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On Codazzi tensors and their application in Einstein and Cotton Gravity

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Alla nonna Cenzi, che mi ha sempre voluto un bene gigantesco,
questa laurea, anche se non in matematica, è anche tua
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Introduction

Harada[12], in 2021, formulated a new theory of gravity: the Cotton gravity. This theory extends Einstein's equations, adding a term to the gravitational potential that becomes relevant at great distances, thus describing the rotational motion of several galaxies without the dark matter contribution.

In this new paper, the energy-momentum tensor is expressed as a function of a Codazzi tensor. It is a (0,2)-symmetric tensor with the covariant property $\nabla_i C_{jk} = \nabla_j C_{ik}$. In this work the properties and applications of Codazzi tensors, both in the theory of general relativity and in Cotton's gravity, are therefore explored.

In the first chapter, their definition and application in the differential geometry, in particular in embedding theory, is presented. In fact, by using the Gauss-Codazzi equations, they describe the curvature tensor of a hypersurface (class 1 embedding) as a function of a Codazzi tensor. Examples are then given in peculiar manifolds, and two theorems are demonstrated, the Goenner theorem and the extension of the Derdzinski-Shen theorem.

In the second chapter, the work of Stephani[32] is presented. He utilizes the embedding class 1 equations to find exact solutions to Einstein's equations. Imposing the Codazzi condition to a perfect and imperfect fluid tensor, he characterizes the embedded spaces-times and makes their metrics explicit.

In the third chapter, we weaken Stephani's assumptions by no longer considering the immersion of space-time. Just using Codazzi equation applied to the perfect fluid tensor, and the current flow tensor, we determine wider solutions than Stephani's, finding a more general metric.

In the last chapter, Harada's theory is presented, together with one of its solution and application. Using the same method as in the previous chapter, solutions of Cotton's gravity are found, explicating the new field source as a function of a Codazzi perfect fluid and Codazzi imperfect fluid tensor.

Chapter 1

Codazzi tensors theory and proprieties

1.1 Embedding Manifolds and Hypersurfaces

Let V_n and V_m be two different manifolds and $\phi : V_m \to V_n$ a map between these two manifolds. If $\phi_{*p} : T_pV_m \to T_{\phi(p)}V_n$ is injective for all x in V_n, then ϕ is called an immersion, where T_pV_m is the tangent space to p.

Moreover, if ϕ is also homeomorphic, it is more accurately called an embedding. Obviously, to admit the existence of an embedding ϕ we must have $n \geq m$.

Let x^a and y^α be two coordinate systems of V_m and V_n respectively $(a = 0, 1...m \ \alpha = 0, 1...n)$, then each point p in V_m is described by the parametric equations:

$$y^\alpha = y^\alpha(x^a) \tag{1.1}$$

Thus it is possible to define the m tangent vectors which span the tangent space to V_m at x:

$$B^\alpha_a = \frac{\partial y^\alpha}{\partial x^a} \tag{1.2}$$

From this relation, it is possible to define the metric and metric tensor of the space V_m starting from those of V_n, as follows:

$$ds^2 = \tilde{g}_{\alpha\beta} dy^\alpha dy^\beta = \tilde{g}_{\alpha\beta} B^\alpha_i B^\beta_j dx^i dx^j \tag{1.3}$$

$$g_{ij} = \tilde{g}_{\alpha\beta} B^\alpha_i B^\beta_j \tag{1.4}$$

where $\tilde{g}$ is the metric tensor of V_n.

The minimum number of extra-dimensions required for embedding is called *embedding class p.* Later on, a Euclidean space will be used as the embedding space, making it useful to state the following:

Theorem 1.1.1. *Let $V_m(s,\ t)$ be a Riemannian space with s-spacelike and t-timelike directions, and E_n (S, T) the pseudo-Euclidean embedding space. Then $V_m(s,\ t)$ can be embedded in E_n (S, T) with $s + t = m$, $S + T = n$, $m \leq n \leq m(m+1)/2$, $s \leq S$, $t \leq T$. Therefore, the embedding class p in a 4-dimensional space-time is between* $0 \leq p \leq 6$.

Definition 1.1.2. *When the embedding class p is equal to one, namely $dim V_n = m + 1$, the manifold V_m is a hypersurface.*

1.2 Gauss-Codazzi equations

From this point going forward, let us restrict to hypersurfaces. We can thus define N as the only vector orthogonal to the hypersurface, so that

$$\tilde{g}_{\alpha\beta} B^{\alpha}_{a} N^{\beta} = 0. \tag{1.5}$$

By imposing the condition that the metric be covariantly constant, and using the equation (1.4):

$$\nabla_k g_{ij} = \nabla_k (\tilde{g}_{\alpha\beta} B^{\alpha}_{i} B^{\beta}_{j}) = \tilde{g}_{\alpha\beta} [(\nabla_k B^{\alpha}_{i}) B^{\beta}_{j} + B^{\alpha}_{i} (\nabla_k B^{\beta}_{j})] = 0 \tag{1.6}$$

Now subtracting the above with all three indices rotated (from (kij) to (ijk) and (jik)) from (1.6) gives:

$$\tilde{g}_{\alpha\beta} B^{\alpha}_{k} \nabla_j B^{\beta}_{i} = 0 \tag{1.7}$$

By comparing with (1.5), it can be seen that $\nabla_j B^{\beta}_{i}$ is proportional to N, i.e. it can be written as:

$$\nabla_j B^{\beta}_{i} = \pm N^{\beta} C_{ij} \tag{1.8}$$

where the coefficients C_{ij} are a (0,2)-symmetric tensor.

In order to meet the integrability conditions of the system and using the equations on B as defined above, starting from the definition of the Riemann tensor, in [16](pag. 279) we read:

$$R_{ijkl} = \tilde{R}_{\alpha\beta\gamma\delta} B^{\alpha}_{i} B^{\beta}_{j} B^{\gamma}_{k} B^{\delta}_{l} \pm (C_{ik} C_{jl} - C_{il} C_{jk}) \tag{1.9}$$

$$\nabla_i C_{jk} - \nabla_j C_{ik} = N^{\beta} \tilde{R}_{\alpha\beta\gamma\delta} B^{\alpha}_{j} B^{\gamma}_{k} B^{\delta}_{i} \tag{1.10}$$

where $\tilde{R}_{\alpha\beta\gamma\delta}$ is the Riemann tensor of V_{m+1} and R_{ijkl} is the Riemann of V_m. These are referred to as *Gauss equation* and *Codazzi equation* respectively.

The two equations are the fundamental equations of embedding theory. They indeed allow the definition of the curvature tensor of the embedded manifold V_m using the curvature tensor of the embedding manifold V_{m+1}. As can be noted, the equations are not completely independent. In fact considering Bianchi's second identity[1] we can find restrictions on C_{jk} (a particular case is treated in section 1.3.2).

If the manifold has constant curvature, then the equation (1.9) can be simplified as :

$$R_{ijkl} = \frac{\tilde{R}}{m(m+1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \pm (C_{ik}C_{jl} - C_{il}C_{jk}) \tag{1.11}$$

Furthermore, if the embedding manifold is pseudo-Euclidean, the term containing $\tilde{R}$ disappears, thus obtaining:

$$R_{ijkl} = C_{ik}C_{jl} - C_{il}C_{jk} \tag{1.12}$$

$$\nabla_i C_{jk} - \nabla_j C_{ik} = 0 \tag{1.13}$$

Definition 1.2.1. *Every (0,2)-symmetric tensor which satisfies (1.13) is called Codazzi tensor.*

Using embedding theory, Hans Stephani derived the metrics of certain space-times [32] by comparing Einstein's equation of general relativity with the equation (1.12):

$$R_{ij} = T_{ij} - \frac{1}{2}g_{ij}T = C_{ij}C^k_{\ k} - C_{ik}C^k_{\ j} \tag{1.14}$$

where T_{ij} is the *energy-momentum tensor* and R_{ij} is the *Ricci tensor*. Ricci is a (0,2) symmetric tensor, derived from the contraction of two indices of the Riemann tensor $R^i_{\ jil} = R_{jl}$, which describes the curvature of a space-time

Throughout the discussion, Stephani proves the following theorem, which states that the Gauss-Codazzi equations, written in the form (1.12) and (1.13), guarantee class one immersion:

Theorem 1.2.2. *If there exists a symmetric tensor C_{ij} which satisfies (1.12), then the embedding is of class one.*

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[1] $\nabla_m R_{ijkl} + \nabla_l R_{ijmk} + \nabla_k R_{ijlm} = 0$

1.3 Codazzi tensors theorems and properties

This section briefly summarizes some specific theorems and properties of Codazzi tensors. With the birth of General Relativity (1915) and the subsequent surge of interest in the field of differential geometry, such tensors were given attention for their connection with the curvature tensor. An excellent compilation of all the results obtained throughout the twentieth century is found in Besse's book [4].

1.3.1 Examples of Codazzi Tensors

Some brief examples of Codazzi's tensors emerging from the study of manifolds are summarized here. The following tensors are *non-trivial*, i.e. not constant multiples of the metric.

(i) Let V_n be a conformally flat manifold (Weyl tensor[2] $= 0$) with $n \geq 4$. Then the tensor

$$C_{ij} = R_{ij} - \tfrac{R}{2n-2} g_{ij}$$

is a Codazzi tensor, since the divergence of the Weyl tensor is given by the formula:

$$\nabla_l C_{ijk}{}^l = \tfrac{n-3}{n-2}(\nabla_i C_{jk} - \nabla_j C_{ik})$$

(ii) A Riemannian manifold has harmonic curvature if $\nabla_l R_{ijk}{}^l = 0$. This occurs if and only if the Ricci tensor is a Codazzi tensor.

(iii) Let V_n be a manifold with constant sectional curvature[3] K and a smooth function $f : V_m \to \mathbf{R}$. Then the tensor

$$C_{ij} = \nabla_i \partial_j f + K f g_{ij}$$

is Codazzi. Furthermore, in 1981 Ferus [8] showed how in any space with constant sectional curvature every Codazzi tensor is, at least locally, of this type.

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[2] Weyl tensor is the traceless component of the Riemann tensor. If it vanishes, then metric is locally conformally flat: for each point of the space-time has a neighborhood that can be mapped to flat space by a conformal transformation.

[3] Sectional curvature is one of the ways to describe the curvature of a manifold, in particular it depends on a 2-dimensional subspace of the space tangent to a point of the manifold.

1.3.2 Goenner's theorem

As mentioned in section (1.2), the Gauss and Codazzi equations describing the embedded hypersurface are not completely independent. In his book, Hans Stephani states how the properties of the Riemann tensor impose specific conditions on the Codazzi tensors.

Reconsidering equations (1.12) and (1.13), Goenner [9] shows that by adding the hypothesis of the invertibility of the symmetric tensor C_{ij}, it results to be Codazzi. This is achieved by narrowing down the discussion to the particular case in which V_{m+1} is pseudo-Euclidean. Thus the equation (1.13) is a direct implication, under those assumptions, of (1.12) [24].

