2+\mu_1^2)(\rho^2+\mu_2^2)(\mu_1-\mu_2)^2(\rho^2+\mu_1\mu_2)^2} \ . \tag{A.17}$$

Is not obvious we can write $g_{\rho\rho}$ in this way starting from (A.14) and (A.15), but starting from (A.13) it easy to verify that it leads back to the known form in (r, y) coordinates.

Appendix B

B.1 Kerr-NUT

Here we provide some steps for the calculation of the horizon energy contribution in the case of the Kerr-NUT solution.
Let d_+ and d_- be defined by $d_+ = (\sigma + m)^2 + (a + n)^2$ and $d_- = (\sigma + m)^2 + (n - a)^2$. Starting from (5.58) we find

$$\begin{aligned}
\chi_+ - \chi_- &= \frac{2d_-(ma - n\sigma) + 2d_+(ma + n\sigma)}{d_+ d_-} = \\
&= \frac{8(m^3 a + mn^2 a + \sigma m^2 a + \sigma n^2 a)}{4[(m^2 + n^2)^2 + 2\sigma m(m^2 + n^2) - a^2 n^2 + \sigma^2 m^2]} = \\
&= \frac{2a(m^2 + n^2)(\sigma + m)}{(m^2 + n^2)(\sigma + m)^2} = \\
&= \frac{2a}{r_H} \, .
\end{aligned}$$

Similarly, we find

$$\chi_+ + \chi_- = -\frac{2n(m^2 + n^2)(\sigma + m)}{(m^2 + n^2)(\sigma + m)^2} = -\frac{2n}{r_H} \, .$$

We used this last formula in (5.65).

B.2 Dyonic-Kerr-Newman-NUT

Here we provide some steps to get the results reported in section 5.5.3 for the rod contributions to the conserved charges.
Consider the horizon rod H; starting from (5.34), the energy contribution is

$$M_H = \frac{\omega_H}{4}(\chi_+ - \chi_-) + \frac{1}{2}\left[\left(A_\varphi \tilde{A}_\varphi\right)_{y=1} - \left(A_\varphi \tilde{A}_\varphi\right)_{y=-1}\right] \, .$$

Let's rewrite in a different way the quantity $\chi_+ - \chi_-$ through the following steps

$$\begin{aligned}
\chi_+ - \chi_- &= \frac{2d_-(ma - n\sigma) + 2d_+(ma + n\sigma)}{d_+ d_-} = \\
&= \frac{4(2m^3 a + 2mn^2 a + 2\sigma m^2 a + 2\sigma n^2 a - mae^2)}{4(m^2 + n^2 + \sigma m - na - e^2/2)(m^2 + n^2 + \sigma m + na - e^2/2)} = \\
&= \frac{4a[2(m^2 + n^2)(\sigma + m) - me^2]}{\nu^4 - 4a^2 n^2} \, ,
\end{aligned}$$

where d_+ and d_- have been defined in the previous section.
Therefore the first term becomes

$$\frac{\omega_H}{4}(\chi_+ - \chi_-) = \frac{\nu^2[2(m^2 + n^2)(\sigma + m) - me^2]}{\nu^4 - 4a^2 n^2} \, .$$

Consider now the second contribution in M_H: it is a difference of two terms. The first is

$$A_\varphi \tilde{A}_\varphi\big|_{y=1} = -\frac{[q(a+n)+p(\sigma+m)]\{pd_+ + 2n[q(a+n)+p(\sigma+m)]\}}{d_+^2} \ .$$

Let's consider the second factor to numerator of the last expression

$$\begin{aligned}
&pd_+ + 2n[q(a+n)+p(\sigma+m)] = \\
&= [p(\sigma+m)^2 + p(a+n)^2 + 2nq(\sigma+m) - 2np(a+n)] = \\
&= p^3 + q^2p - 2m^2p - 2\sigma mp - 2\sigma nq - 2mnq = \\
&= p(p^2+q^2) - 2(pm+qn)(\sigma+m) = \\
&= pe^2 - 2\mu r_H \ ,
\end{aligned}$$

while the denominator can be written as

$$d_+^2 = \left(\nu^2 + 2an\right)^2 \ .$$

The second term is

$$A_\varphi \tilde{A}_\varphi\big|_{y=-1} = \frac{[q(n-a)+p(\sigma+m)]\{pd_- + 2n[q(n-a)+p(\sigma+m)]\}}{d_-^2} \ .$$

