UNIVERSITÀ DEGLI STUDI DI MILANO

FACOLTÀ DI SCIENZE E TECNOLOGIE
DIPARTIMENTO DI FISICA
Corso di Laurea triennale in Fisica (L-30)

Ernst solution generating techniques for Einstein-Maxwell theory in 5d

Relatore : Prof. Luca Guido Molinari
Relatore esterno: Dott. Marco Astorino
Correlatore: Dott. Adriano Viganò

Tesi di Laurea di:
Andrea Corti
Matricola 933332

Anno Accademico 2020/2021

Introduction

General Relativity is the current theory of gravitation universally accepted in modern physics. It was developed by Albert Einstein and published in 1915 becoming the last great classical theory before quantum revolution. Spacetime is identified with a four-dimensional differentiable manifold equipped with a pseudo-Riemannian metric $g_{\mu\nu}$. This tensor can be thought as a field defined on every point of the spacetime and it is described by the famous Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu} \quad , \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, which describes the energy and matter distribution, and R, $R_{\mu\nu}$ are respectively Ricci scalar and Ricci tensor, both defined by the metric $g_{\mu\nu}$. The synthetic form of these equations actually hides the complexity of ten coupled, nonlinear PDEs, and despite the fact that particular solutions were found early after the publication of Einstein's paper (for example Schwarzschild spacetime in 1916), the aim of finding a general one seems out of reach still nowadays. By reason of this complexity, situations with peculiar symmetry are usually considered and different solution generating techniques were developed over the years: in this frame axisymmetric stationary metrics represent a large class of solutions of physical interest. Considering this family, at the end of the sixties Ernst found the way to recast the explicit form of (1) into simpler equations. In his two original papers [2] and [3] he considers respectively the vacuum case and then the more general electrovacuum condition where $T_{\mu\nu}$ brings only the electromagnetic contribution (in this case Einstein equations are coupled to Maxwell equations $\nabla_\mu F^{\mu\nu} = 0$ and we speak of Einstein-Maxwell theory). Although Ernst equations are still difficult to solve, they provide a natural context to develop solution generating methods based on symmetry transformations (as shown for example by Kinnersley in [5]). In principle one can ask why is there any advantage in using Ernst formulation to study symmetries. Indeed, it is possible to perform the analysis also on the raw Einstein equations, but in this case the only transformations one can find are trivial: the metric produced is equivalent to the seed by means of a change of coordinates. In this way we can understand that recasting equations using Ernst method is not just a matter of simplifying the form of the system: it produces non trivial symmetries that can be used to find new solutions.

The first chapter of this thesis deals with Ernst formulation of axially symmetric 4d gravitational problem, showing how to introduce new potentials in order to simplify the equations. In the second chapter we will give a brief introduction to Lie point symmetries as a method to generate new solutions once a seed metric is known. Through the analysis of the action principle that generates Ernst equations, we will derive the full symmetry group characterizing the system. Then, in order to show how to practically use the symmetry transformations, we will use one of them to generate Kerr-Newman metric from Kerr black hole.

Five dimensional gravity was first considered in 1919 by Kaluza aiming to unify gravity with classical electrodynamics. The fundamental idea is that electromagnetism can be described as a purely geometric effect considering a five dimensional manifold. The equivalent four dimensional system (coupled to a new scalar field called "dilaton") can be retrieved afterwards by means of dimensional reduction. In 1926 Klein conjectured that the extra-spatial dimension is real, though giving a quantum interpretation if it and defining what nowadays is known as Kaluza-Klein theory. Despite the quantum interpretation of Klein, the theory was built in a classical frame, circumstance that caused it to be forgotten for some years until the rise of new unification theories, such as string theory, which needed higher dimensional spaces to try a consistent agreement between gravitation and quantum physics. So we can see that studying five or higher dimensional theories of gravity is interesting both from a theoretical point of view (the possible existence of these dimensions need to be tested with particle accelerators) and for the fact that the considered system is equivalent to a four dimensional one.

The third chapter deals with higher dimensional gravity, with particular attention to the five dimensional case. Specifically, we will first modify a transformation presented in [7] in order to add electric charge to Tangherlini black hole (the higher dimensional version of Schwarzschild). However, the main purpose of this section is to show that also in the context of five dimensional gravity it is possible to introduce Ernst potentials to study Lie point symmetries and generate new solutions. After recasting the problem by means of suitable functions, we will show that the formalism is useful to recover and generalize the result used to charge Tangherlini. The last step we will take consists in exploiting the generalized transformation in order to find a new possible metric.

Contents

Chapter 1

Ernst potentials

In this section we will show how to recast Einstein-Maxwell equations in the context of stationary axisymmetric solutions. The method is based on the definition of two complex potentials and was achieved by Ernst in two different papers [2], [3] at the end of the sixties. This formalism is a useful tool both for writing the equations that describe the system in a simpler and more compact form, and for supplying methods that allow to generate new solutions from already known ones. First of all, Einstein-Maxwell theory is represented by the the following couple of equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2\Big[F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}F^{\rho\sigma}\Big] \quad , \tag{1.1}$$

$$\nabla_{\mu} F^{\mu\nu} = 0 \quad , \tag{1.2}$$

where the unknown fields to be found are the metric $g_{\mu\nu}$ and the electromagnetic Faraday tensor $F_{\mu\nu}$, defined by $F_{\mu\nu} := \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$. Inhomogeneous Maxwell equations are represented by (1.2), while the homogeneous ones ($\partial_{[\mu}F_{\nu\rho]} = 0$) are identically satisfied because of the definition of $F_{\mu\nu}$. The right hand side of the first equation shows the explicit expression of the electromagnetic energy-momentum tensor. As a matter of choice, Einstein-Maxwell theory can be alternatively expressed more concisely using the following action functional

$$S[g_{\mu\nu}, F_{\mu\nu}] = \int d^4x \sqrt{-g} \Big[R - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}\Big] \quad . \tag{1.3}$$

These equations can be specified assuming particular symmetries of the physical context we are considering. In this case we are interested in stationary axisymmetric spacetimes, which are characterized by two commuting Killing vectors ∂_t and ∂_{φ}. Under this assumption the most general metric owning these symmetries can be written in the Lewis-Weyl-Papapetrou form

$$\mathrm{d}s^2 = -f(\mathrm{d}t - \omega \mathrm{d}\varphi)^2 + f^{-1}[\rho^2 \mathrm{d}\varphi^2 + e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2)] \quad , \tag{1.4}$$

where f, ω and γ are functions that depend only on the non-Killing variables z, ρ. The most general electromagnetic potential one-form compatible with the spacetime symmetries is given by $A = At(\rho, z)dt + A\varphi(\rho, z)d\varphi$. In term of the functions mentioned above Einstein equations (EE) (1.1) assume the form (actually Einstein equations do not immediately present this simple structure, in appendix A we show how to obtain these formulas)

$$\nabla \cdot \Big[\rho^{-2} f^2 \nabla\omega + 4\rho^{-2} f A_t (\nabla A_\varphi + \omega \nabla A_t)\Big] = 0 \quad , \tag{1.5}$$

$$f\nabla^2 f = (\nabla f)^2 - \rho^{-2} f^4 (\nabla\omega)^2 + 2f\Big[(\nabla A_t)^2 + \rho^{-2} f^2 (\nabla A_\varphi + \omega\nabla A_t)^2\Big] \quad , \tag{1.6}$$

while Maxwell equations (ME) (1.2) become

$$\nabla \cdot \Big[\rho^{-2} f (\nabla A_\varphi + \omega\nabla A_t)\Big] = 0 \quad , \tag{1.7}$$

$$\nabla \cdot \Big[f^{-1}\nabla A_t - \rho^{-2} f\omega(\nabla A_\varphi + \omega\nabla A_t)\Big] = 0 \quad , \tag{1.8}$$

where ∇ and ∇^2 are understood as the usual flat differential operators of multivariable calculus. Exploiting the fact that the functions do not depend on the coordinate φ and expanding the differential operators in cylindrical coordinates it is possible to verify that the following relation is trivially satisfied regardless of the choice of $\hat{A}_\varphi$

$$\nabla \cdot \Big(\rho^{-1}\hat{\varphi} \times \nabla\hat{A}_\varphi\Big) = 0 \quad , \tag{1.9}$$

where $\hat{\varphi}$ is a unit vector in the azimuthal direction. Last equation may be seen as an integrability condition for the existence of a magnetic scalar potential $\hat{A}_\varphi$. If we set $\hat{A}_\varphi$ such that

$$\rho^{-1} f (\nabla A_\varphi + \omega\nabla A_t) =: \hat{\varphi} \times \nabla\hat{A}_\varphi \quad , \tag{1.10}$$

equation (1.7) is necessarily satisfied in virtue of (1.9). Using $\hat{\varphi} \times (\hat{\varphi} \times \nabla\hat{A}_\varphi) = -\nabla\hat{A}_\varphi$, latter formula can be recast as an explicit expression for $\nabla\hat{A}_\varphi$

$$\nabla\hat{A}_\varphi = -\rho^{-1} f \hat{\varphi} \times (\nabla A_\varphi + \omega\nabla A_t) \quad . \tag{1.11}$$

Applying now the divergence operator to the former equation and taking advantage of (1.9) one more time one can find that

$$\nabla \cdot \Big[f^{-1}\nabla\hat{A}_\varphi - \rho^{-1}\omega\hat{\varphi} \times \nabla A_t\Big] = 0 \quad . \tag{1.12}$$

Comparing this equation with (1.8), which assumes the following form in term of $\hat{A}_\varphi$

$$\nabla \cdot \Big[f^{-1}\nabla A_t - \rho^{-1}\omega\hat{\varphi} \times \nabla\hat{A}_\varphi\Big] = 0 \quad , \tag{1.13}$$

it is possible to understand the advantage of introducing the complex potential

$$\mathbf{\Phi} := A_t + i\hat{A}_\varphi \tag{1.14}$$

which allows us to merge (1.12) and (1.13) into one complex equation

$$\nabla \cdot \Big[f^{-1}\nabla\mathbf{\Phi} - i\rho^{-1}\omega\hat{\varphi} \times \nabla\mathbf{\Phi} \Big] = 0 \tag{1.15}$$

that carries all the information contained in ME. Operating with the following vector relations

$$\nabla \cdot \Big(\rho^{-1}\hat{\varphi} \times A_t \nabla\hat{A}_\varphi \Big) = -\nabla\hat{A}_\varphi \times \nabla A_t = \nabla A_t \times \nabla\hat{A}_\varphi = -\nabla \cdot \Big(\rho^{-1}\hat{\varphi} \times \hat{A}_\varphi \nabla A_t \Big) \quad , \tag{1.16}$$

the first Einstein equation (1.5) can be written in the form

$$\nabla \cdot \Big[\rho^{-2} f^2 \nabla\omega - 2\rho^{-1}\hat{\varphi} \times \mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \Big] = 0 \quad , \tag{1.17}$$

which suggests, along the lines of integrability condition (1.9), the introduction of a new scalar potential χ (known as "twist potential") such that

$$\rho^{-1} f^2 \nabla\omega - 2\hat{\varphi} \times \mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}) =: \hat{\varphi} \times \nabla\chi \quad . \tag{1.18}$$

This formula can be treated likewise (1.10) in order to get both an explicit definition of χ

$$\nabla\chi = -\rho^{-1} f^2 \hat{\varphi} \times \nabla\omega - 2\,\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \tag{1.19}$$

and a new equation

$$\nabla \cdot \Big[f^{-2} \Big(\nabla\chi + 2\,\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \Big) \Big] = 0 \tag{1.20}$$

equivalent to (1.5). The definition of χ can be used also to express in another form Maxwell equations: we can also apply divergence to the second term both of (1.12) and (1.13) in order to substitute $\nabla\omega$ with the twist $\nabla\chi$ and express the equations in function of the new potentials χ and $\hat{A}_\varphi$

$$\nabla \cdot \Big[f^{-1}\nabla A_t \Big] - f^{-2}\Big[\nabla\chi + 2(A_t\nabla\hat{A}_\varphi - \hat{A}_\varphi\nabla A_t) \Big] \cdot \nabla\hat{A}_\varphi = 0 \quad , \tag{1.21}$$

$$\nabla \cdot \Big[f^{-1}\nabla \hat{A}_\varphi \Big] - f^{-2}\Big[\nabla\chi + 2(A_t\nabla\hat{A}_\varphi - \hat{A}_\varphi\nabla A_t) \Big] \cdot \nabla A_t = 0 \quad . \tag{1.22}$$

At last, from the square modulus of (1.18) one finds that

$$\rho^{-2} f^4 (\nabla\omega)^2 = \Big[\nabla\chi + 2\,\mathrm{Im}(\Phi^*\nabla\Phi) \Big]^2 \quad , \tag{1.23}$$

while from the definition of $\mathbf{\Phi}$

$$2f\nabla\mathbf{\Phi} \cdot \nabla\mathbf{\Phi}^* = 2f(\nabla A_t)^2 + 2\rho^{-2} f^3 (\nabla A_\varphi - \omega\nabla A_t)^2 \quad . \tag{1.24}$$

These relations allow us to write the second Einstein equation (1.6) as

$$f\nabla^2 f = (\nabla f)^2 - \Big[\nabla\chi + 2\,\mathrm{Im}(\mathbf{\Phi}^*\nabla\mathbf{\Phi})\Big]^2 + 2f\nabla\mathbf{\Phi}\cdot\nabla\mathbf{\Phi}^* \quad . \tag{1.25}$$

If we introduce a second complex function $\boldsymbol{\mathcal{E}}$ such that

$$\boldsymbol{\mathcal{E}} := f - \mathbf{\Phi}\mathbf{\Phi}^* + i\chi \quad , \tag{1.26}$$

equation (1.15) yields

$$(\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \mathbf{\Phi}\mathbf{\Phi}^*)\nabla^2\mathbf{\Phi} = \nabla\mathbf{\Phi}\cdot(\nabla\boldsymbol{\mathcal{E}} + 2\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \quad . \tag{1.27}$$

On the other hand, by means of a substitution of $\boldsymbol{\mathcal{E}}$ and using (1.27), Einstein equations (1.20), (1.25) yield

$$(\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \mathbf{\Phi}\mathbf{\Phi}^*)\nabla^2\boldsymbol{\mathcal{E}} = \nabla\boldsymbol{\mathcal{E}}\cdot(\nabla\boldsymbol{\mathcal{E}} + 2\mathbf{\Phi}^*\nabla\mathbf{\Phi}) \quad . \tag{1.28}$$

The first obvious feature this couple of complex vector differential equation provides is an alternative free-coordinates (yet non-tensorial) representation of the considered system. They can be specified considering the vacuum case characterized by $\mathbf{\Phi} = 0$: equation (1.27) becomes trivial while equation (1.28) turns into

$$(\mathrm{Re}\,\boldsymbol{\mathcal{E}})\nabla^2\boldsymbol{\mathcal{E}} = \nabla\boldsymbol{\mathcal{E}}\cdot\nabla\boldsymbol{\mathcal{E}} \quad , \tag{1.29}$$

which is named after Ernst.
We can also express the two equations of motion by a single action principle for the complex field couple $(\boldsymbol{\mathcal{E}}, \mathbf{\Phi})$

$$S[\boldsymbol{\mathcal{E}},\mathbf{\Phi}] = \int \rho d\rho dz \Big[\frac{(\nabla\boldsymbol{\mathcal{E}} + 2\mathbf{\Phi}^*\nabla\mathbf{\Phi})(\nabla\boldsymbol{\mathcal{E}}^* + 2\mathbf{\Phi}\nabla\mathbf{\Phi}^*)}{(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*)^2} - \frac{2\nabla\mathbf{\Phi}\nabla\mathbf{\Phi}^*}{\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*}\Big] \tag{1.30}$$

which can be specified too to the vacuum case setting $\mathbf{\Phi} = 0$. The two formulas ruling $\boldsymbol{\mathcal{E}}$ and $\mathbf{\Phi}$ actually do not supply a complete description of all potentials since they do not determine γ: this is due to the fact that there are other two non-trivial Einstein equations in addition to (1.5) and (1.6). Expressed in terms of the complex potentials they assume the following form

$$\partial_\rho\gamma(\rho,z) = \frac{\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \mathbf{\Phi}\mathbf{\Phi}^*)^2}\Bigg[\Big(\partial_\rho\boldsymbol{\mathcal{E}} + 2\mathbf{\Phi}^*\partial_\rho\mathbf{\Phi}\Big)\Big(\partial_\rho\boldsymbol{\mathcal{E}}^* + 2\mathbf{\Phi}\partial_\rho\mathbf{\Phi}^*\Big)$$
$$-\Big(\partial_z\boldsymbol{\mathcal{E}} + 2\mathbf{\Phi}^*\partial_z\mathbf{\Phi}\Big)\Big(\partial_z\boldsymbol{\mathcal{E}}^* + 2\mathbf{\Phi}\partial_z\mathbf{\Phi}^*\Big)\Bigg] - \frac{\rho}{\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \mathbf{\Phi}\mathbf{\Phi}^*}\Big(\partial_\rho\mathbf{\Phi}\partial_\rho\mathbf{\Phi}^* - \partial_z\mathbf{\Phi}\partial_z\mathbf{\Phi}^*\Big) \quad , \tag{1.31}$$

$$\partial_z \gamma(\rho, z) = \frac{\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2} \Bigg[\Big(\partial_\rho \boldsymbol{\mathcal{E}} + 2\boldsymbol{\Phi}^* \partial_\rho \boldsymbol{\Phi} \Big) \Big(\partial_z \boldsymbol{\mathcal{E}}^* + 2\boldsymbol{\Phi} \partial_z \boldsymbol{\Phi}^* \Big)$$

$$+ \Big(\partial_z \boldsymbol{\mathcal{E}} + 2\boldsymbol{\Phi}^* \partial_z \boldsymbol{\Phi} \Big) \Big(\partial_\rho \boldsymbol{\mathcal{E}}^* + 2\boldsymbol{\Phi} \partial_\rho \boldsymbol{\Phi}^* \Big) \Bigg] - \frac{\rho}{\mathrm{Re}\,\boldsymbol{\mathcal{E}} + \boldsymbol{\Phi}\boldsymbol{\Phi}^*} \Big(\partial_\rho \boldsymbol{\Phi} \partial_z \boldsymbol{\Phi}^* - \partial_z \boldsymbol{\Phi} \partial_\rho \boldsymbol{\Phi}^* \Big) \quad . \tag{1.32}$$

It is clear that the equations that determine γ are completely decoupled from the previous ones, so as to enable $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ to be found first by solving the system (1.27)-(1.28), leaving γ to be determined by quadrature.