Theorem 1.3.1. *Let R_{ijkl} be a Riemann tensor of the form (1.12) and C_{ij} be invertible, then C_{ij} is a Codazzi tensor.*

Proof. Using Bianchi's second identity, we obtain:

$$\nabla_m(C_{ik}C_{jl} - C_{il}C_{jk}) + \nabla_l(C_{im}C_{jk} - C_{ik}C_{jk}) + \nabla_k(C_{il}C_{jm} - C_{im}C_{jl}) =$$

$$= C_{ik}(\nabla_m C_{jl} - \nabla_l C_{jm}) + C_{jl}(\nabla_m C_{ik} - \nabla_k C_{im}) + C_{il}(\nabla_k C_{jm} - \nabla_m C_{jk}) + C_{jk}(\nabla_l C_{im} - \nabla_m C_{il}) + C_{im}(\nabla_l C_{jk} - \nabla_k C_{jl}) + C_{jm}(\nabla_k Cil - \nabla_l Cik) = 0$$

Multiplying by $(C^{-1})^{jl}$ and knowing that $(C^{-1})^{lj}C_{jk} = \delta^j_k$,

$$(n-3)(\nabla_m C_{ik} - \nabla_k C_{im}) + (C^{-1})^{jl}(C_{ik}(\nabla_m C_{jl} - \nabla_l C_{jm}) + +C_{im}(\nabla_l C_{jk} - \nabla_k C_{jl}) = 0$$

Multiplying once again by $(C^{-1})^{jl}$:

$$2(n-2)(C^{-1})^{ik}(\nabla_m C_{ik} - \nabla_k C_{im}) = 0$$

And finally plugging this result into the previous expression we actually find Codazzi's definition: $\nabla_m C_{ik} - \nabla_k C_{im} = 0$ □

1.3.3 Extension of the Derdzinski-Shen theorem

One of the most interesting developments on the relationship between Codazzi tensors and the curvature tensor of their manifold is certainly the Derdzinski-Shen Theorem [6]. It exposes which conditions are imposed by the existence of non-trivial Codazzi tensors on the structure of the curvature tensor. The theorem states that

Theorem 1.3.2. *Let $C_i^{;j}$ be a Codazzi tensor defined on a Riemannian manifold (V_m,g). Let λ and μ be two eigenvalues of that tensor, with* *eigenspaces V_λ and V_μ in T_xV respectively. Then the subspace $V_\lambda \wedge V_\mu$ is invariant under the curvature operator R_x.*

The previous theorem leverages Codazzi condition to prove the thesis. In 2012 Mantica and Molinari [21] showed that this condition is sufficient but not necessary, in fact it can be replaced by a weaker algebraic condition, and despite this, the theorem holds.

Proposition 1.3.3. *Any Codazzi tensors is Riemann-compatible:*

$$R_{jkl}{}^{m}C_{im} + R_{kil}{}^{m}C_{jm} + R_{ijl}{}^{m}C_{km} = 0 \tag{1.15}$$

Proof. Using (1.13), it can be stated:

$$[\nabla_j, \nabla_k]C_{ij} + [\nabla_k, \nabla_i]C_{lj} + [\nabla_i, \nabla_j]C_{kl} = 0 \tag{1.16}$$

where each commutator, expressing the definition of the Riemann tensor, is:

$$[\nabla_i, \nabla_j]C_{kl} = R_{ijk}{}^{m}C_{ml} + R_{ijl}{}^{m}C_{mk} \tag{1.17}$$

Substituting (1.16) and through Bianchi's first identity we obtain (1.15) □

The Derdzinski-Shen theorem can be extended, replacing the Codazzi tensor with a generic tensor B_{ij} that is Riemann-compatible.

To prove this we must first define the generalized curvature tensor:

Definition 1.3.4. *A generalized curvature tensor is a tensor W which satisfies the Riemann proprieties:*

(i) $W_{ijkl} = -W_{ijlk} = -W_{jikl}$

(ii) $W_{ijkl} = W_{lkji}$

(iii) $W_{ijkl} + W_{kijl} + W_{jkil} = 0$ *(Bianchi's first identity)*

Proposition 1.3.5. *If B_{ij} is Riemann compatible, then $W_{ijkl} = R_{ijmn}B_k^m B_l^n$ is a generalized curvature tensor.*

Then the extended version of the theorem for the Riemann curvature follows:

Theorem 1.3.6. *Let a symmetric tensor B_{ij} satisfying (1.15) be defined on a Riemannian manifold V_m. Let X, Y and Z be three eigenvectors with λ, μ and ν their eigenvalues respectively, with λ and μ necessarily different from ν. Then we can declare that*

$$R_{ijkl}X^iY^jZ^k = 0 \tag{1.18}$$

Proof. The following matrix equation :

$$\begin{bmatrix} 1 & 1 & 1 \\ \lambda & \mu & \nu \\ \lambda\mu & \lambda\nu & \mu\nu \end{bmatrix} \begin{bmatrix} R_{lijk}X^iY^jZ^k \\ R_{ljki}X^iY^jZ^k \\ R_{lkij}X^iY^jZ^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

expresses, respectively, Bianchi's first identity for the Riemann tensor, the (1.15) for the symmetric tensor B_{ij}, and Bianchi's first identity for the tensor $W_{lijk} = R_{limn}B_j^mB_k^n$, being a generalized curvature tensor.

Since the determinant is $(\lambda-\mu)(\lambda-\nu)(\mu-\nu)$, if the eigenvalues are all distinct, it is immediate to see that $R_{lijk}X^iY^jZ^k = 0$. If on the other hand $\lambda = \mu \neq \nu$, unwinding the system again we find $R_{lijk}X^iY^jZ^k = 0$. Due to all the symmetries of the Riemann tensor, it can be stated that the theorem is true for any contraction of three indices out of the four. □

Concluding the extension, Mantica and Molinari showed, by a similar procedure, that this theorem can be applied to a generalized curvature tensor W, defining the tensor B_{ij} as W-compatible.

Definition 1.3.7. *Let W be a generalized curvature tensor, a symmetric (0-2) tensor B_{ij} is called W-compatible if*

$$W_{jkl}{}^mB_{im} + W_{kil}{}^mB_{jm} + W_{ijl}{}^mB_{km} = 0 \tag{1.19}$$

Theorem 1.3.8. *Let W be a generalized curvature tensor and a symmetric tensor B_{ij} satisfying (1.19) be defined on a Riemannian manifold V_m. Let X, Y and Z be three eigenvectors and λ, μ and ν their eigenvalues respectively, with λ and μ necessarily different from ν. Then we can declare that*

$$W_{ijkl}X^iY^jZ^k = 0 \tag{1.20}$$

Proof. Equivalent to proof (1.3.6). □

Chapter 2

Stephani exact solutions

This chapter discusses various forms of Codazzi tensors applied to the theory of general relativity. Indeed, certain forms of tensors, such as perfect fluid or current flow, arise from geometrical considerations. These were utilised by Hans Stephani, using embedding formulas, to solve Einstein's equations and calculate their metrics.

From now on we set ourselves in an n-dimensional Lorentzian manifold, with signature (-,+...+). The Greek indices range over the $(n-1)$ spatial components, the Latin ones over all n components.

2.1 Segre type (0,2) symmetric tensors

Given a n-manifold Lorentzian with signature (1, n-1), let

$$A = (u_i, a_i(2), a_i(3), ...a_i(n))$$

be the orthonormal frame, where u_i is an unit time-like vector, i.e. $u^i u_i = -1$, orthogonal to the unit space-like vectors $a(m)_i$, which satisfy the condition $a(m)_i a(j)^i = \delta_{mj}$.

Thus the metric can be described as follows:

$$g_{ij} = -u_i u_j + \sum_{m=2}^{n} a_i(m) a_j(m) \tag{2.1}$$

or, by changing variables:

$$l_i = \tfrac{u_i + a(2)_i}{\sqrt{2}} \qquad k_i = \tfrac{u_i - a(2)_i}{\sqrt{2}}$$

where l_i and k_i are null vector, i.e. $l_i l^i = k_i k^i = 0$, orthogonal to the vectors $a(m)_i$. Furthermore, it can be easily found that $l_i k^i = -1$. So the metric comes to be:

$$g_{ij} = l_i k_j + k_i l_j + \sum_{m=3}^{n} a_i(m) a_j(m) \tag{2.2}$$

A (0,2)-tensor S_{ab} always defines a linear map which takes a vector a to another vector b. To classify S it is useful to start with the eigenvalues equation

$$S_a^b v^a = \lambda v^b \tag{2.3}$$

Indeed every matrix can be rewritten as a block matrix in the canonical form of Jordan(JCF), i.e. as a quasi-diagonal matrix[28]. Thus, in general, a complex matrix A is similar to:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_n \end{bmatrix}$$

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

where J_i is called Jordan block of A.

Classifying the Ricci-like tensor by applying the equation to the eigenvalues and Jordan's theorem in a 4-dimensional space, Petrov fixed 4 different forms, called *Segre Type*:

1. $S_{ij} = -\lambda u_i u_j + \sum_{m=2}^{n} \rho_m a_i(m) a_j(m)$
2. $S_{ij} = \alpha(u_i a_j(2) + u_j a_i(2)) + \beta(u_i u_j - a_i(2) a_j(2)) + \sum_{m=3}^{n} \rho_m a_i(m) a_j(m) \quad \alpha \neq 0$
3. $S_{ij} = \alpha(l_i k_j + l_j k_b) \pm l_i l_j + \sum_{m=3}^{n} \rho_m a_i(m) a_j(m)$
4. $S_{ab} = \alpha(l_i k_j + l_j k_i + a_i(3) a_j(3)) + (l_i a_j(3) + l_j a_i(3)) + \sum_{m=4}^{n} \rho_m a_i(m) a_j(m)$

We analyze which ones are the simplest tensor forms that emerge from the study of individual Segre Type

2.1.1 Segre Type 1

Three cases are distinguished.

a) Complete degeneration: $\lambda = \rho_m \quad \forall m = 2, ...n$. The Ricci-like tensor can be described as $S_{ij} = \lambda g_{ij}$. So, for example, if $S_{ij} = R_{ij}$, it is an Einstein space.
If S_{ij} satisfies the Codazzi condition , then it is a trivial Codazzi tensor with $\nabla_i \lambda = 0$.

b) Degeneration (1,n-1): $\rho_m = \rho_k = \rho \quad \forall m, k = 2, ...n$. It is locally invertible if $\rho \neq 0$. Using the metric again yields $S_{ij} = (\rho - \lambda)u_i u_j + \rho g_{ij}$. This expression carries interesting implications, indeed if $S_{ij} = R_{ij}$ it is a quasi-Einstein space. If $S_{ij} = T_{ij}$ it is the well-known perfect fluid source $T_{ij} = (\mu + p)u_i u_j + p g_{ij}$(see Appendix A).
If S_{ij} satisfies the Codazzi condition, then it is the *perfect fluid Codazzi tensor*, and is revealed as the simplest form of a Codazzi for a Segre Type 1 tensor.

c) Degeneration (1,1,n-2): $\rho_2 \neq \rho_3 = \rho_4 = ... = \rho_n = \rho$. Analogously, using the metric $S_{ij} = (\rho - \lambda)u_i u_j + (\rho - \rho_2)a_i(2)a_j(2) + \rho g_{ij}$. If $S_{ij} = T_{ij} = (\mu + p_\perp)u_i u_j - p_\perp g_{ij} + (p_r - p_\perp)a_i(2)a_j(2)$ (see Appendix A).
If $S_{ij} = C_{ij}$, and $a_i(2) = \frac{\dot{u}_i}{\sqrt{\eta}}$, where $\dot{u}_i = u^k \nabla_k u_i$ and $\eta = \dot{u}^k \dot{u}_k$, the Codazzi tensor becomes the imperfect fluid tensor:
$C_{ij} = (\rho - \lambda)u_i u_j + \frac{(\rho - \rho_2)}{\eta} a_i(2)a_j(2) + \rho g_{ij}$.