Let's consider the second factor to numerator of the last expression

$$\begin{aligned}
&pd_- + 2n[q(n-a)+p(\sigma+m)] = \\
&= [p(\sigma+m)^2 + p(n-a)^2 + 2nq(\sigma+m) - 2np(n-a)] = \\
&= 2m^2p - p^3 - q^2p + 2\sigma mp + 2\sigma nq + 2mnq = \\
&= \ -p(p^2+q^2) + 2(pm+qn)(\sigma+m) = \\
&= \ -pe^2 + 2\mu r_H \ ,
\end{aligned}$$

while the denominator can be written as

$$d_-^2 = \left(\nu^2 - 2an\right)^2 \ .$$

Therefore we can write

$$\frac{1}{2} A_\varphi \tilde{A}_\varphi\big|_{y=-1}^{y=+1} = \left(\frac{pe^2}{2} - \mu r_H\right)\left[\frac{q(a+n)+pr_H}{(\nu^2+2an)^2} + \frac{q(n-a)+pr_H}{(\nu^2-2an)^2}\right] \ .$$

Finally we arrive at

$$\begin{aligned}
M_H = \ &\frac{\nu^2[2(m^2+n^2)(\sigma+m) - me^2]}{\nu^4 - 4a^2n^2} + \\
&+ \left(\frac{pe^2}{2} - \mu r_H\right)\left[\frac{q(a+n)+pr_H}{(\nu^2+2an)^2} + \frac{q(n-a)+pr_H}{(\nu^2-2an)^2}\right] \ ,
\end{aligned} \tag{B.1}$$

which is the same relation written in (5.71).
Consider now the charge contribution, which, starting from (5.28), is given by

$$Q_H = \frac{\omega_H}{2} \tilde{A}_\varphi\big|_{y=-1}^{y=+1} \ .$$

Firstly, we do not consider the quantity $\omega_H/2$ and we get

$$\begin{aligned}
\tilde{A}_\varphi\big|_{y=-1}^{y=+1} &= \frac{d_+[q(n-a)+p(\sigma+m)] - d_-[q(a+n)+p(\sigma+m)]}{d_+d_-} = \\
&= \frac{2a[qe^2 - 2(m^2q + m\sigma q - n\sigma p - mnp)]}{\nu^4 - 4a^2n^2} = \\
&= -\frac{2a[2r_H(mq-np) - qe^2]}{\nu^4 - 4a^2n^2} \ .
\end{aligned}$$

Therefore we find

$$Q_H = -\frac{2\nu^2[2r_H(mq - np) - qe^2]}{\nu^4 - 4a^2n^2} , \tag{B.2}$$

which is equal to (5.72).
We now calculate the electric potential. Let's define $h_y = q(\sigma + m) - p(ay + n)$ and $d_y = (\sigma + m)^2 + (ay + n)^2$. Then from the definition (5.22) we obtain

$$\begin{aligned}\Phi_H &= -(A_t + \Omega_H A_\varphi) = \\ &= -\left[\frac{h_y}{d_y} + \frac{a}{\nu^2}\frac{pyd_y + [2ny - a(1-y^2)]h_y}{d_y}\right] \\ &= \frac{-h_y\nu^2 - payd_y - [2nay - a^2(1-y^2)]h_y}{\nu^2\, d_y} .\end{aligned}$$

Let's consider the numerator of the last expression

$$\begin{aligned}&- q(\sigma + m)d_y + p(ay + n)d_y - payd_y - 2nayh_y + a^2(1-y^2)h_y = \\ =&- q(\sigma + m)\left[(\sigma + m)^2 + n^2 + a^2y^2 + 2nay\right] + pn\left[(\sigma + m)^2 + n^2 + a^2y^2 + 2nay\right] = \\ =&[pn - q(\sigma + m)]d_y .\end{aligned}$$