As already mentioned, this formalism is suitable for generating new solutions to Einstein-Maxwell equations starting from already known ones. This possibility was first exploited by Ernst in [3] where he rationalized the method used by Newman *et al.* in [4] for finding the electric generalization of the rotating Kerr black hole. Even if it may seem arbitrary to some extention, we now present the argument used by Ernst; in the next chapter it will be proved to be a particular case within a wider generation technique.
Making the additional assumption that $\boldsymbol{\mathcal{E}}$ is an analytic function of $\boldsymbol{\Phi}$ and using the chain rule, equations (1.28)-(1.27) imply

$$\frac{\mathrm{d}^2 \boldsymbol{\mathcal{E}}}{\mathrm{d}\boldsymbol{\Phi}^2} = 0 \quad , \tag{1.33}$$

which shows that $\boldsymbol{\mathcal{E}}$ is a linear function of $\boldsymbol{\Phi}$. Using the asymptotically flat spacetimes boundary conditions $\boldsymbol{\mathcal{E}} \to 1$ and $\boldsymbol{\Phi} \to 0$ we obtain the following relation

$$\boldsymbol{\mathcal{E}} = 1 - \frac{2}{q} \boldsymbol{\Phi} \quad , \tag{1.34}$$

where q is a complex constant. Performing the substitutions

$$\boldsymbol{\mathcal{E}} =: \frac{\boldsymbol{\xi} - 1}{\boldsymbol{\xi} + 1} \quad \Longrightarrow \quad \boldsymbol{\Phi} = \frac{q}{\boldsymbol{\xi} + 1} \quad , \tag{1.35}$$

into equation (1.28) or (1.27) one gets to

$$\Big[\boldsymbol{\xi}\boldsymbol{\xi}^* - (1 - qq^*) \Big] \nabla^2 \boldsymbol{\xi} = 2\boldsymbol{\xi}^* (\nabla \boldsymbol{\xi})^2 \quad , \tag{1.36}$$

which transforms into

$$(\boldsymbol{\xi_0}\boldsymbol{\xi_0}^* - 1) \nabla^2 \boldsymbol{\xi_0} = 2\boldsymbol{\xi_0}^* (\nabla \boldsymbol{\xi_0})^2 \quad , \tag{1.37}$$

once the further substitution $\boldsymbol{\xi} = \boldsymbol{\xi_0}\sqrt{1 - qq^*}$ is carried out. Equation (1.37) is equivalent to the Ernst vacuum equation (1.29) for the potential $\boldsymbol{\mathcal{E}_0}$ through the transformation $\boldsymbol{\mathcal{E}_0} = (\boldsymbol{\xi_0} - 1)/(\boldsymbol{\xi_0} + 1)$. On account of this trick, it is possible to find a vacuum solution of equation (1.29) and then to map it into an electrovacuum one following the scheme

$$\boldsymbol{\mathcal{E}_0} \longmapsto \boldsymbol{\xi_0} \longmapsto \boldsymbol{\xi} \longmapsto (\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}) \quad , \tag{1.38}$$

which can be expressed explicitly as

$$\boldsymbol{\mathcal{E}} = \frac{(a+1)\boldsymbol{\mathcal{E}_0} + (a-1)}{(a-1)\boldsymbol{\mathcal{E}_0} + (a+1)} \quad , \quad \boldsymbol{\Phi} = \sqrt{1-a^2}\left(\frac{1-\boldsymbol{\mathcal{E}_0}}{(a+1)+(a-1)\boldsymbol{\mathcal{E}_0}}\right) \quad , \tag{1.39}$$

where a is a real parameter defined by q via $a \coloneqq \sqrt{1-qq^*}$.

Working out the explicit calculations, Ernst found that the solution $\boldsymbol{\mathcal{E}_0}$ corresponding to Kerr metric is mapped into the couple $(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi})$ that identifies Kerr-Newman black hole. At this point it is not clear if the method involved is just working accidentally by means of wishful thinking or if it can be framed into a more structured generation technique. In the next section we will prove that the analysis of Lie point symmetries associated to Ernst equations (1.28) and (1.27) provides an efficient tool to better understand the transformations used and to control which kind of solution is being generated.

Chapter 2

Symmetries

In this chapter we focus on the method used to determine the transformation symmetries of a set of differential equations (or alternatively of the equivalent action). Then we present how they can be applied in the context of Einstein-Maxwell equations in order to generate new solutions. Standard references are [8] for a general discussion of Lie point symmetries and [9] for their application in the context of General Relativity.

2.1 Lie point symmetries

A Lie point symmetry of the system of PDEs

$$H(x^n, u^\alpha, \partial_n u^\alpha, \partial_{nm} u^\alpha, ...) = 0 \quad , \tag{2.1}$$

where x^n are the independent variables, $u^\alpha(x^n)$ the dependent ones and $\partial_n u^\alpha$ their derivatives respect to x^n, is a mapping

$$\hat{x}^n = \hat{x}^n(x^i, u^\beta; \varepsilon), \quad \hat{u}^\alpha = \hat{u}^\alpha(x^i, u^\beta; \varepsilon) \tag{2.2}$$

which maps solutions into solutions. One remarkable requirement the mapping must satisfy is

$$\hat{x}^n(x^i, u^\beta; 0) = x^n, \quad \hat{u}^\alpha(x^i, u^\beta; 0) = u^\alpha \quad . \tag{2.3}$$

In this case we are assuming the dependence only on one parameter ε but it is possible to consider more general charts with more than one continuous parameter involved. Moreover, due to the fact that under suitable hypothesis these mappings form a one-parameter group, it is possible to compose different symmetries so as to obtain new transformations. In this way, the application of a symmetry leads to a new solution by starting from an already known one. Actually, though from a mathematical point of view this technique provides a method to generate different solutions from the original one, their difference could be not physically significant inasmuch as it can be absorbed into a change of coordinates. In these cases the considered symmetry is just a gauge transformation. Although it is possible to deal with system of PDEs in the form of equation

(2.1), in order to simplify the argument and to show the fundamental ideas of this method, we will consider H independent of the derivatives of the dependent variables u^α. We can now expand equation (2.2) at first order around $\varepsilon = 0$ so as to get (using property (2.3))

$$\hat{x}^n(x^i, u^\beta; \varepsilon) = x^n + \varepsilon\xi^n(x^i, u^\beta) + \cdots = x^n + \varepsilon\boldsymbol{X}x^n + \cdots \ , \tag{2.4}$$

$$\hat{u}^\alpha(x^i, u^\beta; \varepsilon) = u^\alpha + \varepsilon\eta^\alpha(x^i, u^\beta) + \cdots \ = u^\alpha + \varepsilon\boldsymbol{X}u^\alpha + \cdots \ . \tag{2.5}$$

where

$$\xi^n(x^i, u^\beta) := \left.\frac{\partial\hat{x}^n}{\partial\varepsilon}\right|_{\varepsilon=0} \qquad \eta^\alpha(x^i, u^\beta) := \left.\frac{\partial\hat{u}^\alpha}{\partial\varepsilon}\right|_{\varepsilon=0} \quad , \tag{2.6}$$

and

$$\boldsymbol{X} := \xi^n(x^i, u^\beta)\frac{\partial}{\partial x^n} + \eta^\alpha(x^i, u^\beta)\frac{\partial}{\partial u^\alpha} \quad . \tag{2.7}$$

The operator $\boldsymbol{X}$ is called "infinitesimal generator" of the transformation because its iteration produces the finite transformation. Practically, we can obtain the finite transformation by integrating

$$\frac{\partial\hat{x}^n}{\partial\varepsilon} = \xi^n(x^i, u^\beta) \quad , \quad \frac{\partial\hat{u}^\alpha}{\partial\varepsilon} = \eta^\alpha(x^i, u^\beta) \quad , \tag{2.8}$$

with boundary condition represented by equation (2.3). Exploiting the fact that all solutions satisfy

$$H(\hat{x}^n, \hat{u}^\alpha) = 0 \tag{2.9}$$

independently of the value of ε, one can find that

$$0 = \left.\frac{\partial H(\hat{x}^n, \hat{u}^\alpha)}{\partial\varepsilon}\right|_{\varepsilon=0} = \left.\left(\frac{\partial H}{\partial\hat{x}^n}\frac{\partial\hat{x}^n}{\partial\varepsilon} + \frac{\partial H}{\partial\hat{u}^\alpha}\frac{\partial\hat{u}^\alpha}{\partial\varepsilon}\right)\right|_{\varepsilon=0} \quad , \tag{2.10}$$

which is equivalent to

$$\boldsymbol{X}H = 0 \quad . \tag{2.11}$$

At this point we can focus on the problem of finding Lie point symmetries of the system (1.28)-(1.27). Although it is possible to reach this purpose by analyzing directly the system of differential equations, we will follow a slightly different path using the action principle (1.30). Following [9], we see that the considered Lagrangian is in the form (it is actually a property shared by all electrovacuum spacetimes with almost one non-null killing vector) of

$$\mathcal{L} = \sqrt{\gamma}G_{ab}(\varphi^c)\gamma^{AB}\partial_A\varphi^a\partial_B\varphi^b \quad , \tag{2.12}$$

where γ^{AB} is the three dimensional flat metric in a general coordinate form, γ its determinant, φ^c the fields of interest (in this case real and imaginary parts of $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$) and G_{ab} the components of the bilinear form induced by the Lagrangian. The field equations are invariant under an infinitesimal transformation of the form

$$\varphi^c \mapsto \varphi^c + \varepsilon \xi^c(\varphi) \tag{2.13}$$

if ξ^c is an affine vector field, that is

$$\nabla_b \nabla_c \xi_a = R_{abcd} \xi^d \quad . \tag{2.14}$$

In the present context covariant derivatives and the Riemann tensor are pertinent to the metric G_{ab}. One further property that can be proved states that if ξ^a is a Killing vector, namely such that

$$\nabla_{(a}\xi_{b)} = \nabla_a \xi_b + \nabla_b \xi_a = 0 \quad , \tag{2.15}$$

then the Lagrangian (2.12) is left invariant. Moreover, using the first Bianchi identity for Riemann tensor we can check that Killing equation implies (2.14). Practically, we consider the definition of Riemann tensor

$$\nabla_\mu \nabla_\nu \xi_\rho + \nabla_\nu \nabla_\rho \xi_\mu = -R^\lambda{}_{\rho\mu\nu} \xi_\lambda \quad , \tag{2.16}$$

and its cyclic index permutations

$$\nabla_\nu \nabla_\rho \xi_\mu + \nabla_\rho \nabla_\mu \xi_\nu = -R^\lambda{}_{\mu\nu\rho} \xi_\lambda \quad , \tag{2.17}$$

$$\nabla_\rho \nabla_\mu \xi_\nu + \nabla_\mu \nabla_\nu \xi_\rho = -R^\lambda{}_{\nu\rho\mu} \xi_\lambda \quad . \tag{2.18}$$

By summing the first two equations and subtracting the third, we find

$$2\nabla_\nu \nabla_\rho \xi_{mu} = (-R^\lambda{}_{\rho\mu\nu} - R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\nu\rho\mu})\xi_\lambda \quad . \tag{2.19}$$

As already mentioned, using Bianchi identity

$$R^\lambda{}_{\rho\mu\nu} + R^\lambda{}_{\mu\nu\rho} + R^\lambda{}_{\nu\rho\mu} = 0 \quad , \tag{2.20}$$

relation (2.14) is easily verified. This implication represents a pivotal point because it tells us that a priori the symmetries of the action are just a subset of the symmetries of the full system of differential equations. However, considering both the action (1.30) and its relevant field equations (1.27)-(1.28), one can show that in this case the two sets of symmetries are equivalent, so that the difference between the two methods is immaterial. It is straightforward to understand that studying only one scalar function (the Lagrangian that defines the action) is easier than analyzing the full set of differential equations.

2.2 Lie symmetries of Ernst potentials

After these considerations, in order to solve Killing equation, starting with the Lagrangian

$$\mathcal{L} = \frac{(\nabla\boldsymbol{\mathcal{E}} + 2\boldsymbol{\Phi}^*\nabla\boldsymbol{\Phi})(\nabla\boldsymbol{\mathcal{E}}^* + 2\boldsymbol{\Phi}\nabla\boldsymbol{\Phi}^*)}{(\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}^* + 2\boldsymbol{\Phi}\boldsymbol{\Phi}^*)^2} - \frac{2\nabla\boldsymbol{\Phi}\nabla\boldsymbol{\Phi}^*}{\boldsymbol{\mathcal{E}} + \boldsymbol{\mathcal{E}}^* + 2\boldsymbol{\Phi}\boldsymbol{\Phi}^*} \quad , \tag{2.21}$$

we first have to identify G_{ab}. Differential operators are thought in cartesian coordinates in order to avoid the overall term ρ present in (1.30). We see that defining the complex variables $\boldsymbol{\mathcal{E}} =: x + iy$ and $\boldsymbol{\Phi} =: z + iw$ and mapping

$$\sqrt{\gamma}\gamma^{AB}\partial_A\varphi^a\partial_B\varphi^b = \nabla\varphi^a\nabla\varphi^b \longmapsto \mathrm{d}\varphi^a\mathrm{d}\varphi^b \quad , \tag{2.22}$$

where $\{\varphi^i\} = \{x, y, z, w\}$, it is possible to bring the Lagrangian into a four-dimensional potential space with metric G_{ab} such that

$$\begin{aligned}\mathrm{d}s^2 = G_{ab}\mathrm{d}\varphi^a\mathrm{d}\varphi^b = \frac{1}{4(w^2+z^2+x)^2}\Big[&\mathrm{d}x^2 + \mathrm{d}y^2 - 4x(\mathrm{d}z^2 + \mathrm{d}w^2) \\ &- 4w\mathrm{d}y\mathrm{d}z + 4z\mathrm{d}y\mathrm{d}w + 4w\mathrm{d}x\mathrm{d}w + 4z\mathrm{d}x\mathrm{d}z\Big].\end{aligned} \tag{2.23}$$