2.1.2 Segre Type 2

The simplest non-trivial tensor of this form is $S_{ij} = \alpha(u_i a_j(2) + u_j a_i(2))$, where $\alpha \neq 0$. Again, if S_{ij} fulfills the Codazzi condition, and choosing $m_i(2) = \frac{\dot{u}_i}{\sqrt{\eta}}$, we come to: $S_{ij} = \frac{\alpha}{\sqrt{\eta}}(u_i \dot{u}_j + u_j \dot{u}_i)$, which is the relativistic expression of current flow.

2.1.3 Segre Type 3 and 4

The straightforward form of class 3 is

$$S_{ij} = \pm l_i l_j + \rho a_i(2)a_j(2)$$

while, for the 4 class, is

$$S_{ij} = l_i a_j(2) + l_j a_i(2)$$

Both of them describe, in the theory of relativity, a pure radiation source. They are two analogous forms.

2.2 Stephani solutions to General Relativity equations

In this section, we will analyse how Stephani[32] in his work uses the formulas of embedding and immersion theory to solve Einstein's problem. Stephani starts from the simplest forms of the Segre Type, those listed above, and considering the equation (1.14), finds the metrics that solve the Einstein equation.

This section will be conducted entirely in 4 dimensions.

2.2.1 Perfect fluid

The first example makes use of a Codazzi Segre Type 1 tensor. If $C_{ij} = Au_iu_j + Bg_{ij}$, where A and B are scalar fields, then, following from (1.12), we can obtain the Riemann and the Ricci tensors and the scalar curvature:

$$R_{ijkl} = B^2(g_{ik}g_{jl} - g_{il}g_{jk}) + AB(g_{ik}u_ju_l + g_{jl}u_iu_k - g_{il}u_ju_k - g_{jk}u_iu_l) \quad (2.4)$$

$$R_{ij} = B(3B - A)g_{ij} + 2ABu_iu_j \quad (2.5)$$

$$R = 12B^2 - 6AB \quad (2.6)$$

Thanks to these results, we can affirm that the source of this space-time will be a perfect fluid:

$$T_{ij} = (\mu + p)u_iu_j + pg_{ij}$$

$$\mu = 3B^2 \quad p = B(2A - 3B)$$

Proposition 2.2.1. *The space-time with the Riemann (2.4) and the Ricci tensor (2.5) is conformally flat, i.e. the Weyl tensor* $C_{ijkl} = 0$

Proof. The Weyl tensor is

$$C_{ijkl} = R_{ijkl} + \tfrac{g_{il}R_{jk} - g_{jl}R_{ik} + g_{jk}R_{il} - g_{ik}R_{jl}}{2} - R\tfrac{g_{il}g_{jk} - g_{jl}g_{ik}}{6}$$

Considering only the last two terms, we get that:

$$\begin{aligned}
&\frac{g_{il}R_{jk} - g_{jl}R_{ik} + g_{jk}R_{il} - g_{ik}R_{jl}}{2} - R\frac{g_{il}g_{jk} - g_{jl}g_{ik}}{6} \\
&= (3B^2 - AB)(g_{il}g_{jk} - g_{jl}g_{ik}) + AB(u_ju_kg_{il} - u_iu_kg_{jl} + u_iu_lg_{jk} - u_ju_lg_{ik}) \\
&- (2B^2 - AB)(g_{il}g_{jk} - g_{jl}g_{ik}) \\
&= B^2(g_{il}g_{jk} - g_{jl}g_{ik}) + AB(u_ju_kg_{il} - u_iu_kg_{jl} + u_iu_lg_{jk} - u_ju_lg_{ik}) \\
&= -R_{ijkl}
\end{aligned}$$

□

If the tensor C_{ij} is invertible, namely $A \neq B$, thanks to (1.2.2) and Goenner theorem, then C_{ij} is also Codazzi. If $B = 0$, the space-time is flat.

The first solution is almost trivial; by imposing

$$A = 0 \quad T_{ij} = 3B^2 g_{ij} \quad B^2 = \text{cost.}$$

we find a space with constant curvature, which corresponds to the de Sitter space.

From this point going forward, we will make use of the standard decomposition for a time-like unit vector, discussed in Appendix B:

$$\nabla_i u_j = \phi(g_{ij} - u_i u_j) + \omega_{ij} + \sigma_{ij} - u_i \dot{u}_j \tag{2.7}$$

Starting from (2.4), applying to the expression Bianchi's identities, we find:

$$\nabla_i u_j = -u_i \dot{u}_j + \phi(g_{ij} - u_i u_j) \tag{2.8}$$

$$B = B(t) \quad \nabla_t B = \phi A u_0 \tag{2.9}$$

$$\phi = \phi(t) \tag{2.10}$$

$$\nabla_i A = -u_i \dot{A} - \dot{u}_i A \tag{2.11}$$

which means that the velocity field is shear-free and vorticity-free. Furthermore, B and ϕ depend solely on time.

Now we need to distinguish two cases: if the speed is expansion-free or not.

Expansion-free solution

Being expansion-free means that $\phi = 0$. Moreover, the three dimensional space, defined by the orthogonal projector $g_{\mu\nu} + u_\mu u_\nu$, is a space on constant curvature. The final result is the metric[31]:

$$ds^2 = \frac{dr^2}{1 - B^2 r^2} + r^2 d\Omega^2 - (u_0)^2 dt^2 \tag{2.12}$$

$$u_0 = r f_1(t) sin\theta sin\phi + r f_2(t) sin\theta cos\phi + r f_3(t) cos\theta + f_4(t)\sqrt{1 - (Br)^2} - \frac{1}{B}$$

where $f_i(t)$ are four arbitrary functions of time.

We can notice that the expansion-free solution of the conformally flat space-time is a generalization of the interior Schwarzschild solution ($f_1 = f_2 = f_3 = 0$ and f_4 =const).[4]

[4]The Schwarzschild metric describes space-time where the source is a massive, non-rotating, spherically symmetric object. The metric is:

$$ds^2 = r^2 d\Omega^2 + dr^2 (1 - \tfrac{r^2}{R^2})^{-1} - (a - b\sqrt{1 - (\tfrac{r}{R})^2})^2 dt^2$$

Cosmologically, this space-time does not allow dust solutions $p = 0$: this implies that the vector field u is covariantly constant, which is compatible with the curvature tensor if and only if $A = B = 0$, i.e. the space-time is flat.

If $A = B$, we obtain the Einstein universe:

$$ds^2 = \frac{dr^2}{1 - B^2r^2} + r^2 d\Omega^2 - dt^2 \tag{2.13}$$

Non-vanishing expansion solution

In this case, where $\phi \neq 0$, the spatial part $g_{\mu\nu} + u_\mu u_\nu$ of the metric tensor is not time independent, so, using the equation derived from Bianchi's identities and the Riemann (2.4), we get that the hypersurface $t =$const has constant curvature, but the metric is time-dependent. Thus, after a rescaling, the metric is obtained[31]:

$$ds^2 = V^{-2}(dx^2 + dy^2 + dz^2) - \left(\frac{1}{\phi(t)} \frac{\partial_t V}{V}\right)^2 dt^2 \tag{2.14}$$

$$V(x, y, z, t) = V_0(t) + \frac{B^2(t) - \phi^2(t)}{4V_0} \left((x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right) \tag{2.15}$$

where V_0 and x_0, y_0, z_0 are arbitrary functions of time.

This metric express the generalizations of the Robertson-Walker cosmological models, and describes the well-known Stephani Universe.

The characterisation of Stephani universes is that, unlike RW universes, they consider space-time to be anisotropic, due to a spatial variation of the Hubble constant.

Dust solutions can be obtained considering $2A = 3C$; this implies the Friedmann dust models.

2.2.2 Codazzi tensor- imperfect fluid

The second type of solutions gathered by Stephani are calculated from Segre type 1 of the form $C_{ij} = Ag_{ij} + 2Au_iu_j + Bz_iz_j$, where $z_iz^i = 1$, $z_iu^i = 0$ and $AC \neq 0$.

The Riemann and the Ricci tensors and the scalar curvature are:

$$R_{ijkl} = C_{ik}C_{jl} - C_{il}C_{jk} \tag{2.16}$$

$$R_{ij} = (A^2 + AB)g_{ij} + (4A^2 + 2AB)u_ju_l \tag{2.17}$$

$$R = 2AB \tag{2.18}$$

The energy-momentum tensor will be:

$$T_{ij} = (\mu + p)u_i u_j + p g_{ij} \tag{2.19}$$

$$\mu = 2A(B + 2A) \quad p = A^2$$

It is interesting to note, although we initially have an imperfect fluid Codazzi tensor, via the equation (1.14), the source of the gravitational field turns out to be a perfect fluid tensor.

We now demonstrate that Weyl tensor is different to 0.

Proposition 2.2.2. *The Weyl tensor is:*

$$\begin{aligned} C_{ijkl} &= C_{ik}C_{jl} - C_{il}C_{jk} + (A^2 - AB)(g_{il}g_{jk} - g_{jl}g_{ik}) \\ &+ (4A^2 + 2AB)(g_{il}u_j u_k - g_{jl}u_i u_k + g_{jk}u_i u_l - g_{ik}u_j u_k) \end{aligned} \tag{2.20}$$

Proof. We start again with the Weyl tensor equation and calculate its different terms:

$$\begin{aligned} &\frac{g_{il}R_{jk} - g_{jl}R_{ik} + g_{jk}R_{il} - g_{ik}R_{jl}}{2} = \\ &= (g_{jk}g_{il} - g_{ik}g_{jl})(A^2 + AB) \\ &+ (4A^2 + 2AB)(g_{il}u_j u_k - g_{jl}u_i u_k + g_{jk}u_i u_l - g_{ik}u_j u_k) \end{aligned}$$

$$R\frac{g_{il}g_{jk} - g_{jl}g_{ik}}{6} = 3AB(g_{il}g_{jk} - g_{jl}g_{ik})$$

Therefore, adding up all the various addends, we find Weyl in the form(2.20) □

If $A = -2C$, the Codazzi condition (1.13) implies:

$$\nabla_i u_j = z_j p_i \tag{2.21}$$

$$\nabla_i z_j = u_j p_i \tag{2.22}$$

where p is a generic vector orthogonal both at u and at z

The Ricci identity $\nabla_k \nabla_l u_j - \nabla_l \nabla_k u_j = u^i R_{ijkl}$, and the Gauss equation (1.12) imply $C = 0$, i.e. the space-time has null curvature. So we can single out when $A \neq -2C$.