Thus we arrive at

$$\Phi_H = \frac{pn - q(\sigma + m)}{\nu^2} , \tag{B.3}$$

which is the same potential written in (5.75).
Consider now the string rods and remember that $y = 1$ on the rod S_+ while $y = -1$ on the rod S_-. Starting from (5.34), the string energy contribution is

$$\begin{aligned}M_+ &= \frac{\omega_+}{4}\lim_{x\to\infty}\chi\big|_1^x + \frac{1}{2}\lim_{x\to\infty}A_\varphi\tilde{A}_\varphi\big|_1^x = -\frac{\omega_+}{4}\chi_+ - \frac{1}{2}\left(A_\varphi\tilde{A}_\varphi\right)_{x=1} , \\ M_- &= \frac{\omega_-}{4}\lim_{x\to\infty}\chi\big|_x^1 + \frac{1}{2}\lim_{x\to\infty}A_\varphi\tilde{A}_\varphi\big|_x^1 = \frac{\omega_-}{4}\chi_- + \frac{1}{2}\left(A_\varphi\tilde{A}_\varphi\right)_{x=1} ,\end{aligned}$$

where

$$\begin{aligned}\chi_+ &= \frac{2(ma - n\sigma)}{\nu^2 + 2na} , \\ \chi_- &= -\frac{2(ma + n\sigma)}{\nu^2 - 2na} .\end{aligned}$$

The other quantities present in $M_\pm$ were previously calculated, so we immediately arrive at (5.76).
Starting from (5.28), the string charge contribution is

$$\begin{aligned}Q_+ &= \frac{\omega_+}{2}\lim_{x\to\infty}\tilde{A}_\varphi\big|_1^x = -\frac{\omega_+}{2}(\tilde{A}_\varphi)_{x=1} , \\ Q_- &= \frac{\omega_-}{2}\lim_{x\to\infty}\tilde{A}_\varphi\big|_x^1 = \frac{\omega_-}{2}(\tilde{A}_\varphi)_{x=1} ,\end{aligned}$$

so we directly get (5.77) using previous calculations.
We finally calculate the electric potential on the string rods. Consider the rod S_+ and define $h_{x_+} = q(\sigma x + m) - p(a + n)$ and $d_{x_+} = (\sigma x + m)^2 + (a + n)^2$; from the definition (5.22), where $\xi = \partial_t + \Omega_+\partial_\varphi$, we obtain

$$\begin{aligned}\Phi_+ &= -(A_t + \Omega_+ A_\varphi) = \\ &= -\frac{h_{x_+}}{d_{x_+}} + \frac{1}{2n}\frac{pd_{x_+} + 2nh_{x_+}}{d_{x_+}} \\ &= \frac{-2nq(\sigma x + m) + 2np(a + n) + pd_{x_+} + 2nq(\sigma x + m) - 2np(a + n)}{2nd_{x_+}} = \\ &= \frac{p}{2n} .\end{aligned}$$

Consider now the rod S_- and define $h_{x_-} = q(\sigma x + m) - p(n-a)$ and $d_{x_-} = (\sigma x + m)^2 + (n-a)^2$; from the definition (5.22), where $\xi = \partial_t + \Omega_- \partial_\varphi$, we obtain

$$\begin{aligned}
\Phi_- &= -(A_t + \Omega_- A_\varphi) = \\
&= -\frac{h_{x_-}}{d_{x_-}} - \frac{1}{2n}\frac{-pd_{x_-} - 2nh_{x_-}}{d_{x_-}} \\
&= \frac{-2nq(\sigma x + m) + 2np(n-a) + pd_{x_-} + 2nq(\sigma x + m) - 2np(n-a)}{2nd_{x_-}} = \\
&= \frac{p}{2n} = \Phi_+ \,.
\end{aligned}$$

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