The explicit expression of the metric allows to calculate Christoffel symbols and covariant derivatives so as to expand Killing equation in its ten components

$$\begin{cases}
\dfrac{\xi_x}{w^2+x+z^2} + \dfrac{\partial\xi_x}{\partial x} = 0 \\
\dfrac{2\xi_y}{w^2+x+z^2} + \dfrac{\partial\xi_x}{\partial y} + \dfrac{\xi_y}{\partial x} = 0 \\
\xi_w + 2w\xi_x + 2z\xi_y + (w^2+x+z^2)\Big(\dfrac{\partial\xi_x}{\partial w} + \dfrac{\partial\xi_w}{\partial x}\Big) = 0 \\
2z\xi_x - 2w\xi_y + \xi_z + (w^2+x+z^2)\Big(\dfrac{\partial\xi_x}{\partial z} + \dfrac{\partial\xi_z}{\partial x}\Big) = 0 \\
-\dfrac{\xi_x}{w^2+x+z^2} + \dfrac{\partial\xi_y}{\partial y} = 0 \\
-2z\xi_x + 2w\xi_y - \xi_z + (w^2+x+z^2)\Big(\dfrac{\partial\xi_y}{\partial w} + \dfrac{\partial\xi_w}{\partial y}\Big) = 0 \\
2w\xi_x + 2z\xi_y + \xi_w + (w^2+x+z^2)\Big(\dfrac{\partial\xi_y}{\partial z} + \dfrac{\partial\xi_z}{\partial y}\Big) = 0 \\
2\dfrac{w\xi_w - z\xi_z}{w^2+x+z^2} + \dfrac{\partial\xi_w}{\partial w} = 0 \\
4z\xi_w + 4w\xi_z + (w^2+x+z^2)\Big(\dfrac{\partial\xi_w}{\partial z} + \dfrac{\partial\xi_z}{\partial w}\Big) = 0 \\
2\dfrac{-w\xi_w + z\xi_z}{w^2+x+z^2} + \dfrac{\partial\xi_z}{\partial z} = 0
\end{cases} \tag{2.24}$$

The solution of the system produces eight integration constants that correspond to eight different independent infinitesimal transformations. Raising the indices of all ξ_a with the inverse of the metric (2.23) one obtains the components of the most general Killing vector in the four dimensional potential space

$$\begin{aligned}
\xi^x &= 4a_1xy + 2a_2(wx+zy) + 2a_3(xz-wy) + 4a_4x - 2a_6w + 2a_7z \\
\xi^y &= 2a_1(y^2-x^2) + 2a_2(wy-xz) + 2a_3(zy+xw) + 4a_4y + 2a_6z + 2a_7w + 4a_8 \\
\xi^z &= 2a_1(xw+yz) + a_2(4wz-y) + 2a_3(z^2-w^2+x) + 2a_4z - a_5w - a_7 \\
\xi^w &= 2a_1(yw-xz) + 2a_2(w^2-z^2+x) + a_3(4zw+y) + 2a_4w + a_5z + a_6 \ .
\end{aligned} \tag{2.25}$$

Setting seven parameters to zero and the eighth to one, it is possible to find eight different infinitesimal generators which, once integrated, will produce eight nonequivalent finite transformations. Operating the substitution one gets the following vectors

$$\begin{aligned}
\xi_1 &= 4xy\partial_x + 2(y^2-x^2)\partial_y + 2(xw+yz)\partial_z + 2(yw-xz)\partial_w \ , \\
\xi_2 &= 2(xw+yz)\partial_x + 2(yw-xz)\partial_y + (4zw-y)\partial_z + (2w^2-2z^2+x)\partial_w \ , \\
\xi_3 &= 2(xz-yw)\partial_x + 2(xw+yz)\partial_y + (z^2-w^2+x)\partial_z + (4zw+y)\partial_w \ , \\
\xi_4 &= 4x\partial_x + 4y\partial_y + 2z\partial_z + 2w\partial_w \ , \\
\xi_5 &= -w\partial_z + z\partial_w \ , \\
\xi_6 &= -2w\partial_x + 2z\partial_y + \partial_w \ , \\
\xi_7 &= 2z\partial_x + 2w\partial_y - \partial_z \ , \\
\xi_8 &= 4\partial_y \ .
\end{aligned} \tag{2.26}$$

We can now consider all eight generators in order to show how to work out the relevant finite transformations: as already mentioned, equations (2.8) provide the practical method to find them. For example, considering ξ_4, we see that using the boundary conditions $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ and $w(0) = w_0$, the associated system of ODEs can be integrated straightforwardly

$$\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 4x \\
\dfrac{\mathrm{d}y}{\mathrm{d}\varepsilon} = 4y \\
\dfrac{\mathrm{d}z}{\mathrm{d}\varepsilon} = 2z \\
\dfrac{\mathrm{d}w}{\mathrm{d}\varepsilon} = 2w
\end{cases}
\Longrightarrow
\begin{cases}
x(\varepsilon) = e^{4\varepsilon}x_0 \\
y(\varepsilon) = e^{4\varepsilon}y_0 \\
z(\varepsilon) = e^{2\varepsilon}z_0 \\
w(\varepsilon) = e^{2\varepsilon}w_0
\end{cases} \tag{2.27}$$

Turning back to the original definition of the potentials $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$ it is possible to write down the correspondent Lie point symmetry

$$\boldsymbol{\mathcal{E}} = x + iy = b^2\boldsymbol{\mathcal{E}_0} \ , \quad \boldsymbol{\Phi} = z + iw = b\boldsymbol{\Phi_0} \quad , \tag{2.28}$$

where $b := e^{2\varepsilon}$. We can repeat the same process considering now ξ_5

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 0 \\ \dfrac{\mathrm{d}y}{\mathrm{d}\varepsilon} = 0 \\ \dfrac{\mathrm{d}z}{\mathrm{d}\varepsilon} = -w \\ \dfrac{\mathrm{d}w}{\mathrm{d}\varepsilon} = z \end{cases} \Longrightarrow \begin{cases} x(\varepsilon) = x_0 \\ y(\varepsilon) = y_0 \\ z(\varepsilon) = z_0 \cos(\varepsilon) - w_0 \sin(\varepsilon) \\ w(\varepsilon) = z_0 \sin(\varepsilon) + w_0 \cos(\varepsilon) \end{cases} \tag{2.29}$$

This system induces the following transformations on the potentials

$$\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}_0} \quad , \quad \boldsymbol{\Phi} = e^{i\varepsilon}\boldsymbol{\Phi_0} \quad . \tag{2.30}$$

The composition of transformations (2.28) and (2.30) has the evident effect of scaling $\boldsymbol{\mathcal{E}}$ and both scaling and rotating $\boldsymbol{\Phi}$. So this two symmetries can be merged into one by introducing the complex parameter λ, with module b and phase ε:

$$\boldsymbol{\mathcal{E}} = \lambda\lambda^* \boldsymbol{\mathcal{E}_0} \quad , \quad \boldsymbol{\Phi} = \lambda \boldsymbol{\Phi_0} \quad . \tag{2.31}$$

The finite transformations corresponding to the second and the third infinitesimal generators come from more involved equations that deserve a bit more attention. Starting from ξ_3 the system of ODEs reads

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 2(xz - yw) \\ \dfrac{\mathrm{d}y}{\mathrm{d}\varepsilon} = 2(xw + yz) \\ \dfrac{\mathrm{d}z}{\mathrm{d}\varepsilon} = 2[z^2 - w^2 + x] \\ \dfrac{\mathrm{d}w}{\mathrm{d}\varepsilon} = 4zw + y \end{cases} \tag{2.32}$$

By means of the complex (physically meaningful) substitutions $\boldsymbol{\mathcal{E}} = x + iy$ and $\boldsymbol{\Phi} = z + iw$ the system, though still coupled, is significantly simplified

$$\begin{cases} \dfrac{\mathrm{d}\boldsymbol{\mathcal{E}}}{\mathrm{d}\varepsilon} = 2\boldsymbol{\mathcal{E}}\boldsymbol{\Phi} \\ \dfrac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}\varepsilon} = 2\boldsymbol{\Phi}^2 + \boldsymbol{\mathcal{E}} \end{cases} \tag{2.33}$$

In this form the system can be easily decoupled, however, the solution of the consequent second order ODEs is not so easy. The most convenient way to solve them is to first solve the equation

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}\boldsymbol{\mathcal{E}}} = \frac{2\boldsymbol{\Phi}^2 + \boldsymbol{\mathcal{E}}}{2\boldsymbol{\mathcal{E}}\boldsymbol{\Phi}} \quad , \tag{2.34}$$

which belongs to Bernoulli type, and then to substitute $\boldsymbol{\Phi}(\boldsymbol{\mathcal{E}})$ into the first of (2.33) and solve it. As a result we can directly write the finite transformations for the potentials $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\Phi}$:

$$
\begin{aligned}
\boldsymbol{\mathcal{E}} &= \frac{\boldsymbol{\mathcal{E}}_{\mathbf{0}}}{1 - 2\varepsilon\boldsymbol{\Phi}_{\mathbf{0}} - \varepsilon^2\boldsymbol{\mathcal{E}}_{\mathbf{0}}} \quad , \\
\boldsymbol{\Phi} &= \frac{\varepsilon\boldsymbol{\mathcal{E}}_{\mathbf{0}} + \boldsymbol{\Phi}_{\mathbf{0}}}{1 - 2\varepsilon\boldsymbol{\Phi}_{\mathbf{0}} - \varepsilon^2\boldsymbol{\mathcal{E}}_{\mathbf{0}}} \quad .
\end{aligned}
\tag{2.35}
$$

Considering now ξ_2 we see that the associated system is

$$
\begin{cases}
\dfrac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 2(xw + yz) \\
\dfrac{\mathrm{d}y}{\mathrm{d}\varepsilon} = 2(yw + xz) \\
\dfrac{\mathrm{d}z}{\mathrm{d}\varepsilon} = 4zw - y \\
\dfrac{\mathrm{d}w}{\mathrm{d}\varepsilon} = 2[w^2 - z^2 + x]
\end{cases}
\tag{2.36}
$$

In this case the useful substitutions are represented by $\boldsymbol{\eta} = x - iy$ and $\boldsymbol{\gamma} = w + iz$, which yield a system identical to (2.33). Since $\boldsymbol{\eta}$ and $\boldsymbol{\gamma}$ are related to Ernst potential by $\boldsymbol{\mathcal{E}} = \boldsymbol{\eta}^*$ and $\boldsymbol{\Phi} = i\boldsymbol{\gamma}^*$, and remembering that complex conjugation satisfies $(z/w)^* = z^*/w^*$, also in this case we can immediately obtain potential transformations

$$
\begin{aligned}
\boldsymbol{\mathcal{E}}(s) = \eta^* &= \frac{\boldsymbol{\eta}_{\mathbf{0}}^*}{1 - 2\boldsymbol{\gamma}_{\mathbf{0}}^* s - \boldsymbol{\eta}_{\mathbf{0}}^* s^2} = \frac{\boldsymbol{\mathcal{E}}_{\mathbf{0}}}{1 - 2is\boldsymbol{\Phi}_{\mathbf{0}} - s^2\boldsymbol{\mathcal{E}}_{\mathbf{0}}} \quad , \\
\boldsymbol{\Phi}(s) = i\boldsymbol{\gamma}^* &= \frac{is\boldsymbol{\eta}_{\mathbf{0}}^* + i\boldsymbol{\gamma}_{\mathbf{0}}^*}{1 - 2\boldsymbol{\gamma}_{\mathbf{0}}^* s - \boldsymbol{\eta}_{\mathbf{0}}^* s^2} = \frac{is\boldsymbol{\mathcal{E}}_{\mathbf{0}} + \boldsymbol{\Phi}_{\mathbf{0}}}{1 + 2is\boldsymbol{\Phi}_{\mathbf{0}} - s^2\boldsymbol{\mathcal{E}}_{\mathbf{0}}} \quad .
\end{aligned}
\tag{2.37}
$$

Even in this case the last two transformation can be expressed in term of a single map with the introduction of a new complex parameter $\alpha \coloneqq \varepsilon - is$. In this way we can recast (2.36) and (2.37) (with slight change of notation) as

$$
\boldsymbol{\mathcal{E}}' = \frac{\boldsymbol{\mathcal{E}}}{1 - 2\alpha^*\boldsymbol{\Phi} - \alpha\alpha^*\boldsymbol{\mathcal{E}}} \quad , \quad \boldsymbol{\Phi}' = \frac{\alpha\boldsymbol{\mathcal{E}} + \boldsymbol{\Phi}}{1 - 2\alpha^*\boldsymbol{\Phi} - \alpha\alpha^*\boldsymbol{\mathcal{E}}} \quad . \tag{2.38}
$$

Similarly, the examination of the system related to ξ_1 shows the benefit of introducing the complex variable $\boldsymbol{\Gamma} = y - ix$ and using $\boldsymbol{\Phi} = z + iw$

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 4xy \\ \dfrac{\mathrm{d}y}{\mathrm{d}\varepsilon} = 2(y^2 - x^2) \\ \dfrac{\mathrm{d}z}{\mathrm{d}\varepsilon} = 2(xw + yz) \\ \dfrac{\mathrm{d}w}{\mathrm{d}\varepsilon} = 2(yw - xz) \end{cases} \Longrightarrow \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{\Gamma} = 2\mathbf{\Gamma}^2 \\ \dfrac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{\Phi} = 2\mathbf{\Gamma}\mathbf{\Phi} \end{cases} \tag{2.39}$$

The solution of the system is straightforward and brings directly to the final form of the transformation

$$\begin{aligned} \mathbf{\Gamma} = \frac{\mathbf{\Gamma_0}}{1 - 2\varepsilon\mathbf{\Gamma_0}} \quad &\Longrightarrow \quad \boldsymbol{\mathcal{E}} = i\mathbf{\Gamma} = \frac{\boldsymbol{\mathcal{E}}_\mathbf{0}}{1 + ic\boldsymbol{\mathcal{E}}_\mathbf{0}} \quad , \\ \mathbf{\Phi} = \frac{\mathbf{\Phi_0}}{1 - 2\varepsilon\mathbf{\Gamma_0}} &= \frac{\mathbf{\Phi_0}}{1 + ic\boldsymbol{\mathcal{E}}_\mathbf{0}} \quad , \end{aligned} \tag{2.40}$$

where we set $c := -2\varepsilon$. The second transformation that can't be unified with another by introducing a complex parameter is the shift generated by ξ_8. The equations are trivial and lead to

$$\boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}} + ib \quad , \quad \mathbf{\Phi}' = \mathbf{\Phi} \quad . \tag{2.41}$$

Finally, the infinitesimal generators ξ_6 and ξ_7 can be easily integrated and yield

$$\begin{aligned} \boldsymbol{\mathcal{E}}' &= \boldsymbol{\mathcal{E}} + 2it\mathbf{\Phi} - t^2 \quad , \quad \mathbf{\Phi}' = \mathbf{\Phi} + it \quad , \\ \boldsymbol{\mathcal{E}}' &= \boldsymbol{\mathcal{E}} + 2s\mathbf{\Phi} - s^2 \quad , \quad \mathbf{\Phi}' = \mathbf{\Phi} - s \quad , \end{aligned} \tag{2.42}$$

which can be recast in the usual way using a complex parameter defined by $\beta := -s + it$

$$\boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}} - 2\beta^*\mathbf{\Phi} - \beta\beta^* \quad , \quad \mathbf{\Phi}' = \mathbf{\Phi} + \beta \quad . \tag{2.43}$$

In this way we obtained the whole group of Lie point symmetry transformations that leaves invariant the action (1.30) and its equations of motion (1.27)-(1.28): the information enclosed in the eight parameter group is brought by three complex constants (λ, β, α) and two real ones (b, c). For future reference we synthesize here all the finite transformations (dropping the zero as subscript and using apostrophe for transformed potentials):

$$\begin{array}{rll} I) & \boldsymbol{\mathcal{E}} \longmapsto \boldsymbol{\mathcal{E}}' = \lambda\lambda^*\boldsymbol{\mathcal{E}} & \mathbf{\Phi} \longmapsto \mathbf{\Phi}' = \lambda\mathbf{\Phi} \\ II) & \boldsymbol{\mathcal{E}} \longmapsto \boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}} + ib & \mathbf{\Phi} \longmapsto \mathbf{\Phi}' = \mathbf{\Phi} \\ III) & \boldsymbol{\mathcal{E}} \longmapsto \boldsymbol{\mathcal{E}}' = \dfrac{\boldsymbol{\mathcal{E}}}{1 + ic\boldsymbol{\mathcal{E}}} & \mathbf{\Phi} \longmapsto \mathbf{\Phi}' = \dfrac{\mathbf{\Phi}}{1 + ic\boldsymbol{\mathcal{E}}} \\ IV) & \boldsymbol{\mathcal{E}} \longmapsto \boldsymbol{\mathcal{E}}' = \boldsymbol{\mathcal{E}} - 2\beta^*\mathbf{\Phi} - \beta\beta^* & \mathbf{\Phi} \longmapsto \mathbf{\Phi}' = \mathbf{\Phi} + \beta \\ V) & \boldsymbol{\mathcal{E}} \longmapsto \boldsymbol{\mathcal{E}}' = \dfrac{\boldsymbol{\mathcal{E}}}{1 - 2\alpha^*\mathbf{\Phi} - \alpha\alpha^*\boldsymbol{\mathcal{E}}} & \mathbf{\Phi} \longmapsto \mathbf{\Phi}' = \dfrac{\alpha\boldsymbol{\mathcal{E}} + \mathbf{\Phi}}{1 - 2\alpha^*\mathbf{\Phi} - \alpha\alpha^*\boldsymbol{\mathcal{E}}} \end{array} \tag{2.44}$$

As a conclusive remark, we can see at first glance that the first three transformations form a proper subgroup, known as "vacuum group" because it maps vacuum solutions into vacuum solutions. On the other hand, as we will see thoroughly in the next section, transformations $IV)$ and $V)$ operate on the electromagnetic field also when the seed solution presents $\mathbf{\Phi} = 0$.