As above, two cases must be separated: when the acceleration vector field vanishes or not (we recall that $\dot{u}_i = u^j \nabla_j u_i$).

Vanishing acceleration solution

Adding this further hypothesis, the calculations lead to:

$$\nabla_i u_j = a_1 z_i z_j + a_2(g_{ij} + u_i u_j - z_i z_j) \tag{2.23}$$

$$\nabla_i z_j = a_1 u_i z_j + p_j z_i \quad p_i u^i = p_i z^i = 0 \tag{2.24}$$

$$\nabla_i A = u_i((2B + A)a_1 - 2Ba_2) + a_3 z_i + Ap_i \tag{2.25}$$

$$\nabla_i B = 2Ba_2 u_i \tag{2.26}$$

Proposition 2.2.3. *The velocity field u is vorticity-free, but not shear-free*

Proof. We utilise the definitions of vorticity and shear given in Appendix B

$$\omega_{ij} = u_{[i;j]} + \dot{u}_{[i}u_{j]} = 0$$

thus the acceleration vanishes.

$$\begin{aligned}
\sigma_{ij} &= u_{(i;j)} + \dot{u}_{(i}u_{j)} - \phi(g_{ij} + u_i u_j) \\
&= a_1 z_i z_j + a_2(g_{ij} + u_i u_j - z_i z_j) - \frac{\nabla_m u^m}{3}(g_{ij} + u_i u_j) \\
&= a_1 z_i z_j + a_2(g_{ij} + u_i u_j - z_i z_j) - \frac{a_1 + 2a_2}{3}(g_{ij} + u_i u_j) \\
&= z_i z_j(a_1 - a_2) + \frac{1}{3}(a_1 + a_2)(g_{ij} + u_i u_j)
\end{aligned}$$

□

It turns out that the metric depends on two further cases, namely if the two dimensional subspace x-y, which turns out to have constant curvature, is flat or not.

The two metrics are presented here[30]. They describe flat space and non-flat space, respectively.

$$ds^2 = t(dr^2 + r^2 d\phi^2) + V^2 dz^2 - dt^2 \tag{2.27}$$

$$\begin{aligned}
V(r, \phi, z, t) = t\sqrt{t}G_1(z) + \sqrt{t}(G_2(z) r cos\phi + G_3(z) r sin\phi + \frac{3}{4}G_1(z) r^2 \\
+ G_4(z)) + G_5(z)
\end{aligned}$$

Again $G_i(z)$ are arbitrary functions, dependent on z this time.

$$ds^2 = F^2(t)\left(\frac{dr^2}{1 \pm r^2} + r^2 d\phi^2\right) + V^2 dz^2 - dt^2 \tag{2.28}$$

$$\begin{aligned} V(r, \phi, z, t) = G_1(z) \int F^{-1} dt + G_2(z) + F(G_3(z) r cos\phi + G_4(z) r sin\phi \\ \pm G_5(z)\sqrt{1 \pm r^2}) \end{aligned}$$

Non-vanishing acceleration solution

instead, in this case the equations are:

$$\nabla_i u_j = a_1 z_j u_i + a_2 z_i z_j + a_3(g_{ij} + u_i u_j - z_i z_j) \tag{2.29}$$

$$\nabla_i z_j = a_1 u_j u_i + a_2 u_j z_i + \frac{B + 2A}{B} a_1 (g_{ij} - z_j z_i + u_i u_j) \tag{2.30}$$

$$\nabla_i A = 2A a_3 u_i + (2A + C) a_1 z_i \tag{2.31}$$

$$\nabla_i B = (2A + B) a_2 - 2A a_3 u_i + a_4 z_i \tag{2.32}$$

Proposition 2.2.4. *The vorticity is:*

$$\omega = a_1(u_i z_j - u_j z_i) \tag{2.33}$$

and the shear is:

$$\sigma_{ij} = a_1(u_i z_j + u_j z_i) + z_i z_j (a_2 - a_3) + \frac{1}{3}(a_2 + a_3)(g_{ij} + u_i u_j) \tag{2.34}$$

Proof. Following the same calculations as in the above proof, the two equations of the proposition are obtained. □

Not in all conditions are general solutions found: indeed, in this case the solution is only known when the u-field is shear-free: if it vanishes, the metric is static, and in this setting, spherically symmetric[31]. For the purpose of completeness it is given here but will be of little interest in our discussion:

$$ds^2 = k\frac{(a + 2br^2)}{a + br^2} dr^2 + r^2\left(\frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\phi^2\right) - (a + br^2) dt^2 \tag{2.35}$$

These were the two Codazzi tensors mainly studied by Stephani: they give rise via the embedding equations to an energy-momentum tensor T_{ij} of perfect fluid type. This is the reason why he does not investigate a Codazzi tensor of the Segre Type 2 (current flow), since the source of the space-time cannot be traced back to a perfect fluid.

Chapter 3

Codazzi tensors in General relativity

In opposition to Hans Stephani, who related a Codazzi tensor to the geometry of space-time through the hypersurfaces equations, Mantica and Molinari's work[19] shows that even just the fulfilment of the Codazzi condition (1.13) imposes restrictions on the Ricci tensor.

In fact, as proposition (1.15) states, any Codazzi tensor is Riemann-compatible, and therefore, contracting with the metric tensor g^{il}, we obtain:

$$C_{ij} R_k^j = C_{kj} R_i^j \tag{3.1}$$

namely it commutes with the Ricci tensor.

By investigating two forms of tensors (perfect fluid and current flow), we find out which are the conditions for these tensors to be Codazzi. Then we will derive the Ricci tensor, without imposing any conditions on the Riemann tensor, such as the equation (1.9). This tensor will then determine which space-time hosts this Codazzi tensor.

3.1 Perfect fluid tensor

The most immediate tensor to study is obviously the perfect fluid $C_{ij} = Au_i u_j + Bg_{ij}$ with u_i, as always, an unit time-like vector field, and $A \neq 0$ and B two scalar fields.

Theorem 3.1.1. *The perfect fluid tensor $C_{ij} = Au_i u_j + Bg_{ij}$ is Codazzi if and only if:*

$$\nabla_i u_j = \phi(g_{ij} + u_i u_j) - u_i \dot{u}_j \tag{3.2}$$

$$\nabla_i \phi = -u_i \dot{\phi} \tag{3.3}$$

$$\nabla_i A = -u_i \dot{A} - \dot{u}_i A \tag{3.4}$$

$$\nabla_i B = -u_i \dot{B} \tag{3.5}$$

$$\phi = -\frac{\dot{B}}{A} \tag{3.6}$$

$\dot{u}_i$ is the acceleration vector field, which is space-like and orthogonal to the velocity.Its module is the scalar function $\eta = \dot{u}_i \dot{u}^i$.

Proof. We have to demonstrate that $\nabla_i C_{jk} - \nabla_j C_{ik} = 0$

We start by explicating Codazzi condition:

$$\begin{aligned} 0 &= u_k(u_j \nabla_i A - u_i \nabla_j A) + (g_{jk} \nabla_i B - g_{ik} \nabla_j B) \\ &+ A u_k (\nabla_i u_j - \nabla_j u_i) + A(u_j \nabla_i u_k - u_i \nabla_j u_k) \end{aligned} \tag{3.7}$$

Contracting with u^k, and remembering that $u^i \nabla_j u_i = 0$ (acceleration is orthogonal to velocity):

$$0 = -u_j \nabla_i (A - B) + u_i \nabla_j (A - B) - A(\nabla_i u_j - \nabla_j u_i) \tag{3.8}$$

Contracting again with u^j:

$$0 = \nabla_i (A - B) + u_i(\dot{A} - \dot{B}) + A \dot{u}_i \tag{3.9}$$

and, putting it inside the (3.8), being $A \neq 0$:

$$0 = u_j \dot{u}_i - u_i \dot{u}_j - \nabla_i u_j + \nabla_j u_i$$

By considering the standard decomposition (see Appendix B), we find that the perfect fluid is vorticity-free $\omega_{ij} = 0$

Then, by multiplying by the metric tensor g^{jk} the equation (3.7), we get:

$$0 = -\nabla_i A - u_i \dot{A} + (n-1) \nabla_i B - A \dot{u}_i - A u_i \nabla_k u^k \tag{3.10}$$

Contraction with u^i: $(n-1)\dot{B} + A \nabla_k u^k = 0$, by explicating $A\dot{u}_i$ from (3.9) reduces the previous equation to:

$$\begin{aligned} 0 &= -\nabla_i (A - (n-1)B) - u_i(\dot{A} - (n-1)\dot{B}) + \nabla_i (A - B) + u_i(\dot{A} - \dot{B}) \\ &= \nabla_i B + u_i \dot{B} \end{aligned} \tag{3.11}$$

Using this result in (3.9) we find the (3.4).

Contraction with u^i of (3.10), and expressing the decomposition, gives:

$$\phi = -\frac{\dot{B}}{A} \tag{3.12}$$

Using the results we have just come up with, the expression of Codazzi condition (3.7) simply reduces to:

$$u_j \sigma_{ik} = u_i \sigma_{jk} \tag{3.13}$$

which means that the vector field u is also shear-free (3.2).

In order to find the last equation of the theorem, we evaluate the covariant derivative of ϕ:

$$\begin{aligned}
\nabla_i \phi &= -\frac{\nabla_i \dot{B}}{A} + \dot{B}\frac{\nabla_i A}{A^2} \\
&= -\frac{\nabla_i u^k \nabla_k B}{A} + \dot{B}\frac{-\dot{u}_i A - u_i \dot{A}}{A^2} \\
&= -\frac{\dot{u}_i B - u_i \ddot{B}}{A} + \dot{B}\frac{-\dot{u}_i A - u_i \dot{A}}{A^2} \\
&= -u_i \left(\frac{\dot{A}\dot{B}}{A^2} - \frac{\ddot{B}}{A} \right) \\
&= -u_i \dot{\phi}
\end{aligned} \tag{3.14}$$

We have derived all the equations of the theorem; the opposite implication also holds, i.e. if a perfect fluid tensor solves the equations (3.2-3.6), then it is Codazzi. □

A remarkable result to point out is that these equations are the same as those found by Stephani (2.8-2.11), but without imposing the Gauss equation on the Riemann tensor. This leads us to think that we will find ourselves in a more general space-time than the universe of Stephani; this statement will be confirmed when we derive the metric.

3.1.1 Ricci and Energy-momentum tensors

We now deduce the Ricci tensor by means of some propositions.

Proposition 3.1.2. *If C_{ik} perfect fluid tensor is Codazzi, then u_i is Riemann-compatible, and it is an eigenvector of the Ricci tensor $R_{ij}u^j = \gamma u_i$ with $\gamma = (n-1)(\dot{\phi} + \phi^2) - \nabla_k \dot{u}^k$*

Proof. The Riemann compatibility is a directly consequence of (1.15) and Bianchi's identities.

$$\begin{aligned}
R_{jklm}u^m &= \nabla_j \nabla_k u_l - \nabla_k \nabla_j u_l \\
&= \nabla_j [\phi(g_{kl} + u_k u_l) - u_k \dot{u}_l] - \nabla_k [\phi(g_{jl} + u_j u_l) - u_j \dot{u}_l] \\
&= -(\dot{\phi} + \phi^2)(g_{kl}u_j - g_{jl}u_k) - 2u_{[j}\dot{u}_{k]}(\phi u_l - \dot{u}_l) - 2u_{[k}\nabla_{j]}\dot{u}_l
\end{aligned}$$

Contraction with g_{jl} gives:

$$\begin{aligned} R_{km}u^m &= (n-1)(\dot\phi+\phi^2)u_k + \phi\dot u_k - u_k\eta - 2u_{[k}\nabla_{j]}\dot u_j \\ &= [(n-1)(\dot\phi+\phi^2) - \nabla_j \dot u^j]u_k \\ &= \gamma u_k \end{aligned}$$

having replaced through the equation (3.2) $-\dot u^j\nabla_k u_j = -\phi\dot u_k + u_k\eta$.

One can also obtain a useful identity regarding acceleration: subtracting the same expression from the symmetric tensor

$$u^j R_{jklm}u^m = (g_{kl}+u_l u_k)(\dot\phi+\phi^2) + \dot u_k(\phi u_l - \dot u_l) - u_k\ddot u_l - \nabla_k \dot u_l \tag{3.15}$$

with the indices k,l exchanged gives:

$$2\nabla_{[k}\dot u_{l]} = (\phi\dot u_k + \ddot u_k)u_l - u_k(\phi\dot u_l + \ddot u_l) \tag{3.16}$$

□

It is now possible to obtain the Ricci tensor of the space-time which hosts a perfect fluid Codazzi tensor.

Theorem 3.1.3. *The Ricci tensor is:*

$$R_{kl} = \frac{R-n\gamma}{n-1}u_k u_l + \frac{R-\gamma}{n-1}g_{kl} + \Pi_{kl} \tag{3.17}$$

$$\begin{aligned} \Pi_{kl} =& \frac{1}{2}(n-2)[u_k(\phi\dot u_l - \ddot u_l) + u_l(\phi\dot u_k - \ddot u_k) - (\nabla_k\dot u_l + \nabla_l\dot u_k) \\ &+ (n-2)[\dot u_k\dot u_l + E_{kl}] + \frac{n-2}{n-1}(g_{kl}+u_k u_l)\nabla_p\dot u^p \end{aligned} \tag{3.18}$$

Π_{kl} is symmetric and traceless and $\Pi_{kl}u^l = 0$.

$E_{kl} = u^j u^m C_{jklm}$ is the electric tensor, which is symmetric, traceless and $E_{kl}u^l = 0$.

Proof. Recalling the generic expression on the Weyl tensor:

$$C_{jklm} = R_{jklm} + \frac{g_{jm}R_{kl} - g_{km}R_{jl} + g_{kl}R_{jm} - g_{jl}R_{km}}{n-2} - R\frac{g_{jm}g_{kl} - g_{k,}g_{jl}}{(n-1)(n-2)}$$

the double contraction with u^j and u^m yields, substituting (3.15):

$$\begin{aligned} E_{kl} =& (g_{kl}+u_l u_k)(\dot\phi+\phi^2) + \dot u_k(\phi u_l - \dot u_l) - u_k\ddot u_l - \nabla_k\dot u_l \\ &- \frac{R_{kl} + 2\gamma u_k u_l + \gamma g_{kl}}{n-2} + R\frac{g_{kl}+u_k u_l}{(n-1)(n-2)} \end{aligned}$$

Thus it is found, as the thesis states, that Ricci tensor is:

$$R_{kl} = \frac{R - n\gamma}{n-1} u_k u_l + \frac{R - \gamma}{n-1} g_{kl} + \Pi_{kl}$$

where we use the identity (3.16). □

To fully describe a space-time, the energy-momentum tensor must be made explicit. It is linked to the Ricci tensor via the Einstein equation.

Proposition 3.1.4. *The energy-momentum tensor is:*

$$T_{kl} = \frac{R - n\gamma}{n-1} u_k u_l + \frac{(3-n)R - 2\gamma}{2(n-1)} g_{kl} + \Pi_{kl} \tag{3.19}$$

The field source is therefore an imperfect fluid, with Π_{kl} being the dispersion and anistotropy term, while perpendicular pressure and density are:

$$p_\perp = \left(\frac{(3-n)R - 2\gamma}{2(n-1)} \right) \tag{3.20}$$

$$\mu = \left(\frac{R - n\gamma}{n-1} \right) - p_\perp \tag{3.21}$$

3.1.2 Metric of the space-time

From the above equations, in particular from (3.2), which describes the derivative of the velocity vector field, we have restrictions on space-time. In fact, since the velocity is vorticity-free, we know that it will be *doubly twisted*, i.e. the metric is[20]:

$$ds^2 = -b^2(t, \mathbf{x})dt^2 + a^2(t, \mathbf{x})\tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.22}$$

The vector $\mathbf{x}$ denotes the spatial coordinates, e $\tilde{g}_{\mu\nu}$ is a Riemann metric tensor.

The Christoffels symbols are:

$$\Gamma^0_{00} = \frac{\partial_t b}{b} \qquad \Gamma^0_{\mu 0} = \frac{\partial_\mu b}{b} \qquad \Gamma^0_{\mu\nu} = \frac{\partial_t a}{ab^2} g_{\mu\nu} \qquad \Gamma^\nu_{0\mu} = \frac{\partial_t a}{a} \delta^\nu_\mu$$

Comparing with the equations (3.2):

$$\dot{u}_0 = -b(t, \mathbf{x}) \qquad \dot{u}_0 = 0 \qquad u_\mu = 0 \qquad \dot{u}_\mu = \frac{\partial_\mu b(t, \mathbf{x})}{b(t, \mathbf{x})} \qquad \phi = \frac{\partial_t a(t, \mathbf{x})}{b(t, \mathbf{x}) a(t, \mathbf{x})}$$

We can state that the scalar function ϕ only depends on time; defining $a = \frac{1}{V(t,\mathbf{x})}$, the metric becomes:

$$ds^2 = -\left(\frac{\partial_t V}{\phi V}\right)^2 dt^2 + \frac{\tilde{g}_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu}{V^2} \tag{3.23}$$

As we can observe, this metric is a general case of the metric found by Stephani and describes in equation (2.14). This space-time is denominated *Generalized Stephani universe.*

Proposition 3.1.5. *If the acceleration field is close* $(\nabla_i \dot{u}_j - \nabla_j \dot{u}_i) = 0$*, then* $b(t,\mathbf{x})$ *can be factorized in* $b(t,\mathbf{x}) = b(t)b(\mathbf{x})$*. It also holds that* $\ddot{u}_k = \eta u_k - \phi \dot{u}_k$

Proof. If $\dot{u}$ is closed, then

$$\begin{aligned} 0 &= \nabla_0 \dot{u}_\mu - \nabla_\mu \dot{u}_0 \\ &= \partial_t \dot{u}_\mu - \Gamma^\nu_{o\mu} \dot{u}_n u + \Gamma^\nu_{\mu 0} \dot{u}_\nu \\ &= \partial_t (\partial_\mu log b) \end{aligned}$$

i.e. it is independent on time, thus it can be factorized. Furthermore, equation (3.16) reduces to $(\phi \dot{u}_k + \ddot{u}_k)u_l - u_k(\phi \dot{u}_l + \ddot{u}_l) = 0$. Contraction with u^l gives $\ddot{u}_k = \eta u_k - \phi \dot{u}_k$. □

By setting additional conditions on the variables, two special cases can be obtained, which are interesting to investigate:

- If $\dot{u}_k = 0$, $\nabla_i A = -u_k \dot{A}$, then $b(t,\mathbf{x})$ depends only on time. Condition (3.3) state that $\partial_\mu \phi = 0$, so a is only a function of time. It suggests a *warped metric*, which describes an *Generalized Robertson-Walker* space-time[22]:

$$ds^2 = -dt^2 + a^2(t)\tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.24}$$

  The theorem proved by Merton[26], which states that a perfect fluid Codazzi tensor such that $h^{ij}\nabla_j C^m_{\ m} = 0$ is a necessary and sufficient condition for the metric to be warped, is in total agreement with the results obtained.

- If $B = 0$, $\nabla_i u_j = -u_i \dot{u}_j$. The function a is not dependent on time since $\phi = 0$ (3.6), so it can be absorbed in the Riemann metric:

$$ds^2 = -b^2(t,\mathbf{x})dt^2 + \tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.25}$$

  It is the same of (2.12), the generalization of the interior Schwarzschild solution. Also here, we can obtain a static universe, as in (2.13), imposing the acceleration to be closed:

$$ds^2 = -b(\mathbf{x})dt^2 + \tilde{g}_{\mu\nu} dx^\mu dx^\nu \tag{3.26}$$

3.2 Current flow Codazzi tensor

In this section we investigate, by analogy with the previous section, a particular form of the symmetric tensor $C_{jk} = \lambda(u_j\dot{u}_k + u_k\dot{u}_j)$ and impose the Codazzi condition on it. We then study which space-time hosts such a tensor, and some special examples, also known from literature.

The tensor $C_{jk} = \lambda(u_j\dot{u}_k + u_k\dot{u}_j)$ is the relativistic expression of the current-flow, λ is a scalar vector field, u_j is the velocity vector field, and $\dot{u}_j$ is the acceleration, which is closed.