2.3 Group action on Ernst potential

Once studied the symmetries and worked out the group (2.44), it is interesting to analyze the effects of the transformations on metric functions and on electromagnetic potentials when a particular seed metric is chosen. This process is equivalent to understand which solutions to Einstein-Maxwell equations can be generated from known ones. The first general feature shared by every single element of the group consists in leaving invariant equations (1.31) and (1.32), so that γ does not change when a new solution is generated. The second characteristic shared by $I)$, $II)$ and $IV)$ can be investigated without considering a specific seed metric: indeed these transformations turn out to be gauge ones that do not give rise to any new physical nonequivalent solution. As shown by Kinnersley in [5], transformation $I)$ produces both a scaling of the potentials that can be absorbed by the change of coordinates $t \to t$, $\varphi \to |\lambda|\varphi$, and a rotation of $\mathbf{\Phi}$ equivalent to duality transformation which is well-known from special relativity. Transformation $II)$ affects only twist potential by the mapping $\chi \to \chi + b$ but this does not affect ω because, as shown by (1.18), it is related to χ up to a constant. Analogously transformation $IV)$ comes from the definition of $\boldsymbol{\mathcal{E}}$, $\mathbf{\Phi}$ and the arbitrariness of integration constants in (1.12) and (1.13).

2.3.1 Generating Kerr-Newman from Kerr

We can now focus on transformation $V)$ and show by direct computation that it can generate new metrics. It is named after Harrison, who first discovered it, and can be used to add electric charge to vacuum solutions. In the present case we will consider Kerr black hole as seed metric and we will show how to generate Kerr-Newman metric. First of all, Kerr metric can be written in Boyer-Lindquist coordinates as follows

$$\begin{aligned} \mathrm{d}s^2 = -\frac{\Delta_0(r)}{\Sigma_0(r,\theta)}(\mathrm{d}t - a\sin^2\theta\mathrm{d}\varphi)^2 + \Sigma_0(r,\theta)\Big(\frac{\mathrm{d}r^2}{\Delta_0(r)} + \mathrm{d}\theta^2\Big) + \\ + \frac{\sin^2\theta}{\Sigma_0(r,\theta)}\Big[a\mathrm{d}t - (r^2+a^2)\mathrm{d}\varphi\Big]^2 \quad , \end{aligned} \tag{2.45}$$

where $\Sigma_0(r,\theta) \coloneqq r^2 + a^2\cos^2\theta$, $\Delta_0(r) \coloneqq r^2 - 2mr + a^2$ and the electromagnetic one-form A is identically zero. The first necessary step to identify Ernst potential is to compare Kerr metric to (1.4) in order to find f_0, e^{γ_0}, ω_0. Despite the fact that the general metric and the seed are expressed in different coordinate systems, the transformation between Boyer-Lindquist coordinates and Weyl coordinates does not involve t and φ. This property allows us to consider the block $t-\varphi$ and to compare the coefficients of the two metrics. When quadratic differentials and mixed terms are factored in (2.45), one can clearly identify f_0 as g_{tt}. Therefore we get

$$f_0(r,\theta) = \frac{\Delta_0(r) - a^2\sin^2\theta}{\Sigma_0(r,\theta)} = 1 - \frac{2m}{r^2 + a^2\cos^2\theta} \quad . \tag{2.46}$$

Since the function ω_0 must satisfy $f_0\omega_0 = g_{t\varphi}$, knowing f_0 we can find that

$$\omega_0(r,\theta) = -\frac{2mar\sin^2\theta}{r^2 - 2mr + a^2\cos^2\theta} \quad . \tag{2.47}$$

The component $g_{\varphi\varphi}$ of Kerr metric, once considered the relation $g_{\varphi\varphi} = \rho^2/f_0 - f_0\omega_0^2$, allows us to define the relation between Weyl radial coordinate ρ and (r, θ). Explicitly we find that

$$\rho(r,\theta) = \sin\theta\sqrt{\Delta_0(r)} \quad . \tag{2.48}$$

Considering a second coordinate transformation

$$z(r,\theta) = \cos\theta(r-m) \tag{2.49}$$

and setting $x \coloneqq \cos\theta$, LWP line element (1.4) assumes the convenient form

$$\mathrm{d}s^2 = -f_0(\mathrm{d}t+\omega_0\mathrm{d}\varphi)^2 - f_0^{-1}\left[e^{2\gamma_0}\Big((r-m)^2-(m^2-a^2)x^2\Big)\Big(\frac{\mathrm{d}r^2}{\Delta_0(r)}+\frac{\mathrm{d}x^2}{1-x^2}\Big)+\rho^2\mathrm{d}\varphi^2\right] . \tag{2.50}$$

At this point we can identify the last metric potential e^{γ_0} from the equation

$$f_0^{-1}e^{2\gamma_0}\Big[(r-m)^2 - (m^2-a^2)x^2\Big] = \Sigma_0 \quad , \tag{2.51}$$

which yields

$$e^{2\gamma_0} = \frac{r^2 - 2mr + a^2x^2}{(r-m)^2 - (m^2-a^2)x^2} \quad . \tag{2.52}$$

Since we want to build Ernst gravitational potential, we first need to find the twist χ_0. Using $\mathbf{\Phi}_0 = 0$ and the explicit gradient operator associated to a three dimensional flat space via Boyer-Lindquist coordinates

$$\nabla\Omega(r,x) = \frac{1}{\sqrt{(r-m)^2 - (m^2-a^2)x^2}}\Big[\hat{r}\sqrt{\Delta_0(r)}\frac{\partial}{\partial r}\Omega(r,x) + \hat{x}\sqrt{1-x^2}\frac{\partial}{\partial x}\Omega(r,x)\Big], \tag{2.53}$$

it is possible to expand in components equation (1.18)

$$\begin{cases} \partial_r\chi_0 = -\dfrac{f_0^2}{\Delta_0}\partial_x\omega_0 = -\dfrac{4amrx}{(r^2+a^2x^2)^2} \\[2ex] \partial_x\chi_0 = \dfrac{f_0^2}{1-x^2}\partial_r\omega_0 = \dfrac{2m(ar^2 - a^3x^2)}{(r^2+a^2x^2)^2} \end{cases} \tag{2.54}$$

These equations can be easily integrated: starting from the first, χ_0 is defined up to a function of x which turns out to be zero once the second equation is considered. Then the solution is

$$\chi_0(r,x) = \frac{2amx}{r^2 + a^2x^2} \quad . \tag{2.55}$$

After the computation of all functions, using $\boldsymbol{\mathcal{E}}_0 = f_0 - \boldsymbol{\Phi}_0\boldsymbol{\Phi}_0^* + i\chi_0$, we can now express Ernst gravitational potential

$$\boldsymbol{\mathcal{E}}_0 = 1 - \frac{2m}{r + iax} \quad . \tag{2.56}$$

The application of Harrison transformation leads both to a new gravitational potential and to a non-zero electromagnetic potential

$$\boldsymbol{\mathcal{E}} = \frac{\boldsymbol{\mathcal{E}}_0}{1 - |\alpha|^2\boldsymbol{\mathcal{E}}_0} \quad , \quad \boldsymbol{\Phi} = \frac{\alpha\boldsymbol{\mathcal{E}}_0}{1 - |\alpha|^2\boldsymbol{\mathcal{E}}_0} = \alpha\boldsymbol{\mathcal{E}} \quad , \tag{2.57}$$

where $\alpha = b + ic$. Performing calculations one gets

$$\boldsymbol{\mathcal{E}} = \frac{1}{1 - |\alpha|^2}\left[1 - \frac{2m}{r(1 - |\alpha|^2) + 2m|\alpha|^2 + i(1 - |\alpha|^2)ax}\right], \tag{2.58}$$

which assumes a more compact form by defining $R \coloneqq r(1 - |\alpha|^2) + 2m|\alpha|^2$

$$\boldsymbol{\mathcal{E}} = \frac{1}{1 - |\alpha|^2}\left(1 - \frac{2m}{R + ia(1 - |\alpha|^2)x}\right) \quad . \tag{2.59}$$

Then the electromagnetic potential is

$$\boldsymbol{\Phi} = \frac{\alpha}{1 - |\alpha|^2}\left(1 - \frac{2m}{R + ia(1 - |\alpha|^2)x}\right) \quad . \tag{2.60}$$

Electromagnetic components can be calculated straightforwardly considering real and imaginary part of $\boldsymbol{\Phi}$

$$\hat{A}_\varphi(R,x) = \mathrm{Im}(\boldsymbol{\Phi}) = -\frac{c(2m - R)R - 2ab(1 - |\alpha|^2)mx + a^2c(1 - |\alpha|^2)x^2}{(1 - |\alpha|^2)R^2 + a^2(1 - |\alpha|^2)^3x^2} \quad , \tag{2.61}$$

$$A_t(R,x) = \mathrm{Re}(\boldsymbol{\Phi}) = -\frac{b(2m - R)R + 2ac(1 - |\alpha|^2)mx + a^2b(1 - |\alpha|^2)x^2}{(1 - |\alpha|^2)R^2 + a^2(1 - |\alpha|^2)^3x^2} \quad . \tag{2.62}$$

Twist potential can be easily found too

$$\chi(R,x) = \mathrm{Im}(\boldsymbol{\mathcal{E}}) = \frac{2max}{R^2 + a^2(1 - |\alpha|^2)x^2} \quad , \tag{2.63}$$

and the metric function f follows from the definition of $\boldsymbol{\mathcal{E}}$

$$f = \boldsymbol{\mathcal{E}} + |\boldsymbol{\Phi}|^2 - i\chi = \frac{(2m - R)(2m|\alpha|^2 - R) + a^2(1 - |\alpha|^2)x^2}{(1 - |\alpha|^2)^2R^2 + a^2(1 - |\alpha|^2)^4x^2} \quad . \tag{2.64}$$

Since the definition of ω involves the specific form of gradient operator, at this point it is useful to evaluate how non-Killing part of the metric transforms. In order to perform this calculation, we have to remember that $e^{2\gamma_0}$ is invariant under the full group of symmetry: we just need to operate the change of coordinates from r to R.

$$f^{-1}e^{2\gamma_0}(\mathrm{d}\rho^2+\mathrm{d}\theta^2)=\Big[R^2+a^2(1-|\alpha|^2)x^2\Big]\Big(\frac{\mathrm{d}R^2}{\Delta(R)}+\frac{\mathrm{d}x^2}{1-x^2}\Big)\quad , \tag{2.65}$$

where $\Delta(R)\coloneqq R^2-2m(1+|\alpha|^2)R+4m^2\alpha^2+a^2(1-|\alpha|^2)^2$. The definition of gradient operator is completely analogous to (2.53) and allows us to write in explicit form equation (1.18)

$$\begin{cases}\partial_x\omega=-\dfrac{\Delta}{f^2}\partial_R\chi+2(\hat{A}_\varphi\partial_R A_t-A_t\partial_R\hat{A}_\varphi)\\[2ex] \partial_R\omega=\dfrac{1-x^2}{f^2}\partial_x\chi+2(\hat{A}_\varphi\partial_x A_t-A_t\partial_x\hat{A}_\varphi)\quad .\end{cases} \tag{2.66}$$

The substitution of all the potentials in the second line leads to the final form

$$\partial_R\omega=-\frac{2am(|\alpha|^2-1)^3\,(x^2-1)\,\Big(a^2x^2\left(|\alpha|^2-1\right)^2\left(|\alpha|^2+1\right)+R\left(|\alpha|^2(4m-R)-R\right)\Big)}{\Big[a^2x^2\left(|\alpha|^2-1\right)^2+(2m-R)\left(2m|\alpha|^2-R\right)\Big]^2} \tag{2.67}$$

Integration is straightforward and produces an indefinite function of x. Substituting the result in the first line of the system, just like we did for χ_0, one finds that this function must be constant, so we set it to zero. We find that

$$\omega(R,x)=\frac{2am\,(x^2-1)\left(|\alpha|^2-1\right)^3\Big[|\alpha|^2(2m-R)-R\Big]}{a^2x^2\left(|\alpha|^2-1\right)^2+(2m-R)\left(2m|\alpha|^2-R\right)}\quad . \tag{2.68}$$

As a concluding remark on these quadratures, we can specify that we chose a particular integration order because it was the only one to produce constant integration function. If we had inverted the order, we would have had to solve another non-trivial differential equation so as to define the specific form of the integration function. A_ϕ is the last function we need to find to determine the system. In order to do so we will use the explicit expression of equation (1.10)

$$\begin{cases}\partial_x A_\varphi=\dfrac{\Delta}{f}\partial_R\hat{A}_\varphi-\omega\partial_x A_t\\[2ex] \partial_R A_\varphi=\dfrac{x^2-1}{f}\partial_x\hat{A}_\varphi-\omega\partial_R A_t\quad .\end{cases} \tag{2.69}$$

Substituting all the terms in the second line we get

$$\partial_R A_\varphi=-\frac{2am\,(x^2-1)\left(|\alpha|^2-1\right)^2\Big(a^2bx^2\left(|\alpha|^2-1\right)^2+2acRx\left(|\alpha|^2-1\right)-bR^2\Big)}{a^4x^4\left(|\alpha|^2-1\right)^4+2a^2R^2x^2\left(|\alpha|^2-1\right)^2+R^4} \tag{2.70}$$

We can integrate this expression and generate an arbitrary function of x, then differentiating it respect to x and comparing with the first line of the system we see that this function must assume the form $k(x) = -2b^2cmx - 2c^3mx + 2cmx$. Substituting it in A_φ we obtain

$$A_\varphi(R,x) = \frac{2m\left(|\alpha|^2-1\right)\left[a^2cx\left(|\alpha|^2-1\right)^2 + abR\left(x^2-1\right)\left(|\alpha|^2-1\right) + cR^2x\right]}{a^2x^2\left(|\alpha|^2-1\right)^2 + R^2} \quad . \tag{2.71}$$

At this point, defined all the potentials that characterize both the metric and the electromagnetic one-form, we can note that the following substitutions bring an evident simplification to the system. Indeed

$$\begin{aligned} a' &:= a(1-|\alpha|^2) \quad , \\ m' &:= m(1+|\alpha|^2) \quad , \\ q := -2mb \quad &, \quad p := 2mc \quad , \end{aligned} \tag{2.72}$$

transform the metric like

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{1}{(1-|\alpha|^2)^2}\Big(1+\frac{q^2+p^2-2m'R}{R^2+a'^2x^2}\Big)\Big[\mathrm{d}t^2+2\omega(R,x)\mathrm{d}t\mathrm{d}\varphi+\omega(R,x)^2\mathrm{d}\varphi^2\Big]+ \\ &+\Big(R^2+a'^2x^2\Big)\Big[\frac{\mathrm{d}R^2}{\Delta(R)}+\frac{\mathrm{d}x^2}{1-x^2}\Big]+(1-|\alpha|^2)^2(R^2+a'^2x^2)\mathrm{d}\varphi^2 \quad , \end{aligned} \tag{2.73}$$

where ω staisfies

$$\omega(R,x) = (1-|\alpha|^2)^2\frac{2a'(1-x^2)(q^2+p^2-2m'R)}{\Delta(R)} \tag{2.74}$$

and $\Delta(R) = R^2 - 2m'R + a'^2x^2 + q^2 + p^2$ after performing transformations (2.72). The electromagnetic one-form reads instead

$$A = \frac{1}{1-|\alpha|^2}\Big(b+\frac{qR-pa'x}{R^2+a'^2x^2}\Big)\mathrm{d}t + (1-|\alpha|^2)\frac{px(a'^2+R^2)-qa'R(1-x^2)}{R^2+a'^2x^2}\mathrm{d}\varphi \tag{2.75}$$