Theorem 3.2.1. *The current fluid tensor C_{jk} is Codazzi if and only if:*

$$\nabla_j u_k = -\frac{\dot{\lambda}\dot{u}_j\dot{u}_k}{\lambda\eta} - u_j\dot{u}_k \tag{3.27}$$

$$\nabla_j \dot{u}_k = -\eta u_j u_k - \frac{\dot{\lambda}}{\lambda}(\dot{u}_k u_j + \dot{u}_j u_k) + \dot{u}_j\dot{u}_k\frac{\dot{u}^m\nabla_m\eta}{2\eta^2} \tag{3.28}$$

$$\nabla_j\lambda = -u_j\dot{\lambda} - \lambda\dot{u}_j\left(2 + \frac{\dot{u}^m\lambda_m\eta}{2\eta^2}\right) \tag{3.29}$$

Proof. As above, we explicit the Codazzi condition $\nabla_i C_{jk} - \nabla_j C_{ik} = 0$, and then we contract with g^{jk}:

$$\begin{aligned}
0 &= \nabla^k[\lambda(u_i\dot{u}_k + u_k\dot{u}_i)] \\
&= u_i(\dot{u}^m\nabla_m\lambda + \lambda\nabla_k\dot{u}^k) + \dot{u}_i(\dot{\lambda} + \lambda\nabla_k u^k) + \lambda\ddot{u}_i + \lambda\dot{u}^k\nabla_k u_i
\end{aligned}$$

Contraction with u^i gives:

$$\dot{u}^m\nabla_m\lambda + \lambda\nabla_k\dot{u}^k + \lambda\eta = 0$$

Contraction with $\dot{u}^i$ gives:

$$\dot{\lambda} + \lambda\nabla_k u^k = 0$$

Using these two relations in the previous equation:

$$-\eta u_i + \ddot{u}_i + \dot{u}^k\nabla_k u_i = 0 \tag{3.30}$$

Contracting the Codazzi condition with u^k:

$$\begin{aligned}
0 &= \nabla_i(u^k C_{jk}) - C_{jk}\nabla_i u^k - \nabla_j(u^k C_{ik}) + C_{ik}\nabla_j u^k \\
&= \nabla_i(-\lambda\dot{u}_j) - \lambda u_j\dot{u}_k\nabla_i u^k - \nabla_j(-\lambda\dot{u}_i) - \lambda u_i\dot{u}_k\nabla_j u^k \\
&= -(\nabla_i\lambda)\dot{u}_j + \lambda u_j\ddot{u}_i + (\nabla_j\lambda)\dot{u}_i - \lambda u_i\ddot{u}_j
\end{aligned}$$

Contracting again with u^i:

$$\lambda\ddot{u}_j = \dot{\lambda}\dot{u}_j + \lambda\eta u_j$$

Using this in the preceding expression, it gives:

$$\nabla_i\lambda = -\dot{\lambda}u_i + \dot{u}_i\frac{\dot{u}^m\nabla_m\lambda}{\eta} \tag{3.31}$$

We are now rewriting Codazzi condition, including the equations found so far:

$$\begin{aligned} 0 &= (\dot{u}_i u_j - u_i\dot{u}_j)(\dot{\lambda}u_k + \dot{u}_k\frac{\dot{u}^m\nabla_m\lambda}{\eta}) + 2\lambda(\nabla_{[i}u_{j]})\dot{u}_k \\ &+ \lambda(u_j\nabla_i\dot{u}_k - u_i\nabla_j\dot{u}_k) + \lambda(\dot{u}_j\nabla_i u_k - \dot{u}_i\nabla_j u_k) \end{aligned} \tag{3.32}$$

Contracting with u^i:

$$\begin{aligned} 0 &= \dot{u}_j\left(\dot{\lambda}u_k + \dot{u}_k\frac{\dot{u}^m\nabla_m\lambda}{\eta}\right) + 2\lambda\dot{u}_j\dot{u}_k + \lambda\eta u_j u_k + \dot{\lambda}\dot{u}_k u_j + \lambda\nabla_j\dot{u}_k \\ \lambda\nabla_j\dot{u}_k &= -\dot{u}_j\dot{u}_k\left(\frac{\dot{u}^m\nabla_m\lambda}{\eta} + 2\lambda\right) - \lambda\eta u_j u_k - \dot{\lambda}(\dot{u}_j u_k + u_j\dot{u}_k) \end{aligned} \tag{3.33}$$

Then, contracting first with $\dot{u}^k$, and then with $\dot{u}^j$, we obtain:

$$\frac{\dot{u}^m\nabla_m\lambda}{\eta} = -2\lambda - \lambda\frac{\dot{u}^j\nabla_j\eta}{2\eta^2} \tag{3.34}$$

Utilizing this relation in (3.31) and (3.33), we achieve the (3.28) and (3.29).

In order to find out the last equation of the theorem we contract with $\dot{u}^k$ the (3.32), specifying then $\ddot{u}_k$ and $\nabla_k\eta$:

$$\begin{aligned} 0 &= (\dot{u}_i u_j - u_i\dot{u}_j)(\dot{u}^m\nabla_m\lambda) + 2\lambda\eta\nabla_{[i}u_{j]} \\ &+ \frac{\lambda}{2}(u_j\nabla_i\eta - u_i\nabla_j\eta) + \lambda(\dot{u}_j\ddot{u}_i - \dot{u}_i\ddot{u}_j) \\ &= -(\dot{u}_i u_j - u_i\dot{u}_j) + (\nabla_i u_j - \nabla_j u_i) \end{aligned}$$

This relation means that the velocity is vorticity-free, i.e. $\omega_{ij} = 0$

We still contract the (3.32), but this time with $\dot{u}^i$.

$$0 = u_j(\dot{\lambda}\eta u_k + \dot{u}_k\dot{u}^m\nabla_m\lambda) + \lambda\eta u_j\dot{u}_k$$
$$+ \lambda(u_j\dot{u}^i\nabla_i\dot{u}^k + \dot{u}_j\dot{u}^i\nabla_i u_k - \eta\nabla_j u_k)$$

Knowing that $\dot{u}^i\nabla_k\dot{u}^i = \frac{\nabla_k\eta}{2} = -\frac{\dot{\lambda}}{\lambda}\eta\dot{u}_k - \dot{u}_k\left(2\eta + \frac{\dot{u}^m\nabla_m\lambda}{\lambda}\right)$ and (3.30):

$$\lambda\eta\nabla_j u_k = u_j(\dot{u}_k\dot{u}^m\nabla_m\lambda + \lambda\eta\dot{u}_k) - u_j\dot{u}_k(2\eta\lambda + u^m\nabla_m\lambda) + \dot{\lambda}\dot{u}_j\dot{u}_k$$
$$\dot{\lambda}\dot{u}_j\dot{u}_k = -\lambda\eta u_j\dot{u}_k$$
$$\nabla_j u_k = -\frac{\dot{\lambda}\dot{u}_j\dot{u}_k}{\lambda\eta} - u_j\dot{u}_k \tag{3.35}$$

We have obtained all the equations presented in the theorem. By simply carrying out the calculations, it is easy to prove the way back.

□

Even if Stephani did not study the current flow tensor, there are many examples in literature to compare it with. To begin with, we find the metric of such a space-time.

3.2.1 Metric of the space-time

Since the velocity is not shear-free

$$\sigma_{jk} = \frac{\dot{\lambda}}{\lambda}\left(\frac{g_{jk} + u_j u_k}{n-1} - \frac{\dot{u}_j\dot{u}_k}{\eta}\right) \tag{3.36}$$

the metric structure is as follows[7]:

$$ds^2 = -b(t,\mathbf{x})dt^2 + \tilde{g}_{\mu\nu}(t,\mathbf{x})dx^\mu dx^\nu \tag{3.37}$$

The Christoffel symbols are:

$$\Gamma^0_{00} = \frac{\partial_t b}{b} \qquad \Gamma^0_{\mu 0} = \frac{\partial_\mu b}{b} \qquad \Gamma^\mu_{00} = \tilde{g}^{\mu\nu} b\partial_\nu b$$

$$\Gamma^0_{\mu\nu} = \frac{\partial_t\tilde{g}_{\mu\nu}}{2b^2} \qquad \Gamma^\mu_{0\nu} = \frac{\tilde{g}^{\mu\rho}\partial_t\tilde{g}_{\nu\rho}}{2} \qquad \Gamma^\rho_{\mu\nu} = \tilde{\Gamma}^\rho_{\mu\nu}$$

Comparing the Christoffel symbols with the theorem (3.2.1), the equations for u and $\dot{u}$ becomes:

$$u_0 = -b(t, \mathbf{x}) \qquad u_\mu = 0 \qquad \dot{u}_0 = 0 \qquad \dot{u}_\mu = \frac{\partial_\mu b}{b} \tag{3.38}$$

Furthermore, explicating these relations $\nabla_\mu u_\nu = -\frac{\dot{\lambda}\dot{u}_\mu \dot{u}_\nu}{\lambda\eta}$ and $\nabla_0 u_\mu = -\frac{\dot{\lambda}u_0\dot{u}_\mu}{\lambda}$, we obtain important results, as a starting point for analysing specific cases:

$$-b\partial_t \tilde{g}_{\mu\nu} = \frac{\dot{\lambda} b \partial_\mu b \partial_n u}{\lambda\eta} \qquad \partial_t \dot{u}_\mu - \dot{u}_\nu \frac{\tilde{g}^{\nu\rho}\partial_t \tilde{g}_{\mu\rho}}{2} = -\frac{\dot{\lambda} b \dot{u}_\mu}{\lambda} \tag{3.39}$$

3.2.2 Static space-time

We impose an important condition: $\dot{\lambda} = 0$. The equation (3.39) tells us that both $\tilde{g}_{\mu\nu}$ and $\dot{u}_\mu$ are time independent. Thus, we can factorize the function $b(t, \mathbf{x}) = b_1(t)b_2(\mathbf{x})$; we obtain again the static metric (3.26) and the new Codazzi equations for the current flow tensor are:

$$\nabla_j u_k = -u_j \dot{u}_k \tag{3.40}$$

$$\nabla_j \dot{u}_k = -\eta u_j u_k + \dot{u}_j \dot{u}_k \frac{\dot{u}^m \nabla_m \eta}{2\eta^2} \tag{3.41}$$

$$\nabla_j \lambda = -\lambda \dot{u}_j \left(2 + \frac{\dot{u}^m \nabla_m \eta}{2\eta^2}\right) \tag{3.42}$$

Ricci and energy-momentum tensor

In this situation, the Ricci tensor can be easily derived, as the vector fields u and $\dot{u}$ are bound to it by the following statement:

Proposition 3.2.2. *In a static space-times, the vector fields u and $\dot{u}$, with $\dot{u}$ closed, are eigenvectors of the Ricci tensor with the same eigenvalue*

Proof. For convenience we call $\epsilon = \dot{u}^m \nabla_m \eta$.

In the proof of theorem (3.2.1), we have obtained an useful relation $\nabla_i \eta = -2\frac{\dot{\lambda}\eta u_i}{\lambda} + \frac{\dot{u}_i \epsilon}{\eta}$. In the static metric , it becomes $\nabla_i \eta^2 = 2\epsilon \dot{u}_i$.

$$
\begin{aligned}
0 &= \nabla_k \nabla_j \eta^2 - \nabla_j \nabla_k \eta^2 \\
&= 2\dot{u}_j \nabla_k \epsilon + 2\epsilon \nabla_k \dot{u}_j - (2\dot{u}_k \nabla_j \epsilon + 2\epsilon \nabla_j \dot{u}_k) \\
&= \dot{u}_j \nabla_k \epsilon - \dot{u}_k \nabla_j \epsilon \\
&= \nabla_k \epsilon - \dot{u}_k \frac{\dot{u}^m \nabla_m \epsilon}{\eta}
\end{aligned} \tag{3.43}
$$

Now, utilizing the definition of Riemann tensor and the equation (3.40) and (3.41), it gives:

$$
\begin{aligned}
R_{jklm} u^m &= \nabla_j \nabla_k u_l - \nabla_k \nabla_j u_l \\
&= (u_j \dot{u}_k - u_k \dot{u}_j)\dot{u}_l - u_k \nabla_j \dot{u}_l + u_j \nabla_k \dot{u}_l \\
&= \dot{u}_l (u_j \dot{u}_k - u_k \dot{u}_j)(1 + \frac{\epsilon}{2\eta^2})
\end{aligned}
$$