We can now scale Killing coordinates t and φ in order to delete the overall $(1-|\alpha|^2)^2$ factors that appear here and there. Applying the scaling

$$\begin{aligned} t = \tau(1-|\alpha|^2) &\Longrightarrow \mathrm{d}t = (1-|\alpha|^2)\mathrm{d}\tau \\ \varphi = \phi(1-|\alpha|^2)^{-1} &\Longrightarrow \mathrm{d}\varphi = (1-|\alpha|^2)^{-1}\mathrm{d}\phi \end{aligned} \tag{2.76}$$

and dropping both the irrelevant constant term in A_t and the apexes on a', m', after some algebra we finally get to Kerr-Newman metric

$$\begin{aligned} \mathrm{d}s^2 = &-\frac{\Delta(R)}{\Sigma(R,\theta)}(\mathrm{d}\tau - a\sin^2\theta\mathrm{d}\phi)^2 + \Sigma(R,\theta)\Big(\frac{\mathrm{d}R^2}{\Delta(R)}+\mathrm{d}\theta^2\Big)+ \\ &+\frac{\sin^2\theta}{\Sigma(R,\theta)}\Big[a\mathrm{d}\tau-(R^2+a^2)\mathrm{d}\phi\Big]^2 \quad , \end{aligned} \tag{2.77}$$

where $\Sigma(R,\theta) \coloneqq R^2 + a^2\cos^2\theta$, with the associated electromagnetic one-form

$$A = \frac{qR - pa\cos\theta}{R^2 + a^2\cos^2\theta}\mathrm{d}\tau + \frac{p\cos\theta(a^2+R^2) - qaR\sin^2\theta}{R^2 + a^2\cos^2\theta}\mathrm{d}\phi \quad . \tag{2.78}$$

The two parameters q and p are to be understood as electric charge and magnetic monopole, while a, like in Kerr metric, is connected to angular momentum. We can indeed see this by setting to zero one parameter at time. When $a = 0$ it is possible to recognize Reissner-Nordström metric (including magnetic monopole), when $q = 0$ and $p = 0$ Kerr metric is retrieved, while when both electromagnetic charges and the angular momentum are turned off, we get back to Schwarzschild black hole. The combination of these parameters can be summarized into the following table

	Not rotating ($a = 0$)	Rotating ($a \neq 0$)
Uncharged ($q = 0$, $p = 0$)	Schwarzschild	Kerr
Charged ($q \neq 0$, $p \neq 0$)	Reissner-Nordström	Kerr-Newman

2.3.2 More on Harrison transformation

In the first chapter we hinted at the fact that transformation (1.39), implicitly used by Ernst, is substantially an Harrison transformation. By means of easy algebra we can recast it in the more suitable form

$$\boldsymbol{\mathcal{E}} = \frac{\boldsymbol{\mathcal{E}_0} - \frac{1-a}{1+a}}{1 - \frac{1-a}{a+1}\boldsymbol{\mathcal{E}_0}} \quad , \quad \boldsymbol{\Phi} = \sqrt{\frac{1-a}{1+a}}\frac{1 - \boldsymbol{\mathcal{E}_0}}{1 - \frac{1-a}{a+1}\boldsymbol{\mathcal{E}_0}} \quad , \tag{2.79}$$

Since the two mappings (Ernst and Harrison) are in a slightly different form, we will use also gauge transformations $I)$ and $IV)$ in order to get the desired result. However, we remark that the charging property is carried only by $V)$ and the necessity of using gauge transformations comes from the free-coordinate form we will be working with in this section.

First of all, the most general composition $I) \circ IV) \circ V)$ (assuming real parameters and $\boldsymbol{\Phi_0} = 0$) reads

$$\boldsymbol{\mathcal{E}} = \lambda^2 \frac{\left[\boldsymbol{\mathcal{E}}_0(1 + \beta^2\alpha^2 - 2\beta\alpha) - \beta^2\right]}{1 - \alpha^2\boldsymbol{\mathcal{E}}_0} \quad , \tag{2.80}$$

$$\boldsymbol{\Phi} = \lambda \frac{\boldsymbol{\mathcal{E}}_0(\alpha - \beta\alpha^2) + \beta}{1 - \alpha^2\boldsymbol{\mathcal{E}}_0} \quad . \tag{2.81}$$

Comparing the denominators in $\boldsymbol{\mathcal{E}}$ we immediately obtain

$$\alpha = \sqrt{\frac{1-a}{1+a}} \quad , \tag{2.82}$$

while the additive term in the numerator of $\boldsymbol{\mathcal{E}}$ fixes

$$\lambda^2\beta^2 = \frac{1-a}{1+a} \quad . \tag{2.83}$$

Comparing the coefficients of $\boldsymbol{\mathcal{E}_0}$ in the numerator of $\boldsymbol{\Phi}$ it is possible to get the further condition

$$\lambda(\alpha - \beta\alpha^2) = -\sqrt{\frac{1-a}{1+a}} \quad , \tag{2.84}$$

which, together with (2.83), defines β and λ

$$\beta = -\frac{\sqrt{(1-a)(1+a)}}{2a} \quad , \quad \lambda = -\frac{2a}{a+1} \quad . \tag{2.85}$$

This set of parameters satisfies also the condition

$$\lambda^2(1 + \beta^2\alpha^2 - 2\beta\alpha) = 1 \quad , \tag{2.86}$$

so Ernst transformation is anything different from a gauge-Harrison mapping with null initial electromagnetic potential. Moreover, the arbitrary hypothesis that $\boldsymbol{\mathcal{E}}$ must be a function of $\boldsymbol{\Phi}$ naturally comes out from Harrison transformation when the seed solution is uncharged ($\boldsymbol{\Phi} = \alpha\boldsymbol{\mathcal{E}}$). In this way we see that it is a general property of every spacetime that can be charged from a vacuum metric by means of V).

2.4 Discrete symmetries

Lie point symmetries do not represent the only type of transformations that generates new nonequivalent solutions: for example, the requirement that maps (2.2) must depend continuously on ϵ automatically excludes all discrete symmetries. Within this family of transformations, Wick rotation is doubtless of particular use in the context of axially symmetric solutions. In particular we will consider the double Wick rotation that maps time coordinate to the angular one and vice versa:

$$\begin{aligned} t &\longmapsto i\varphi \quad , \\ \varphi &\longmapsto it \quad . \end{aligned} \tag{2.87}$$

The application of this complex transformation to

$$\mathrm{d}s^2 = -f(\mathrm{d}t - \omega\mathrm{d}\varphi)^2 + f^{-1}[\rho^2\mathrm{d}\varphi^2 + e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2)] \tag{2.88}$$

produces the following metric

$$\mathrm{d}s^2 = f(\mathrm{d}\varphi - \omega\mathrm{d}t)^2 + f^{-1}[-\rho^2\mathrm{d}t^2 + e^{2\gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2)] \tag{2.89}$$

which still solves Einstein equations, but belongs to a different family from the first one. In this way it is possible to compose double Wick rotation to other continuous symmetries in order to generate new solutions. In the next chapter, working with higher dimensional generalizations of these metrics, we will show a detailed example about how it works.

Chapter 3

Higher dimensional gravity

Higher-dimensional theories of gravity can be thought of as a natural extension of their four-dimensional counterpart. The main aim of this chapter is to show that the solution generating method based on symmetries and potentials can be used fruitfully also in this context. In particular we will deal firstly with n-dimensional gravity, showing a particular transformation able to add electric charge to Schwarzschild-Tangherlini black hole (higher dimensional generalization of four-dimensional Schwarzschild) and then, considering five dimensions, we will prove that this mapping can be recovered and generalized by means of Ernst potentials and Lie point symmetries.

3.1 Charging transformations for Tangherlini

As already outlined in chapter introduction, Schwarzschild solution in arbitrary dimension $n \geq 4$ was first found by Tangherlini in [10]. It represents an asymptotically flat, spherically symmetric black hole, where the metric assumes the form

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2_{n-2} \quad , \tag{3.1}$$

with $\mathrm{d}\Omega^2_{n-2}$ as line element on the $(n-2)$-dimensional sphere with unitary radius, and f is such that

$$f = 1 - \frac{\mu}{r^{n-3}} \quad . \tag{3.2}$$

The parameter μ is related to the physical mass M by

$$M = \frac{(n-2)\pi^{\frac{n-3}{2}}}{8\Gamma(\frac{n-1}{2})}\mu = \frac{(n-2)\Omega_{n-2}}{16\pi}\mu \quad , \tag{3.3}$$

where $\Gamma(s)$ is Euler Gamma function and Ω_{n-2} is the hypersphere surface defined in $n-1$ spatial dimensions. A fast check shows that for $n = 4$ we have $\mu = 2M$

as expected. In this context it is useful to introduce an hypersphere parametrization slightly different from the most common one: it is almost analogous to the standard diffeomorphism as far as the first $n-4$ coordinates are concerned, but it differs in the last two one. It can be written as

$$\begin{cases} x_1 = \cos\theta \cos\psi_1 \\ x_2 = \cos\theta \sin\psi_1 \cos\psi_2 \\ \quad\vdots \\ x_{n-5} = \cos\theta \sin\psi_1 \ldots \sin\psi_{n-6} \cos\psi_{n-5} \\ x_{n-4} = \cos\theta \sin\psi_1 \ldots \sin\psi_{n-6} \sin\psi_{n-5} \\ x_{n-3} = \sin\theta \cos\varphi \\ x_{n-2} = \sin\theta \sin\varphi \end{cases} \tag{3.4}$$

where $\theta \in [0, \pi/2]$, $\varphi \in [0, 2\pi]$, $\psi_{n-5} \in [0, 2\pi]$ and the other $\psi_i \in [0, \pi]$. With such coordinates the spherical line element can be expressed in function of the coordinates φ, θ and the standard line element of a lower dimensional hypersphere, so that it assumes the form $\mathrm{d}\Omega^2_{n-2} = \cos^2\theta \mathrm{d}\Omega^2_{n-4} + \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$. Substituting it in (3.1) one obtains

$$\mathrm{d}s^2 = -f\mathrm{d}t^2 + f^{-1}\mathrm{d}r^2 + r^2\cos^2\theta \mathrm{d}\Omega^2_{n-4} + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta \mathrm{d}\varphi^2 \ , \tag{3.5}$$

which will be used as seed metric for the charging transformation. As described by Ortaggio in [7], when we consider a solution admitting one spacelike Killing vector ∂_φ within pure Einstein-Maxwell gravity, it is possible to choose a set of coordinates $\{x^i,\varphi\}$, $i = 1, ..., n-1$, with $g_{i\varphi} = 0 = A_i$. In this case the general form of the solution is

$$\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j + V\mathrm{d}\varphi^2 \quad , \tag{3.6}$$

$$F = \partial_i A_\varphi \mathrm{d}x^i \wedge \mathrm{d}\varphi \Longleftrightarrow A = A_\varphi \mathrm{d}\varphi \quad , \tag{3.7}$$

where all functions depend only on the set of non-Killing coordinates $\{x^i\}$. Assuming these properties, one can show that the following transformation leaves invariant Einstein-Maxwell action, generating non-equivalent solution respect to the seed:

$$\begin{cases} \Lambda = \Big(1 + \dfrac{n-3}{n-2}BA_\varphi\Big)^2 + \dfrac{1}{2}\dfrac{n-3}{n-2}B^2V \\ g'_{ij} = \Lambda^{2/(n-3)}g_{ij} \quad , \quad V' = \Lambda^{-2}V \\ A'_\varphi = \Lambda^{-1}\Big[A_\varphi + B\Big(\dfrac{1}{2}V + \dfrac{n-3}{n-2}A^2_\varphi\Big)\Big] \end{cases} \tag{3.8}$$

where B is the constant that will add the relevant charge to the seed spacetime. In [7] this transformation is used to add magnetic field to (3.5): the identification of potentials is quick and leads to the new solution

$$\mathrm{d}s^2 = \Lambda^{2/(n-3)}[-f\mathrm{d}t^2+f^{-1}\mathrm{d}r^2+r^2\cos^2\theta\mathrm{d}\Omega^2_{n-4}+r^2\mathrm{d}\theta^2]+\Lambda^{-2}r^2\sin^2\theta\mathrm{d}\varphi^2 \ , \tag{3.9}$$

$$A = \frac{1}{2}\Lambda^{-1}Br^2\sin^2\theta\mathrm{d}\varphi \quad , \tag{3.10}$$

where f is the same of (3.5) and

$$\Lambda = 1 + \frac{1}{2}\frac{n-3}{n-2}B^2r^2\sin^2\theta \quad . \tag{3.11}$$

This is the generalization of four-dimensional black hole immersed in Melvin universe, a particular spacetime surrounded by parallel non-vanishing magnetic lines. Its properties, and in particular its stability were studied extensively by Melvin and Thorne in [6] and [11] hoping to set a conceivable context to gravitational collapse in strong magnetic fields.

We will now use transformation (3.8) together with double Wick rotation in order to generate n-dimensional Reissner-Nordström black hole. First of all, performing double Wick rotation (2.87) on Tangherlini metric (3.5) we find

$$\mathrm{d}s^2 = f\mathrm{d}\varphi^2 + f\mathrm{d}r^2 + r^2\cos^2\theta\mathrm{d}\Omega^2_{n-4} + r^2\mathrm{d}\theta^2 - r^2\sin^2\theta\mathrm{d}t^2 \ , \tag{3.12}$$

while electromagnetic four potential A is still zero. At this point, exploiting the explicit form of f, the application of (3.8) yields

$$\begin{aligned}\Lambda = 1 + \frac{1}{2}\frac{n-3}{n-2}B^2\Big(1-\frac{\mu}{r^{n-3}}\Big) &= \\ &= \frac{1}{2r^{n-3}}\frac{n-3}{n-2}\Big[\Big(\frac{2(n-2)}{n-3}+B^2\Big)r^{n-3}-\mu B^2\Big] \quad ,\end{aligned} \tag{3.13}$$

which suggests the following change of coordinates

$$\alpha R^{n-3} = \Big(\frac{2(n-2)}{n-3}+B^2\Big)r^{n-3} - \mu B^2 \quad , \tag{3.14}$$

where α is a generic constant that will be determined later on the basis of convenience. Electromagnetic potential can be calculated as

$$A'_\varphi = \frac{BV}{2\Lambda} = \frac{B(n-2)}{2(n-2)+(n-3)B^2} - \frac{2B(n-2)^2\mu}{(n-3)[2(n-2)+(n-3)B^2]}\frac{1}{\alpha R^{n-3}} \ , \tag{3.15}$$

where the first constant term is unphysical and can be dropped. The Killing component of the metric transforms in such way that

$$\begin{aligned}V' = \frac{V}{\Lambda^2} = \Big(\frac{2(n-2)}{n-3}\Big)^2\frac{1}{(\frac{2(n-2)}{n-3}+B^2)^2}\Bigg[1 - \frac{\mu(\frac{2(n-2)}{n-3}-B^2)}{\alpha}\frac{1}{R^{n-3}} - \\ - \frac{2(n-2)}{n-3}\frac{\mu^2B^2}{\alpha^2}\frac{1}{R^{2(n-3)}}\Bigg] \quad .\end{aligned} \tag{3.16}$$

All the other metric functions such as g_{tt} can be found considering transformations in the form

$$g'_{tt} = \Lambda^{2/(n-3)} g_{tt} = \Big(\frac{n-3}{2(n-2)}\Big)^{2/(n-3)} R^2 \alpha^{2/(n-3)} \sin^2\theta \quad , \tag{3.17}$$

which shows the convenience of choosing $\alpha = \Big(\frac{2(n-2)}{n-3}\Big)$, that simplifies g'_{tt} into

$$g'_{tt} = R^2 \sin^2\theta \quad , \tag{3.18}$$

V' into

$$\begin{aligned} V' = \Big(\frac{2(n-2)}{n-3}\Big)^2 \frac{1}{(\frac{2(n-2)}{n-3} + B^2)^2} \Bigg[1 - \mu\Big(1 - \frac{n-3}{2(n-2)} B^2\Big) \frac{1}{R^{n-3}} - \\ - \frac{(n-3)\mu^2 B^2}{2(n-2)} \frac{1}{R^{2(n-3)}} \Bigg] \quad , \end{aligned} \tag{3.19}$$

and the electromagnetic potential A'_φ to

$$A'_\varphi = -\frac{2(n-2)}{(n-3)\Big(\frac{2(n-2)}{n-3} + B^2\Big)} \frac{\mu B}{2R^{n-3}} \quad . \tag{3.20}$$

At this point we can see that scaling the azimuthal coordinate is advantageous to delete overall factors in V' and A'_φ. Indeed

$$\Bigg[\frac{2(n-2)}{(n-3)\Big(\frac{2(n-2)}{n-3} + B^2\Big)} \Bigg] \varphi \longmapsto \phi \tag{3.21}$$

leads to

$$V' \mathrm{d}\varphi^2 \mapsto \Bigg[1 - \mu\Big(1 - \frac{n-3}{2(n-2)} B^2\Big) \frac{1}{R^{n-3}} - \frac{(n-3)\mu^2 B^2}{2(n-2)} \frac{1}{R^{2(n-3)}} \Bigg] \mathrm{d}\phi^2 \quad , \tag{3.22}$$

$$A'_\varphi \mathrm{d}\varphi \longmapsto A_\phi \mathrm{d}\phi = -\frac{\mu B}{2} \frac{1}{R^{n-3}} \mathrm{d}\phi \tag{3.23}$$

The other terms that show r^2 as an overall factor transform like g_{tt} and can be written simply replacing r^2 with R^2. The second component of the non-Killing block, included the transformation of the differential, is mapped like

$$f^{-2} \mathrm{d}r^2 \longmapsto \Lambda^{2/(n-3)} f^{-2} \mathrm{d}r^2 = \Big(1 - \frac{\mu}{r^{n-3}}\Big)^{-1} \mathrm{d}R^2 = \tag{3.24}$$

$$= \Bigg[1 - \mu\Big(1 - \frac{n-3}{2(n-2)} B^2\Big) \frac{1}{R^{n-3}} - \frac{(n-3)\mu^2 B^2}{2(n-2)} \frac{1}{R^{2(n-3)}} \Bigg]^{-1} \mathrm{d}R^2 \,. \tag{3.25}$$

At this point we can apply for the second time double Wick rotation, which leads to the final form of both the metric and the electromagnetic potential

$$\mathrm{d}s^2 = -\left[1 - \frac{m}{R^{n-3}} + \frac{2}{(n-3)(n-2)}\frac{q^2}{R^{2(n-3)}}\right]\mathrm{d}t^2 + \\ + \left[1 - \frac{m}{R^{n-3}} + \frac{2}{(n-3)(n-2)}\frac{q^2}{R^{2(n-3)}}\right]^{-1}\mathrm{d}R^2 + R^2\cos^2\theta\mathrm{d}\Omega^2_{n-4} \\ + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 \quad , \quad (3.26)$$

$$A = A_\phi \mathrm{d}t = -\frac{q}{n-3}\frac{1}{R^{n-3}}\mathrm{d}t \quad , \quad (3.27)$$

where we performed the following substitutions

$$m := \mu\Big(1 - \frac{n-3}{2(n-2)}B^2\Big) \quad , \quad (3.28)$$

$$q := \frac{i\mu B}{2} \quad . \quad (3.29)$$

We want to highlight that in (3.27) A_ϕ, though keeping its old name, after Wick rotation refers to time component of the electromagnetic potential. Moreover last formula clearly shows that B must be taken imaginary. The presented solution is the arbitrary-dimensional generalization of Reissner-Nordström 4d black hole: by imposing $n = 4$ one can easily retrieve the better known RN solution.

3.2 Ernst potentials in five dimensions

In this section we define Ernst potentials for Einstein-Maxwell theory in five dimensions assuming only one electromagnetic component. We also show that this formulation allows to study symmetries, which are formally very similar to the four dimensional ones, giving the possibility to exploit Lie transformations in order to generate new solutions. First of all, we consider the following ansatz

$$\mathrm{d}s^2 = -e^{S-T}(\mathrm{d}t - \omega\mathrm{d}\psi)^2 + e^{-S-T}\rho^2\mathrm{d}\psi^2 + e^{-S-T+2\Gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + e^{2T}\mathrm{d}\varphi^2 \ , \quad (3.30)$$

$$A = A_\varphi \mathrm{d}\varphi \quad , \quad (3.31)$$

where S, S, Γ, ω, A_φ are functions of non-Killing coordinates (ρ, z). Under these assumptions the only non-trivial Maxwell equation is given by $\nabla_\mu F^{\mu\varphi} = 0$. Explicitly it reads

$$2(\partial_\rho A_\varphi \partial_\rho T + \partial_z A_\varphi \partial_z T) - (\partial_{\rho\rho}A_\varphi + \partial_{zz}A_\varphi + \frac{1}{\rho}\partial_\rho A_\varphi) = 0 \quad . \quad (3.32)$$

We can clearly recognize the scalar product between gradients in the first term and the cylindrical Laplacian in the second one. Multiplying the equation by e^T we obtain

$$e^T\nabla^2 A_\varphi = \nabla e^T \nabla A_\varphi \quad . \quad (3.33)$$

Einstein equations come in a more elaborate form, however, by means of linear combinations one can find more suitable relations that still describe the system. In order to ease the dissertation, the explicit calculation to get to the following formulas is reported in appendix B. The first equation reads

$$2(\partial_\rho S \partial_\rho \omega + \partial_z S \partial_z \omega) + \partial_{\rho\rho}\omega + \partial_{zz}\omega - \frac{1}{\rho}\partial_\rho \omega = 0 \quad , \tag{3.34}$$

and it is equivalent to the vector form

$$\nabla \cdot \left(e^{2S}\rho^{-2}\nabla\omega\right) = 0 \quad . \tag{3.35}$$

We can notice that this formula is formally identical to equation (1.17) when $\mathbf{\Phi} = 0$, so we can define analogously a twist potential χ

$$\rho^{-1}e^{2S}\nabla\omega =: \hat{\varphi} \times \nabla\chi \quad , \tag{3.36}$$

such that (3.35) is trivially satisfied. Equivalently to (1.20), the new equation is

$$\nabla \cdot \left(e^{-2S}\nabla\chi\right) = 0 \quad . \tag{3.37}$$

The second useful Einstein equation assumes the explicit form

$$\frac{e^{2S}}{\rho^2}\left[(\partial_\rho\omega)^2 + (\partial_z\omega)^2\right] + \partial_{\rho\rho}S + \partial_{zz}S + \frac{1}{\rho}\partial_\rho S = 0 \quad , \tag{3.38}$$

that is equivalent to

$$e^S\nabla^2 e^S = (\nabla e^S)^2 - (\nabla\chi)^2 \quad . \tag{3.39}$$

Last relation again involves electromagnetic component A_φ

$$3\left(\partial_{\rho\rho}T + \partial_{zz}T + \frac{1}{\rho}\partial_\rho T\right) + 4e^{-2T}\left[(\partial_\rho A_\varphi)^2 + (\partial_z A_\varphi)^2\right] = 0 \quad , \tag{3.40}$$

and can be written in terms of flat differential operators too

$$3\nabla^2 T + 4e^{-2T}(\nabla A_\varphi)^2 = 0 \quad . \tag{3.41}$$

At this point we can see the advantage of introducing two complex functions $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ defined by

$$\boldsymbol{\mathcal{E}}_1 := e^T + i\frac{2}{\sqrt{3}}A_\varphi \quad , \tag{3.42}$$

$$\boldsymbol{\mathcal{E}}_2 := e^S + i\chi \quad , \tag{3.43}$$

which allow to recast all the relevant PDEs into a very symmetric form: equations (3.33) and (3.41) can be expressed respectively as imaginary and real part of

$$\mathrm{Re}(\boldsymbol{\mathcal{E}}_1)\nabla^2\boldsymbol{\mathcal{E}}_1 = \nabla\boldsymbol{\mathcal{E}}_1 \cdot \nabla\boldsymbol{\mathcal{E}}_1 \quad , \tag{3.44}$$

while equations (3.37) and (3.39) correspond to the imaginary and real part of

$$\mathrm{Re}(\boldsymbol{\mathcal{E}}_2)\nabla^2\boldsymbol{\mathcal{E}}_2 = \nabla\boldsymbol{\mathcal{E}}_2 \cdot \nabla\boldsymbol{\mathcal{E}}_2 \quad . \tag{3.45}$$

Last two formulas are formally identical and can be recognized as the vacuum Ernst equation (1.29) already known from 4d formulation. However, in the present context potentials are defined differently, and the twist is used only in connection to the gravitational function χ, while electromagnetic component A_φ appears in $\boldsymbol{\mathcal{E}}_1$ without considering any mediation with new functions. Like the four-dimensional counterparts, the equations for γ are completely uncoupled from the others and can be expressed in terms of $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$

$$\begin{aligned}\partial_\rho\Gamma = \frac{\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}}_2)^2}(\partial_\rho\boldsymbol{\mathcal{E}}_2\partial_\rho\boldsymbol{\mathcal{E}}_2^* - \partial z\boldsymbol{\mathcal{E}}_2\partial_z\boldsymbol{\mathcal{E}}_2^*)+ \\ + \frac{3\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}}_1)^2}(\partial_\rho\boldsymbol{\mathcal{E}}_1\partial_\rho\boldsymbol{\mathcal{E}}_1^* - \partial_z\boldsymbol{\mathcal{E}}_1\partial_z\boldsymbol{\mathcal{E}}_1^*) \quad ,\end{aligned} \tag{3.46}$$

$$\partial_z\Gamma = \frac{2\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}}_2)^2}\partial_\rho\boldsymbol{\mathcal{E}}_2\partial_z\boldsymbol{\mathcal{E}}_2^* + \frac{6\rho}{4(\mathrm{Re}\,\boldsymbol{\mathcal{E}}_1)^2}\partial_\rho\boldsymbol{\mathcal{E}}_1\partial_z\boldsymbol{\mathcal{E}}_1^* \quad . \tag{3.47}$$

These results are also shown in [12] by Yazadjiev in the context of inverse scattering method. The equations of motion (3.44) and (3.45) for the fields $\boldsymbol{\mathcal{E}}_1$, $\boldsymbol{\mathcal{E}}_2$ are characterized as the minimum of the effective action:

$$S[\boldsymbol{\mathcal{E}}_1,\boldsymbol{\mathcal{E}}_2] = \int \rho d\rho dz \left[\frac{\nabla\boldsymbol{\mathcal{E}}_1\nabla\boldsymbol{\mathcal{E}}_1^*}{(\boldsymbol{\mathcal{E}}_1+\boldsymbol{\mathcal{E}}_1^*)^2} + \frac{\nabla\boldsymbol{\mathcal{E}}_2\nabla\boldsymbol{\mathcal{E}}_2^*}{(\boldsymbol{\mathcal{E}}_2+\boldsymbol{\mathcal{E}}_2^*)^2}\right] \quad . \tag{3.48}$$

In this case the proof is straightforward if we consider that the least action principle

$$0 = \frac{\delta S}{\delta\varphi^i} \quad , \quad S[\varphi^i] = \int \mathcal{L}(\varphi^i, \nabla_\mu\varphi^i)\sqrt{|g|}\mathrm{d}^n x \tag{3.49}$$

gives rise to Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\varphi^i} - \nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\varphi^i)}\right) = 0 \quad . \tag{3.50}$$

In this specific case we can consider $\boldsymbol{\mathcal{E}}_1^*$ and $\boldsymbol{\mathcal{E}}_2^*$ as unknown fields of the theory (they are not independent of $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$, so the choice of one pair of fields or their conjugates returns equivalent equations), in this way we find

$$\nabla_\mu\left(\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\boldsymbol{\mathcal{E}}_1^*)}\right) = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\boldsymbol{\mathcal{E}}_1^*)}\right) = \tag{3.51}$$

$$= \frac{1}{\rho}\Big[\frac{\partial_\rho \boldsymbol{\mathcal{E}}_1 + \rho\partial_{\rho\rho}\boldsymbol{\mathcal{E}}_1 + \rho\partial_{zz}\boldsymbol{\mathcal{E}}_1}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^2} - 2\rho\frac{\partial_\rho\boldsymbol{\mathcal{E}}_1}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^3}\Big(\partial_\rho\boldsymbol{\mathcal{E}}_1 + \partial_\rho\boldsymbol{\mathcal{E}}_1^*\Big) - \\ - 2\rho\frac{\partial_z\boldsymbol{\mathcal{E}}_1}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^3}\Big(\partial_z\boldsymbol{\mathcal{E}}_1 + \partial_z\boldsymbol{\mathcal{E}}_1^*\Big)\Big] =$$

$$= \frac{\nabla^2\boldsymbol{\mathcal{E}}_1}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^2} - \frac{2}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^3}(\nabla\boldsymbol{\mathcal{E}}_1\nabla\boldsymbol{\mathcal{E}}_1 + \nabla\boldsymbol{\mathcal{E}}_1\nabla\boldsymbol{\mathcal{E}}_1^*) \quad ,$$

where covariant derivatives and g are referred to cylindrical metric in flat space (in this context we could have substituted covariant derivatives with vector operators, but they are basically the same). The other term of Euler-Lagrange equations is

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{\mathcal{E}}_1^*} = -2\frac{\nabla\boldsymbol{\mathcal{E}}_1\nabla\boldsymbol{\mathcal{E}}_1^*}{(\boldsymbol{\mathcal{E}}_1 + \boldsymbol{\mathcal{E}}_1^*)^3} \quad . \tag{3.52}$$

So we can see that the effective Lagrangian (3.48) retrieves correctly the equation for the field $\boldsymbol{\mathcal{E}}_1^*$. Because of the action principle's symmetrical form, the calculation respect to the variation of $\boldsymbol{\mathcal{E}}_2^*$ is identical and returns (3.45).

3.3 5d Lie Symmetries

Having set the right equations and their correspondent action, we can now apply methods from chapter 2 in order to find all the infinitesimal generators of Lie point symmetries. Every single term of (3.48) represents the particular case of $\boldsymbol{\Phi} = 0$ in the four dimensional Einstein-Maxwell action (1.30), so we expect that symmetry transformations on $\boldsymbol{\mathcal{E}}_1$ and $\boldsymbol{\mathcal{E}}_2$ form a subgroup of (2.44). In order to prove it we just need to check that these mappings do not mix $\boldsymbol{\mathcal{E}}_1$ with $\boldsymbol{\mathcal{E}}_2$. As in chapter two, considering the components of complex potentials $\boldsymbol{\mathcal{E}}_1 = x + iy$ and $\boldsymbol{\mathcal{E}}_2 = u + iv$, we can associate a metric to action (3.48) and find its Killing vector fields. In this case we find

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{4x^2} + \frac{\mathrm{d}u^2 + \mathrm{d}v^2}{4u^2} \quad , \tag{3.53}$$

whose Killing vectors (a priori we must assume $\xi_i = \xi_i(x, y, u, v)$) are solutions of the following system of ten PDEs (we split it into blocks in order to ease its visualization)

$$\begin{cases} \partial_u\xi_x + \partial_x\xi_u = 0 \\ \partial_v\xi_x + \partial_x\xi_v = 0 \\ \partial_u\xi_y + \partial_y\xi_u = 0 \\ \partial_v\xi_y + \partial_y\xi_v = 0 \end{cases} \Longrightarrow \begin{cases} \partial_u(\partial_x\xi_y - \partial_y\xi_x) = 0 \\ \partial_v(\partial_x\xi_y - \partial_y\xi_x) = 0 \\ \partial_x(\partial_u\xi_v - \partial_v\xi_u) = 0 \\ \partial_y(\partial_u\xi_v - \partial_v\xi_u) = 0 \end{cases} \tag{3.54}$$

$$\begin{cases} \dfrac{2\xi_y}{x} + \partial_y \xi_x + \partial_x \xi_y = 0 \\ \partial_x \xi_x + \dfrac{\xi_x}{x} = 0 \\ \partial_y \xi_y - \dfrac{\xi_x}{x} = 0 \end{cases} \tag{3.55}$$

$$\begin{cases} \dfrac{2\xi_v}{u} + \partial_v \xi_u + \partial_u \xi_v = 0 \\ \partial_u \xi_u + \dfrac{\xi_u}{u} = 0 \\ \partial_v \xi_v - \dfrac{\xi_u}{u} = 0 \end{cases} \tag{3.56}$$

We can start by solving block (3.55). The equation for ξ_x is already uncoupled and gives

$$\xi_x = \frac{C(y,u,v)}{x} \quad , \tag{3.57}$$

where $C(y,u,v)$ is an integration function. Differentiating the first equation of the block respect to y, the third respect to x and subtracting them, one finds that $\partial_{yy}C(y,u,v) = 0$, which means that C is linear in y. We can specify ξ_x further by writing

$$\xi_x = \frac{D(u,v) + B(u,v)y}{x} \quad . \tag{3.58}$$

Integrating now the third equation of the block, we find that ξ_y depends on a new function $E(x,u,v)$ according to

$$\xi_y = \frac{1}{x^2}\left[D(u,v)y + \frac{B(u,v)}{2}y^2\right] + E(x,u,v) \quad . \tag{3.59}$$