Contracting with g^{jl}:

$$
R_{km} u^m = \left(\eta + \frac{\epsilon}{2\eta}\right) u_k \tag{3.44}
$$

Then we find the eigenvalue of $\dot{u}$:

$$
\begin{aligned}
R_{jklm} u^m &= \nabla_j \nabla_k \dot{u}_l - \nabla_k \nabla_j \dot{u}_l \\
&= \nabla_j (-\eta u_l u_k + \dot{u}_l \dot{u}_k \frac{\epsilon}{2\eta^2}) - \nabla_k (-\eta u_j u_l + \dot{u}_j \dot{u}_l \frac{\epsilon}{2\eta^2}) \\
&= -u_l (u_k \nabla_j \eta - u_j \nabla_k \eta) + \eta u_l (u_j \dot{u}_k - u_k \dot{u}_j) \\
&+ \frac{\epsilon}{2\eta^2} (\dot{u}_k \nabla_j \dot{u}_l - \dot{u}_j \nabla_k \dot{u}_l) + \dot{u}_l (\dot{u}_k \nabla_j \frac{\epsilon}{2\eta^2} - \dot{u}_j \nabla_k \frac{\epsilon}{2\eta^2}) \\
&= u_l (u_j \dot{u}_k - u_k \dot{u}_j)(\eta + \frac{\epsilon}{\eta}) + \frac{\epsilon}{2\eta^2} (\dot{u}_k \nabla_j \dot{u}_l - \dot{u}_j \nabla_k \dot{u}_l)
\end{aligned}
$$

Contracting with g^{jl}:

$$
R_{km} \dot{u}^m = \left(\eta + \frac{\epsilon}{2\eta}\right) \dot{u}_k \tag{3.45}
$$

□

Theorem 3.2.3. *The Ricci tensor is:*

$$
\begin{aligned}
R_{kl} = &\left(\frac{R}{n-1} + 2\epsilon + \frac{\epsilon}{\eta}\right) u_k u_l + \left(\frac{R}{n-1} + \epsilon + \frac{\epsilon}{2\eta}\right) g_{kl} \\
&- (n-2)\left(E_{kl} + \dot{u}_k \dot{u}_l \left(1 + \frac{\epsilon}{2\eta^2}\right)\right)
\end{aligned} \tag{3.46}
$$

where E_{kl} is the electric tensor, and it holds

$$E_{kl}\dot{u}^l = \left(\frac{R}{n-1} - (n-4)\left(\eta + \frac{\epsilon}{2\eta}\right)\right)\dot{u}^k \tag{3.47}$$

Proof. In (3.2.2) we found that $R_{jklm}u^m = \dot{u}_l(u_j\dot{u}_k - u_k\dot{u}_j)(1+\frac{\epsilon}{2\eta^2})$.

Since $u^j u^m R_{ijkl} = -\dot{u}_k\dot{u}_l(1+\frac{\epsilon}{2\eta^2})$, contracting the Weyl expression with $u^j u^l$, we get:

$$u^j u^m C_{jklm} = E_{kl} = -\dot{u}_k\dot{u}_l(1+\frac{\epsilon}{2\eta^2}) - \frac{R_{kl} - (g_{kl} + 2u_k u_l)(\eta + \epsilon/(2\eta))}{n-2} + R\frac{g_{kl} + u_k u_l}{(n-1)(n-2)}$$

Therefore, as we aimed to prove, we have found such a Ricci tensor. Utilizing the Ricci eigenvalue equation (3.45) with $\dot{u}^l$, we find the eigenvalue of $\dot{u}^l$ with respect to E_{kl}. □

As in the previous section, we intend to fully describe the Einstein equation, so we calculate the momentum-energy tensor below.

Proposition 3.2.4. *The energy-momentum tensor is:*

$$T_{kl} = \left(\frac{R}{n-1} + 2\epsilon + \frac{\epsilon}{\eta}\right)u_k u_l + \left(\frac{(3-n)R}{2(n-1)} + \epsilon + \frac{\epsilon}{2\eta}\right)g_{kl} - (n-2)\left(E_{kl} + \dot{u}_k\dot{u}_l\left(1 + \frac{\epsilon}{2\eta^2}\right)\right) \tag{3.48}$$

The third Codazzi condition (3.42) with $\epsilon = 0$ was obtained by Rao and Rao[29] in order to describe the relativistic generalisation of the uniform Newton force onto a spatial hypersurface in a static universe.

3.2.3 Spherically symmetrical static space

We analyze the static space with the condition of spherical symmetry and then list some examples of this space-time. The metric is:

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + f_2^2(r)d\Omega_{n-2} \tag{3.49}$$

Due to symmetry, $\dot{u}$ is radial and $\dot{u}_r = \frac{b'(r)}{b(r)}$ (prime indicates a derivative in r).

$$\eta(r) = \frac{b'^2(r)}{f_1^2(r)b^2(r)} \tag{3.50}$$

$$\lambda(r) = \frac{kf_1(r)}{b'(r)b(r)} \tag{3.51}$$

k is a constant.

Since $\dot{u}$ is radial vector. the angular component are $\nabla_{a_i}\dot{u}_{a_j} = 0$, where a_i, a_j enumerate the n-2 angles. Thus $\Gamma^r_{a_i a_j}\dot{u}_r = 0$.

Therefore, the metric of a static, spherically symmetric space-time, which hosts a Codazzi current flow tensor, is reduced by the condition given in the appendix in [23] $\frac{df_2}{dr} = 0$:

$$ds^2 = -b^2(r)dt^2 + f_1^2(r)dr^2 + K^2(r)d\Omega_{n-2} \tag{3.52}$$

where K is a positive constant.

Again, in [23] the electric tensor and the scalar curvature of the space manifold are found:

$$E_{jk} = \frac{n-3}{(n-2)f_1^2}\left(\frac{f_1^2}{K^2} + \frac{b'f_1'}{bf_1} - \frac{b''}{b}\right)\left(\frac{\dot{u}_j\dot{u}_k}{\eta} - \frac{g_{jk} + u_j u_k}{n-1}\right) \tag{3.53}$$

$$R^* = \frac{(n-2)(n-3)}{K^2} \tag{3.54}$$

Below are some examples of such metrics. They have the same Ricci tensor (3.46) and electric tensor (3.53) and a current flow Codazzi tensor with non-zero components $C_{0r} = k\frac{f_1(r)}{b}$

(i) Bertotti-Robinson space-times are conformally flat of the source-free Einstein-Maxwell equations with non null e.m. fields. The metric and the Ricci tensor are:

$$ds^2 = \frac{r_0^2}{r^2}(-dt^2 + dr^2 + r^2 d\Omega_{n-2}) \tag{3.55}$$

$$R_{jk} = \frac{2u_j u_k + g_{jk}}{r_0^2} - 2\dot{u}_k\dot{u}_j \tag{3.56}$$

Equations (3.40) and (3.41) imply that Ricci is also Codazzi. Hence, in the Bertotti-Robinson space-time exist two Codazzi tensor: the Ricci tensor and

$$C_{jk} = -k\frac{r^2}{r_0^2}(u_j\dot{u}_k + u_k\dot{u}_j)$$

In [25], through to a coordinate change, the metric becomes, with $b(\rho) = 1 + \rho^2/r_0^2$

$$ds^2 = -bd\tau^2 + \frac{1}{b}d\rho^2 + r_0^2 d\Omega_2$$

(ii) Metin Gurses, in 1992, calculated[11] the metric of black holes their outer horizons, using a string correction. It describes an non-singular and homogeneous space-time. It is also solution of Einstein-Maxwell equations.

$$ds^2 = -(ar^2 + br + c)dt^2 + \frac{dr^2}{ar^2 + br + c} + K^2 d\Omega_2 \tag{3.57}$$

(iii) In [17] the Bertotti-Robinson black holes, namely black hole static and homogeneous, in which the curvature is uniform, are described by the metric

$$ds^2 = -\left(\frac{r^2}{l^2} + \frac{J^2}{r^2} - M\right) dt^2 + \left(\frac{r^2}{l^2} + \frac{J^2}{r^2} - M\right)^{-1} dr^2 + K^2 d\Omega_{n-2} \tag{3.58}$$

M is the mass, J the angular momentum and l^2 is proportional to cosmological constant.

3.3 Yang Pure space-times

We conclude the chapter by describing a brief example known in literature where Codazzi condition has a great deal of relevance: Yang Pure space-times[10].

The Yang Pure space-time (YPS) must satisfy the following equation

$$\nabla_i R_{jk} = \nabla_j R_{ik} \tag{3.59}$$

or, equivalently,

$$\nabla^l R_{ijkl} = 0 \tag{3.60}$$

From these last two equations, we can immediately state:

Proposition 3.3.1. *Every Lorentzian manifold (V, g), which is*

- *an Einstein space*
- *conformally flat with constant curvature*

is a Yang Pure space-times.

Proof. The proof is a direct implication of the Weyl tensor gradient equation::

$$\nabla_i C^i{}_{jkl} = \nabla_{[k} R_{l]j} + \frac{1}{6}(g_{j[l} \nabla_{k]} R) \tag{3.61}$$

□

The Yang equation is not a replacement of the Einstein equation, but rather an additional condition that determines peculiar solutions. Thus combining $T_{ij} = R_{ij} - \frac{1}{2}g_{ij}R$ with the equation (3.59):

$$\nabla_{[i}T_{j]k} = 0 \tag{3.62}$$

Applying the conservation law $\nabla^i T_{ij} = 0$, the scalar tensor $T = g^{ij}T_{ij}$ will be constant.

Considering again the perfect fluid as source $T_{ij} = (\mu + \rho)u_i u_j + p g_{ij}$ with $u_i u^i = -1$, straightforwards calculations give:

$$p = \frac{1}{3}\mu + c \tag{3.63}$$

$$\nabla_j u_i = \phi(g_{ij} + u_i u_j) \tag{3.64}$$

$$\dot{u}_i = 0 \tag{3.65}$$

$$\nabla_j \mu = 3\phi(\mu + p)u_j \tag{3.66}$$

assuming $\mu + p \neq 0$ (c is a constant). In fact, if $\mu + p = 0$, then μ and p are constant, giving a Einstein space.

Conditions (3.64) and (3.65) mean that the velocity is geodesic, vorticity-free and shear-free. As already pointed out, these are necessary and sufficient conditions for Robertson-Walker geometry [15].

The result can be summarized by the following theorem:

Theorem 3.3.2. *A perfect fluid space-time* (V, g) *with* $\mu+p \neq 0$ *is a YPS if and only if* (V, g) *is a Robertson-Walker space-time with* $p = \frac{1}{3}\mu + c$.

Chapter 4

Cotton gravity

4.1 Harada's Theory

Recently, Harada published an article[12] in which he outlined a new theory of gravity, extending Einstein's. He denominated it *Cotton Gravity* since the Einstein tensor $G_{jk} = R_{jk} - \frac{1}{2}Rg_{jk}$ is replaced by the Cotton tensor.

The Cotton tensor $\mathcal{C}_{jkl}$ is a (0-3) tensor [5] describing the curvature of a manifold, in particular $\mathcal{C}_{jkl} = 0$ is a necessary condition for a space with more than 4 dimensions to be conformally flat (but not sufficient).