Substituting the solution in the first equation, which specifies $E(x,u,v) = A(u,v)/x^2 - B(u,v)/2$, we find that ξ_y must be in the form

$$\xi_y = \frac{1}{x^2}\left[D(u,v)y + \frac{B(u,v)}{2}(y^2 - x^2) + A(u,v)\right] \quad . \tag{3.60}$$

We can now consider the first block (3.54) and notice that the combination $\partial_x \xi_y - \partial_y \xi_x$ must be independent both of u and v. Since it turns out to be a polynomial in the variables x and y, the independence can be satisfied only when all coefficients $D(u,v)$, $B(u,v)$ and $A(u,v)$ are constant functions. Block (3.56) is formally identical to the system we have already solved, so it is possible to write its solution just by replacing the subscripts. That said, contravariant components (indices are raised with the inverse of (3.53)) of the most general

Killing vector field are given by

$$
\begin{aligned}
\xi^x &= 4x(d+by) \quad , \\
\xi^y &= 4\Big[dy + \frac{b}{2}(y^2 - x^2) + a\Big] \quad , \\
\xi^u &= 4u(e+fv) \quad , \\
\xi^v &= 4\Big[ev + \frac{f}{2}(y^2 - x^2) + h\Big] \quad ,
\end{aligned}
\tag{3.61}
$$

which correspond, considering in turn just one non-zero parameter, to six independent generators

$$
\begin{aligned}
\xi_1 &= 4xy\partial_x + 2(y^2 - x^2)\partial_y \, , \\
\xi_2 &= 4x\partial_x + 4y\partial_y \, , \\
\xi_3 &= 4\partial_y \, , \\
\xi_4 &= 4uv\partial_u + 2(v^2 - u^2)\partial_v \, , \\
\xi_5 &= 4u\partial_u + 4v\partial_v \, , \\
\xi_6 &= 4\partial_v \, .
\end{aligned}
\tag{3.62}
$$

We are now sure that our initial guess about symmetry transformations that do not mix x, y with u, v, then $\boldsymbol{\mathcal{E}}_1$ with $\boldsymbol{\mathcal{E}}_2$ is true. The present infinitesimal transformations are two copies of the four dimensional vacuum case: as such they were already integrated in more general form in chapter 2. In particular, the integration of ξ_1 and ξ_4 is a particular case of (2.39). The relevant finite transformations, expressed in terms of complex potentials, read

$$
\begin{cases}
\boldsymbol{\mathcal{E}}_1 \longmapsto \boldsymbol{\mathcal{E}}'_1 = \lambda\lambda^* \boldsymbol{\mathcal{E}}_1 \\
\boldsymbol{\mathcal{E}}_1 \longmapsto \boldsymbol{\mathcal{E}}'_1 = \boldsymbol{\mathcal{E}}_1 + ia \\
\boldsymbol{\mathcal{E}}_1 \longmapsto \boldsymbol{\mathcal{E}}'_1 = \dfrac{\boldsymbol{\mathcal{E}}_1}{1 + ib\boldsymbol{\mathcal{E}}_1}
\end{cases}
\qquad
\begin{cases}
\boldsymbol{\mathcal{E}}_2 \longmapsto \boldsymbol{\mathcal{E}}'_2 = \gamma\gamma^* \boldsymbol{\mathcal{E}}_2 \\
\boldsymbol{\mathcal{E}}_2 \longmapsto \boldsymbol{\mathcal{E}}'_2 = \boldsymbol{\mathcal{E}}_2 + ic \\
\boldsymbol{\mathcal{E}}_2 \longmapsto \boldsymbol{\mathcal{E}}'_2 = \dfrac{\boldsymbol{\mathcal{E}}_2}{1 + i\ell\boldsymbol{\mathcal{E}}_2}
\end{cases}
\tag{3.63}
$$

The first two mappings are gauge transformations, while the third is known as "Ehlers transformation" and can be used to generate new nonequivalent solutions. For example, as shown in [1], in the context of 4d Einstein-Maxwell theory, it can be used to add NUT charge to the seed metric. Because of the different definition of potentials, in this five dimensional formalism the transformation acts differently respect to the four dimensional formulation, adding two dissimilar charges depending on whether it is used on $\boldsymbol{\mathcal{E}}_1$ or $\boldsymbol{\mathcal{E}}_2$. Moreover, before considering specific seed solutions and computing how the transformation acts on them, we cannot be able to define the physical significance of the couple of parameters. We will now show that Ehlers transformation, once applied to $\boldsymbol{\mathcal{E}}_1$, is equivalent to (3.8) in 5d. In light of the calculations done with (3.8), we will identify the symmetry as a magnetizing/charging transformation. First of all, when $n = 5$ we have

$$
\begin{cases}
\Lambda = \Big(1+\dfrac{2}{3}BA_\varphi\Big)^2 + \dfrac{1}{3}B^2V \\[2ex]
g'_{ij} = \Lambda g_{ij} \quad , \qquad V' = \Lambda^{-2}V \\[2ex]
A'_\varphi = \Lambda^{-1}\Big[A_\varphi + B\Big(\dfrac{1}{2}V + \dfrac{2}{3}A_\varphi^2\Big)\Big] \quad .
\end{cases}
\tag{3.64}
$$

Evaluating $\boldsymbol{\mathcal{E}}_1/(1+ib\boldsymbol{\mathcal{E}}_1)$ explicitly using metric functions we can check that there exists a value for b that satisfies (3.64).

$$
\boldsymbol{\mathcal{E}}'_1 = \frac{e^T + \frac{2i}{\sqrt{3}}A_\varphi}{1 + ibe^T - \frac{2b}{\sqrt{3}}A_\varphi} =
$$

$$
= \frac{1}{(\sqrt{3}-2bA_\varphi)^2 + 3b^2e^{2T}}\Big[3e^T + i(2\sqrt{3}A_\varphi - 4bA_\varphi^2 - 3be^{2T})\Big] \quad . \tag{3.65}
$$

Taking real and imaginary parts of $\boldsymbol{\mathcal{E}}'_1$, it is possible to find the new potentials $e^{T'}$ and A'_φ.

$$
V' = e^{2T'} = \mathrm{Re}(\boldsymbol{\mathcal{E}}'_1)^2 = \frac{e^{2T}}{\Big[\Big(1 - \frac{2b}{\sqrt{3}}A_\varphi\Big)^2 + b^2e^{2T}\Big]^2} \tag{3.66}
$$

We see that $V' = \Lambda^{-2}V$ holds if we set $b = -B/\sqrt{3}$. Analogously we can work out A'_φ from the imaginary part $\boldsymbol{\mathcal{E}}'_1$. Substituting $b = -B/\sqrt{3}$ we have

$$
A'_\varphi = -i\frac{\sqrt{3}}{2}\,\mathrm{Im}(\boldsymbol{\mathcal{E}}'_1) =
$$

$$
= \frac{1}{2\sqrt{3}}\frac{1}{\Big(1+\frac{2B}{3}A_\varphi\Big)^2 + \frac{1}{3}B^2e^{2T}}\Big(2\sqrt{3}A_\varphi + \frac{4}{\sqrt{3}}BA_\varphi^2 + \sqrt{3}Be^{2T}\Big) =
$$

$$
= \frac{1}{\Lambda}\Big(A_\varphi + \frac{2}{3}BA_\varphi^2 + \frac{1}{2}Be^{2T}\Big) \quad , \tag{3.67}
$$

which is the same of the correspondent transformation in (3.64). Considering the invariance of S and Γ under this mapping, we see that the transformation of the other components g_{ij} of the metric follows e^{-T}. Since $e^{-T'} = \Lambda e^{-T}$, all non-Killing components are mapped to $g'_{ij} = \Lambda g_{ij}$.

We can now propose the same argument, yet considering $\boldsymbol{\mathcal{E}}_2 = e^S + i\chi$ as seed potential. Exploiting its explicit form and separating real and imaginary part, we can write transformation laws for e^S and χ

$$
e^{S'} = \frac{e^S}{(1-\ell\chi)^2 + \ell^2e^{2S}} \quad , \tag{3.68}
$$

$$
\chi' = \frac{\chi - \ell\chi^2 - \ell e^{2S}}{(1-\ell\chi)^2 + \ell^2e^{2S}} \quad . \tag{3.69}
$$

If we define $\Omega := (1-\ell\chi)^2 + \ell^2 e^{2S}$ the full symmetry group acting both on the metric and the electromagnetic potential can be written as

$$\begin{cases} e^{S-T} \longmapsto e^{S'-T'} = \dfrac{\Lambda}{\Omega} e^{S-T} \quad , \\ e^{-S-T} \longmapsto e^{-S'-T'} = \Lambda\Omega e^{-S-T} \quad , \\ e^{2T} \longmapsto e^{2T'} = \dfrac{1}{\Lambda^2} e^{2T} \quad , \\ \omega \longmapsto \omega' \quad , \\ A_\varphi \longmapsto A'_\varphi = \Lambda^{-1}\Big[A_\varphi + B\Big(\dfrac{1}{2}V + \dfrac{2}{3}A_\varphi^2\Big)\Big] \quad , \end{cases} \tag{3.70}$$

where ω' satisfies

$$\begin{cases} \partial_\rho \omega' = \rho e^{-2S'} \partial_z \chi' \\ \partial_z \omega' = -\rho e^{-2S'} \partial_\rho \chi' \quad . \end{cases} \tag{3.71}$$

3.4 Ehlers

In this section we will apply Ehlers transformation on the second potential $\boldsymbol{\mathcal{E}}_2$ starting again from Tangherlini black hole. Introducing the new coordinate $x = \sin\theta$ we can write the seed metric as

$$\mathrm{d}s^2 = -\Big(1 - \frac{\mu}{r^2}\Big)\mathrm{d}t^2 + \Big(1 - \frac{\mu}{r^2}\Big)^{-1}\mathrm{d}r^2 + \frac{r^2}{1-x^2}\mathrm{d}x^2 + r^2(1-x^2)\mathrm{d}\psi^2 + r^2x^2\mathrm{d}\varphi^2 \quad . \tag{3.72}$$

As we have already done in various contexts, it is possible to compare this particular metric with the general ansatz (3.30) in order to identify the various metric potentials. First of all we can see that $\omega = 0$ and recognize T and S

$$e^T = rx \quad , \tag{3.73}$$

$$e^{S-T} = 1 - \frac{\mu}{r^2} \Longrightarrow e^S = \frac{(r^2-\mu)x}{r} \quad . \tag{3.74}$$

The component $g_{\psi\psi}$ supplies the change of coordinates between ρ and the couple (r, x)

$$e^{-S-T}\rho^2 = r^2(1-x^2) \Longrightarrow \rho = xr\sqrt{r^2-\mu}\sqrt{1-x^2} \quad . \tag{3.75}$$

At this point, applying Ehlers transformation in the form of (3.68) and (3.69) to the seed $\boldsymbol{\mathcal{E}}_2 = e^S$, we obtain

$$e^{S'} = \frac{rx(r^2-\mu)}{r^2 + \ell^2(\mu - r^2)^2x^2} \quad , \tag{3.76}$$

$$\chi' = -\frac{\ell(\mu - r^2)x^2}{r^2 + \ell^2(\mu - r^2)x^2} \quad . \tag{3.77}$$

Since we know χ' we may calculate ω' using (3.71). However, having expressed the potentials in function of the couple (r, x) intead of (ρ, z), it is convenient to convert differential operators as we did in chapter 2 while charging Kerr. Whereas in four dimensions the change of coordinates for z is well known, in this case we need to find a suitable one asking the metric be diagonal also in coordinates (r, x). Assuming the coordinate $z(r, x)$ in the form $z(r, x) = f(r)g(x)$ and calculating $\mathrm{d}\rho^2 + \mathrm{d}z^2$ in function of $\mathrm{d}r$ and $\mathrm{d}x$, one can see that diagonal terms vanish only if the following differential equation is satisfied

$$rx(\mu - r^2)(2x^2 - 1) + f(r)g(x)\frac{\mathrm{d}}{\mathrm{d}r}f(r)\frac{\mathrm{d}}{\mathrm{d}x}g(x) = 0 \quad . \tag{3.78}$$

The equation is separable and can be easily integrated. The integration constants are chosen in order to have

$$\begin{cases} f(r) = r^2 - \dfrac{\mu}{2} \\ g(x) = x^2 - \dfrac{1}{2} \end{cases} \Longrightarrow z(r, x) = \Big(r^2 - \frac{\mu}{2}\Big)\Big(x^2 - \frac{1}{2}\Big) \quad . \tag{3.79}$$

We can now express the line element in the new coordinate system

$$\mathrm{d}\rho^2 + \mathrm{d}z^2 = \frac{(r^2 - \mu x^2)\Big[r^2 - \mu(1 - x^2)\Big]}{r^2}\left[\Big(1 - \frac{\mu}{r^2}\Big)^{-1}\mathrm{d}r^2 + \frac{r^2}{1 - x^2}\mathrm{d}x^2\right] \quad , \tag{3.80}$$

which allows to write the explicit form of gradient operator

$$\begin{aligned} \nabla\Omega(r, x) = \frac{1}{\sqrt{(r^2 - \mu x^2)[r^2 - \mu(1 - x^2)]}}\Big[&\hat{r}\sqrt{r^2 - \mu}\frac{\partial}{\partial r}\Omega(r, x) + \\ &+ \hat{x}\sqrt{1 - x^2}\frac{\partial}{\partial x}\Omega(r, x)\Big] \quad . \end{aligned} \tag{3.81}$$

Now it is possible to write (3.71) in terms of the new coordinates

$$\begin{cases} \partial_r\omega' = \rho(r, x)e^{-2S'}\sqrt{\dfrac{1 - x^2}{r^2 - \mu}}\partial_x\chi' = -2\ell r(1 - x^2) \\ \\ \partial_x\omega' = -\rho(r, x)e^{-2S'}\sqrt{\dfrac{r^2 - \mu}{1 - x^2}}\partial_r\chi' = 2\ell x(\mu + r^2) \, . \end{cases} \tag{3.82}$$

The integration of the system is straightforward and yields

$$\omega' = \ell\Big[\mu x^2 - r^2(1 - x^2)\Big] \quad . \tag{3.83}$$

The last potential Γ can be easily found when Weyl line element is converted with equation (3.80): the condition

$$\frac{e^{-S-T-2\Gamma}}{r^2}(r^2-\mu x^2)\Big[r^2-\mu(1-x^2)\Big]=1 \quad , \tag{3.84}$$

defines

$$e^{2\Gamma}=\frac{r^2x^2\,(r^2-\mu)}{(r^2-\mu x^2)\,[(r^2-\mu\,(1-x^2)]} \quad . \tag{3.85}$$

Then we can write the final metric

$$\begin{aligned}\mathrm{d}s^2=&-\frac{r^2-\mu}{r^2-\ell^2(r^2-\mu)^2x^2}\Big[\mathrm{d}t-\ell[\mu x^2-r^2(1-x^2)]\mathrm{d}\psi\Big]^2+\\&+\left(1+\frac{\ell^2(r^2-\mu)^2x^2}{r^2}\right)\left[\left(1-\frac{\mu}{r^2}\right)^{-1}\mathrm{d}r^2+\frac{r^2}{1-x^2}\mathrm{d}x^2\right]+\\&+(1-x^2)\Big[r^2-\ell^2(r^2-\mu)x^2\Big]\mathrm{d}\psi^2+r^2x^2\mathrm{d}\varphi^2 \quad .\end{aligned} \tag{3.86}$$

Appendix **A**

4d Einstein equations

As anticipated in Chapter 1, we show how to recast Einstein equations in order to introduce Ernst potentials in four dimensions. Maxwell equations (1.7) and (1.8) comes straightforwardly from $\nabla_\mu F^{\mu\varphi} = 0$ and $\nabla_\mu F^{\mu t} = 0$, so it is possible to bring about the computation by writing more explicitly

$$\nabla_\mu F^{\mu\varphi} = 0 = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\varphi}) \quad , \tag{A.1}$$

and expanding all the terms. In this formula $g = -e^{4\gamma} r^2/f^2$ is the determinant of the metric (1.4) and the electromagnetic strength tensor is defined by $F^{\mu\nu} = g^{\mu\lambda} g^{\nu\sigma} F_{\lambda\sigma}$, with $F_{\lambda\sigma} = \partial_{[\lambda} A_{\sigma]}$ such that

$$F_{\lambda\sigma} = \begin{pmatrix} 0 & \partial_\rho A_t & \partial_z A_t & 0 \\ -\partial_\rho A_t & 0 & 0 & -\partial_\rho A_\varphi \\ -\partial_z A_t & 0 & 0 & -\partial_z A_\varphi \\ 0 & \partial_\rho A_\varphi & \partial_z A_\varphi & 0 \end{pmatrix} \quad . \tag{A.2}$$

The case of Einstein equations is a bit more complicated because (1.5) and (1.6) do not correspond to the explicit form gravitational equations (1.1) assume. Because of this, in order to bring them in a more suitable form, we need to consider linear combinations that allow to simplify similar terms. In order to accomplish this task, it can be convenient to take the trace of Einstein equations (1.1) so as to simplify R. Indeed, since the electromagnetic energy-momentum tensor is traceless, we find that $R = 0$, which allows to write Einstein equations as

$$H_{\mu\nu} := R_{\mu\nu} - 2T_{\mu\nu} = 0 \quad . \tag{A.3}$$

Explicitly, the various non-trivial components read (removing factorized overall terms)

$$\begin{aligned} H_{tt} = -2f^3 \Big[&(\partial_\rho A_\varphi)^2 + (\partial_z A_\varphi)^2 + \omega^2 \Big((\partial_\rho A_t)^2 + (\partial_z A_t)^2 \Big) - 2\omega (\partial_z A_t \partial_z A_\varphi + \\ &+ \partial_\rho A_t \partial_\rho A_\varphi) \Big] - r^2 \Big[(\partial_z f)^2 + (\partial_\rho f)^2 \Big] + f^4 \Big[(\partial_z \omega)^2 + (\partial_\rho \omega)^2 \Big] + \\ &+ rf \Big[\partial_\rho f + r \partial_{\rho\rho} f + r \partial_{zz} f - 2r \big((\partial_\rho A_\varphi)^2 + (\partial_z A_\varphi)^2 \big) \Big] = 0 \ , \end{aligned} \tag{A.4}$$

This component, once divided by r, directly corresponds to the second Einstein equation (1.6). Basically this is the reason why we choose to use (A.3): otherwise the component H_{tt} would have carried derivatives of γ coupled to the other potentials. The second useful relation is

$$\begin{aligned}
H_{t\varphi} = -2f^3\omega\Big[(\partial_\rho A_\varphi)^2 + (\partial_z A_\varphi)^2 + \omega^2\Big((\partial_\rho A_t)^2 + (\partial_z A_t)^2\Big) - \\
- 2\omega\Big(\partial_\rho A_t \partial_\rho A_\varphi + \partial_z A_t \partial_z A_\varphi\Big)\Big] + rf^2(-\partial_\rho\omega + \partial_{\rho\rho}\omega + \partial_{zz}\omega) + \\
+ rf\omega\left[\partial_\rho f + r\partial_{\rho\rho} f + r\partial_{zz} f + 2r\Big[(\partial_\rho A_t)^2 + (\partial_z A_t)^2\Big]\right] - \\
- r^2\omega\Big[(\partial_\rho f)^2 + (\partial_z f)^2\Big] + f^4\omega\Big[(\partial_\rho\omega)^2 + (\partial_z\omega)^2\Big] = 0\,. \quad \text{(A.5)}
\end{aligned}$$

These two components are enough to retrieve the first Einstein equation (1.5). As a matter of fact, we can combine them trying to remove the dependence on the second derivatives of f, perhaps the most direct way to do it is to consider

$$h_1 := \frac{H_{tt}\cdot\omega - H_{t\varphi}}{r^4} = 0 \quad , \qquad \text{(A.6)}$$

which corresponds, in vector notation, to the following equation

$$h_1 = \nabla\cdot(\rho^{-2}f^2\nabla\omega) + 4\rho^{-2}f(\nabla A_t)\cdot(\nabla A_\varphi + \omega\nabla A_t) = 0 \quad . \qquad \text{(A.7)}$$

Summing the first Maxwell equation (scaled by $4A_t$) to h_1 we find Einstein equation (1.5)

$$\begin{aligned}
0 = h_1 + 4A_t\nabla\cdot\Big[\rho^{-2}f(\nabla A_\varphi + \omega\nabla A_t)\Big] = \\
= \nabla\cdot\Big[\rho^{-2}f^2\nabla\omega + 4\rho^{-2}fA_t(\nabla A_\varphi + \omega\nabla A_t)\Big] \quad . \qquad \text{(A.8)}
\end{aligned}$$

The other non-trivial components $H_{\rho\rho}$, H_{zz} and $H_{\rho z}$ define γ, while $H_{\varphi\varphi}$ is equivalent to the couple H_{tt}, $H_{t\varphi}$. The equations that allow to integrate γ once the other potentials are known are supplied by

$$\begin{aligned}
H_{\rho z} = \frac{\partial_z\gamma}{r} + \frac{a\partial_z A_t\partial_\rho A_t}{f} - \frac{2f(\omega\partial_z A_t - \partial_z A_\varphi)(\omega\partial_\rho A_t - \partial_\rho A_\varphi)}{r^2} - \\
- \frac{\partial_z f\partial_\rho f}{2f^2} + \frac{f^2\partial_z\omega\partial_\rho\omega}{2r^2} = 0 \quad , \qquad \text{(A.9)}
\end{aligned}$$

and by the the difference

$$\begin{aligned}
H_{\rho\rho} - H_{zz} = \frac{\partial_\rho\gamma}{r} + \frac{(\partial_\rho A_t)^2 - (\partial_z A_t)^2}{f} + \frac{f}{r^2}\bigg[(\partial_z A_\varphi)^2 - (\partial_\rho A_\varphi)^2 + \\
+ \omega^2\Big[(\partial_z A_t)^2 - (\partial_\rho A_t)^2\Big] + 2\omega(\partial_\rho A_t\partial_\rho A_\varphi - \partial_z A_t\partial_z A_\varphi)\bigg] + \\
+ \frac{(\partial_z f)^2 - (\partial_\rho f)^2}{f^2} + \frac{f^2}{r^2}\Big[(\partial_z\omega)^2 - (\partial_\rho\omega)^2\Big] = 0 \quad . \qquad \text{(A.10)}
\end{aligned}$$

Appendix B

5d Einstein equations

Similarly to what we did in the previous appendix, we recast the raw Einstein equation in five dimensions so as to give them an appropriate form to introduce Ernst potentials. First of all we define

$$H_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - 2\Big[F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big] \quad , \tag{B.1}$$

so that Einstein equations reduce to

$$H_{\mu\nu} = 0 \quad . \tag{B.2}$$

They can be written explicitly once we consider the ansatz

$$\mathrm{d}s^2 = -e^{S-T}(\mathrm{d}t - \omega\mathrm{d}\psi)^2 + e^{-S-T}\rho^2\mathrm{d}\psi^2 + e^{-S-T+2\Gamma}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + e^{2T}\mathrm{d}\varphi^2 \ , \tag{B.3}$$

$$A = A_\varphi \mathrm{d}\varphi \quad . \tag{B.4}$$

Non trivial components of $H_{\mu\nu} = 0$ read (minus overall terms which do not influence the solution):

$$H_{tt} = \Big(-4e^{2T}(\partial_z A_\varphi)^2 - (\partial_z S)^2 - 3(\partial_z T)^2 + \frac{3e^{2S}(\partial_z\omega)^2}{\rho^2} - 4e^{2T}(\partial_\rho A_\varphi)^2 - (\partial_\rho S)^2 - \\ - 3(\partial_\rho T)^2 + \frac{3e^{2S}(\partial_\rho\omega)^2}{\rho^2} + 4(\partial_{zz}S + \frac{\partial_\rho S}{\rho} + \partial_{\rho\rho}S) - 4(\partial_{zz}\gamma + \partial_{\rho\rho}\gamma)\Big) = 0 \tag{B.5}$$

$$H_{t\psi} = \Big[-4(\partial_z S\partial_z\omega + \partial_\rho S\partial_\rho\omega) - 2(\partial_{zz}\omega - \frac{\partial_\rho\omega}{\rho} + \partial_{\rho\rho}\omega) + \omega\Big(-4e^{2T}(\partial_z A_\varphi)^2 - \\ - (\partial_z S)^2 - 3(\partial_z T)^2 + \frac{3e^{2S}(\partial_z\omega)^2}{\rho^2} - 4e^{2T}(\partial_\rho A_\varphi)^2 - (\partial_\rho S)^2 - 3(\partial_\rho T)^2 + \\ + \frac{3e^{2S}(\partial_\rho\omega)^2}{\rho^2} + 4(\partial_{zz}S + \frac{\partial_\rho S}{\rho} + \partial_{\rho\rho}S) - 4(\partial_{zz}\gamma + \partial_{\rho\rho}\gamma)\Big)\Big] = 0 \tag{B.6}$$

$$H_{\rho\rho} = \Big(4e^{2T}(\partial_z A_\varphi)^2 + (\partial_z S)^2 + 3(\partial_z T)^2 - \frac{3e^{2S}(\partial_z \omega)^2}{\rho^2} + 4e^{2T}(\partial_\rho A_\varphi)^2 + (\partial_\rho S)^2 - \\ - 3(\partial_\rho T)^2 + \frac{3e^{2S}(\partial_\rho \omega)^2}{\rho^2} + \frac{4\partial_\rho \gamma}{\rho}\Big) = 0 \tag{B.7}$$

$$H_{\rho z} = \Big(\frac{2\partial_z \gamma}{\rho} - 4e^{-2T}\partial_z A_\varphi \partial_\rho A_\varphi - \partial_z S \partial_\rho S - 3\partial_z T \partial_\rho T + \frac{e^{2S}\partial_z \omega \partial_\rho \omega}{\rho^2}\Big) = 0 \tag{B.8}$$

$$H_{\psi\psi} = \Big[(\rho^2 - e^{2S}\omega^2)\Big((\partial_z S)^2 + (\partial_z T)^2\Big) + 8e^{2S}\omega(\partial_z S \partial_z \omega + \partial_\rho S \partial_\rho \omega) + e^{2S}\omega^2 + \\ + \frac{3e^{4S}\omega^2(\partial_z \omega)^2}{\rho^2} + 4e^{2S}\omega^2 \partial_{zz} S + 4\rho^2 \partial_{zz}\gamma - 4e^{2S}\omega^2\partial_{zz}\gamma + 4e^{2S}\omega\partial_{zz}\omega + \\ + 4e^{-2T}(\rho^2 - e^{2S}\omega^2)\Big((\partial_z A_\varphi)^2 + (\partial_\rho A_\varphi)^2\Big) + \frac{4e^{2S}\omega^2\partial_\rho S}{\rho} + \rho^2(\partial_\rho S)^2 - \\ - e^{2S}\omega^2(\partial_\rho S)^2 + 3\rho^2(\partial_\rho T)^2 - 3e^{2S}\omega^2(\partial_\rho T)^3 - \frac{4e^{2S}\omega^2\partial_z \omega}{\rho} + \\ + 8e^{2S}\omega\partial_\rho S \partial_\rho \omega + e^{2S}(\partial_\rho \omega)^2 + \frac{3e^{4S}\omega^2(\partial_z \omega)^2}{\rho^2} + 4e^{2S}\omega^2\partial_{\rho\rho} S + \\ + 4\rho^2\partial_{\rho\rho}\gamma - 4e^{2S}\omega^2\partial_{\rho\rho}\gamma + 4e^{2S}\omega\partial_{\rho\rho}\omega\Big] = 0 \tag{B.9}$$

$$H_{\varphi\varphi} = \Big[-4(\partial_z A_\varphi)^2 - 4(\partial_\rho A_\varphi)^2 + \frac{1}{\rho^2}e^{2T}\Big(\rho^2(\partial_z S)^2 + 3\rho^2(\partial_z T)^2 - e^{2S}(\partial_z \omega)^2 - \\ - 2\rho^2\partial_{zz} S - 6\rho^2\partial_{zz} T + 4\rho^2\partial_{\rho\rho} S - 6\rho^2\partial_{\rho\rho} T + 4\rho^2\partial_{\rho\rho}\gamma\Big)\Big] = 0 \tag{B.10}$$

The last non trivial component is H_{zz}, however, since it is related to $H_{\rho\rho}$ by the simple relation $H_{\rho\rho} = -H_{zz}$, it does not add any relevant information to the system. At this point we can focus on H_{tt} and $H_{t\psi}$ in order to delete the dependence on γ. By performing

$$0 = H_{tt} \cdot \omega + H_{t\psi} =: h_1 \tag{B.11}$$

only the first terms of $H_{t\psi}$ survive, then we get

$$2(\partial_\rho S \partial_\rho \omega + \partial_z S \partial_z \omega) + \partial_{\rho\rho}\omega + \partial_{zz}\omega - \frac{1}{\rho}\partial_\rho \omega = 0 \quad , \tag{B.12}$$

which is equation (3.34). Considering

$$0 = \frac{\rho^2 \cdot H_{\psi\psi} + 4\rho^2\omega e^{2S}h_1 + (\rho^4 - \rho^2\omega^2 e^{2S})H_{tt}}{4\rho^2} =: h_2 \quad , \tag{B.13}$$

we find equation (3.38):

$$\frac{e^{2S}}{\rho^2}\Big[(\partial_\rho\omega)^2+(\partial_z\omega)^2\Big]+\partial_{\rho\rho}S+\partial_{zz}S+\frac{1}{\rho}\partial_\rho S=0 \quad . \tag{B.14}$$

As last step, we can perform the linear combination

$$-\left(H_{tt}+e^{-2T}H_{\varphi\varphi}\right)+h_2 \tag{B.15}$$

in order to retrieve equation (3.40)

$$3\Big(\partial_{\rho\rho}T+\partial_{zz}T+\frac{1}{\rho}\partial_\rho T\Big)+4e^{-2T}\Big[(\partial_\rho A_\varphi)^2+(\partial_z A_\varphi)^2\Big]=0 \quad . \tag{B.16}$$

As a conclusive remark we highlight that this process allows to uncouple the equations that define ω, T, S, A_φ from the ones that define γ. Once solved the first set, one can consider $H_{\rho\rho}$, $H_{\rho z}$ and find γ by simple quadratures.

Bibliography

[1] M. Astorino. “Enhanced Ehlers Transformation and the Majumdar-Papapetrou-NUT Spacetime”. In: *Journal of High Energy Physics* 2020.123 (2020). DOI: https://doi.org/10.1007/JHEP01(2020)123.

[2] F. J. Ernst. “New Formulation of the Axially Symmetric Gravitational Field Problem”. In: *Physical Review* 167.5 (1968), pp. 1175–1178. DOI: https://doi.org/10.1103/PhysRev.167.1175.

[3] F. J. Ernst. “New Formulation of the Axially Symmetric Gravitational Field Problem. II”. In: *Physical Review* 168.5 (1968), pp. 1415–1417. DOI: https://doi.org/10.1103/PhysRev.168.1415.

[4] E. T. Newman *et al.* “Metric of a Rotating, Charged Mass”. In: *Journal of Mathematical Physics* 6.6 (1965), pp. 918–919. DOI: https://doi.org/10.1063/1.1704351.

[5] W. Kinnersley. “Symmetries of the stationary Einstein–Maxwell field equations. I”. In: *Journal of Mathematical Physics* 18 (1977), pp. 1529–1537. DOI: https://doi.org/10.1063/1.523458.

[6] M. A. Melvin. “Dynamics of Cylindrical Electromagnetic Universes”. In: *Physical Review* 139.1B (1965), pp. 225–243. DOI: https://doi.org/10.1103/PhysRev.139.B225.

[7] M. Ortaggio. “Higher dimensional black holes in external magnetic fields”. In: *Journal of High Energy Physics* 2005.05 (2005). DOI: DOI:10.1088/1126-6708/2005/05/048.

[8] H. Stephani. *Differential equations. Their solutions using symmetries*. Cambridge University Press, 1989.

[9] H. Stephani. *Exact Solutions of Einstein’s Field Equations*. Cambridge University Press, 2009.

[10] F. R. Tangherlini. “Schwarzschild field in n dimensions and the dimensionality of space problem”. In: *Il Nuovo Cimento* 27 (1963), pp. 636–651. DOI: https://doi.org/10.1007/BF02784569.

[11] K. S. Thorne. “Absolute stability of Melvin’s magnetic universe”. In: *Physical Review* 139.1B (1965), pp. 244–254. DOI: https://doi.org/10.1103/PhysRev.139.B244.

[12] S. S. Yazadjiev. “5D Einstein-Maxwell solitons and concentric rotating dipole black rings”. In: *Physical Review D* 78.064032 (2008). DOI: https://doi.org/10.1103/PhysRevD.78.064032.