The general expression of the Cotton tensor is:

$$\mathcal{C}_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} - \frac{g_{kl}\nabla_j R - g_{jl}\nabla_k R}{2(n-1)} \tag{4.1}$$

Furthermore, three proprieties of Cotton tensor are:

- $\mathcal{C}_{jkl} = -\mathcal{C}_{ikj}$ and therefore, $\mathcal{C}_{[jkl]} = 0$
- $g^{jk}\mathcal{C}_{jkl} = 0$
- It is related to the Weyl tensor $\mathcal{C}_{jkl} = -\frac{n-2}{n-3}\nabla^m C_{jklm}$

The fundamental equation of Cotton gravity, replacing Einstein's is:

$$\mathcal{C}_{jkl} = 16\pi G \nabla_i T^i_{\ jkl} \tag{4.2}$$

where the tensor T_{jklm} is

$$T_{jklm} = \frac{1}{2}(g_{jl}T_{km} - g_{kl}T_{jm} - g_{jm}T_{kl} + g_{km}T_{jl}) - \frac{1}{2}(g_{jl}g_{km} - g_{kl}g_{jm})T \tag{4.3}$$

It can be easily shown that every solution that satisfies Einstein equation, also with cosmological constant, $G_{jk} + \Lambda g_{jk} = T_{jk}$ satisfies also the (4.2)

The equation (4.2) can be derived from Einstein equations. Indeed, this is a criticism that has been raised against Harada's theory, as it would make it merely a different formulation, rather than an extension of the theory of relativity [2].

We consider the well-known decomposition of the Riemann tensor

$$R_{jklm} = C_{jklm} + E_{jklm} + \frac{R}{12} g_{jklm}$$

where C_{jklm} is the Weyl tensor, $E_{jklm} = g_{jkp[l} S^p{}_{m]}$ $S^p{}_m = R^p{}_m - \frac{R}{4}\delta^p{}_m$ and $g_{jklm} = 2g_{i[k} g_{m]j}$ Applying the Bianchi identities $\nabla^m R_{jklm} = 0$ and using Einstein's field equations, it gives:

$$\nabla^m C_{jklm} = E_{jklm} + g_{jklm} \frac{\nabla^m R}{12}$$
$$\mathcal{C}_{jkl} = (T_{j[k;l]} - \frac{1}{3} g_{j[k} T_{,l]})$$

which is exactly the (4.2).

Actually, Harada demonstrated how his equations are not a derived identity of the equations of general relativity, but rather they are a new class of equations obtained from a variational principle of action[12][14]. This specific action is composed by Weyl's action and the analogue of the source term of Maxwell's theory.:

$$\mathcal{I} = \mathcal{I}_w + \mathcal{I}_s = \frac{1}{2} \int d^4x \sqrt{-g} C^{jklm} C_{jklm} - \int d^4x \sqrt{-g} T^{jklm} R_{jklm} \quad (4.4)$$

We now explicate the variational principle, emphasising that we vary the action with respect to the derivative of the metric, namely the connection, keeping the metric itself fixed.

$$0 = \delta\mathcal{I} = \delta\mathcal{I}_w + \delta\mathcal{I}_s = \int d^4x \sqrt{-g} \mathcal{C}^{jkl} \delta\Gamma_{jkl} - \int d^4x \sqrt{-g} \nabla_m T^{jklm} \delta\Gamma_{jkl}$$

Thus we obtain exactly the equation of Cotton gravity.

In the same paper, Harada finds exact solutions in vacuum, described by $\mathcal{C}_{jkl} = 0$, a generalisation of $R_{jk} = 0$ of general relativity; every solution in Einstein's vacuum will also be a solution for Harada, as is obvious from (4.1).

The resulting metric is:

$$ds^2 = -\left(1 - \frac{2M}{r} + \gamma r - \frac{1}{3}\Lambda r^2\right) dt^2 + \left(1 - \frac{2M}{r} + \gamma r - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 + r^2 d\Omega_2^2 \quad (4.5)$$

It adds the term γr to the Schwarzschild metric.

It is remarkable the similarity of this metric with another result, again of vacuum solution, obtained in conformal gravity[18] [27]:

$$ds^2 = \left(1 - 3\beta\gamma - \frac{-\beta(2-3\beta\gamma)}{r} + \gamma r - Kr^2\right) dt^2 + \left(1 - 3\beta\gamma - \frac{-\beta(2-3\beta\gamma)}{r} + \gamma r - Kr^2\right)^{-1} dr^2 + r^2 d\Omega_2^2 \quad (4.6)$$

The conformal gravity is an alternative theory of gravitation, an alternative to Einstein's theory. It refers to those theories which are invariant under conformal transformations of the type $\tilde{g}_{ij} \to \Omega^2(x) g_{ij}$, where $\Omega(x)$ is a function on space-time. Its action is the same action used by Harada, the Weyl action (4.4) , with the distinction that the variation is made with respect to the metric, thus leading to the equations:

$$T_{kl} = -4(2\nabla^j\nabla^m C_{jklm} + R^{jm} C_{jklm}) \quad (4.7)$$

$B_{kl} = 2\nabla^j\nabla^m C_{jklm} + R^{jm} C_{jklm}$ is the Bach tensor [1]. It is symmetric, traceless, its divergence is $\nabla^k B_{kl} = 0$, thus the conservation of energy is geometrically respected $\nabla^l T_{kl} = 0$. Furthermore, most relevantly, under conformal transformation, it is proportional to

$$\tilde{B}_{kl} = \Omega^{-2} B_{kl} \quad (4.8)$$

Therefore it is called conformal gravity, because the Bach tensor does not depend on further terms under conformal transformation [3].

While the Cotton tensor, as shown in [12] (equation 36), under conformal transformation depends on:

$$\tilde{\mathcal{C}}_{jkl} = \mathcal{C}_{jkl} + \Omega^{-1}\partial_m W^m_{\ jkl} \quad (4.9)$$

Thus, the Cotton tensor is conformally invariant if and only if the Weyl tensor vanishes. This is one of the main differences between the two alternative theories of gravity, even though they derive from the same action.

The first application of the Cotton gravity developed by Harada was the description of the rotational motion of 84 galaxies: using the new gravitational potential derived from the vacuum metric (4.5), the rotation curves were fitted with good results, and with an interesting development: in fact, the γr term allowed an accurate description without having to use dark matter[13].

The utilized potential is $\phi = -G\frac{M}{r} + \frac{\gamma}{2}r^2$, where the factor $\frac{\gamma}{2}r^2$ cannot be obtain from general relativity, and becomes very relevant at large distances: it is this additional term that replaces the contribution of dark matter.This is one of the reasons why Harada's theory is so intriguing. M and γ are two physical integration constants that are determined for each specific galaxy.

4.2 Codazzi tensor describing Cotton Gravity

In the development of the Cotton gravity, Codazzi tensors naturally emerge. They characterise space-time, as shown in the previous section, and allow us to derive the Ricci tensor and thus obtain the energy-momentum tensor.

By expressing the equation (4.2) through the (4.3):

$$\mathcal{C}_{jkl} = \nabla_j T_{kl} - \nabla_k T_{jl} - \frac{g_{kl}\nabla_j T - g_{jl}\nabla_k T}{(n-1)} \tag{4.10}$$

and subtracting the (4.1) gives the following system:

$$C_{jk} = R_{jk} - T_{jk} - g_{jk}\frac{R - 2T}{2(n-1)} \tag{4.11}$$

$$\nabla_i C_{jk} = \nabla_j C_{ik} \tag{4.12}$$

These two equations are equivalent to the Harada's descriptions of Cotton gravity (4.2). The third-order character in Cotton's tensor is replaced in this new formulation, reducing it to second order, by an additional tensor C_{jk}, which can be interpreted as either a modification of space-time or a modification of the source of the gravitational field. If it is equal to 0, or $C_{jk} = Bg_{jk}$, the equations of general relativity are re-established, with the addition of a cosmological constant. If instead $C_{jk} \neq Bg_{jk}$, we obtain a new expression of the Ricci tensor (or momentum energy tensor):

$$R_{jk} = T_{jk} + C_{jk} - g_{jk}C^m{}_m + \frac{1}{2}g_{jk}R \tag{4.13}$$

$$T_{jk} = R_{jk} - C_{jk} + g_{jk}C^m{}_m - \frac{1}{2}g_{jk}R \tag{4.14}$$

These two new equations can be interpreted as a generalization of Einstein's equations, with a new formulation of the energy-momentum tensor (or Ricci tensor) adding the Codazzi term C_{ij}.

Based on the study of hypersurfaces, and having analysed Stephani's solution methods, we can state the theorem:

Theorem 4.2.1. *Every solution of embedding class 1 is also solution of Cotton gravity*

Proof. A solution of embedding class 1 satisfies the Gauss-Codazzi equation (1.12) and (1.13). Thus, constructing the energy-momentum tensor of the form (4.14), and substituting the Gauss equation:

$$\begin{aligned} T_{jk} &= R_{jk} - C_{jk} + g_{jk}C^m_{\ m} - \frac{1}{2}R - C_{jm}C^m_{\ k} \\ &= C_{jk}(C^m_{\ m} - 1) + g_{jk}(C^m_{\ m} - \frac{1}{2}R) - C_{jm}C^m_{\ k} \end{aligned}$$

and the condition (4.12) is precisely the Codazzi condition (1.13). □

Perfect Fluid

The process is similar to the one shown in the previous section. We impose the Codazzi condition on the symmetric tensor C_{jk} in the formula (4.14). It imposes certain restrictions on space-time, from which we can then derive the Ricci tensor. Finally, the Harada energy-momentum tensor (4.14) is made explicit.

We shall start with the perfect fluid $C_{jk} = Au_ju_k + Bg_{jk}$. Obviously, Codazzi condition is satisfied by the same equations (3.2-3.6). The metric of the space-time and the Ricci tensor are respectively (3.22) and (3.17).

Proposition 4.2.2. *The energy-momentum tensor, considering the Cotton gravity, is:*

$$T_{jk} = \left(\frac{R - n\gamma}{n-1} - A\right)u_ku_j + \left(\frac{(3-n)R - 2\gamma}{2(n-1)} + (n-1)B - A\right)g_{jk} + \Pi_{jk} \tag{4.15}$$

The term Π_{jk} denotes dissipation and anisotropy and is the same as in the analogous case (3.19), while the pressure terms $p_\perp$ and energy density μ are:

$$p_\perp = \left(\frac{(3-n)R - 2\gamma}{2(n-1)} + (n-1)B - A\right) \tag{4.16}$$

$$\mu = \left(\frac{R - n\gamma}{n-1} - A\right) - p_\perp \tag{4.17}$$

Harada's theory in this case simply leads to a modification of the characteristic terms of the field source, but without substantially changing its nature.

Imperfect fluid

The last case studied is the imperfect fluid $C_{jk} = Ag_{jk} + 2Au_j u_k + Bz_j z_k$. We compare the results with Stephani's solutions (section 2.2.2.).

The Ricci tensor and the curvature scalar, since they express the curvature of space-time, will still be of the form (2.17) and (2.18).

Proposition 4.2.3. *The energy-momentum tensor, considering the Cotton gravity, is:*

$$T_{jk} = (A^2 + A + B)g_{jk} + (4A^2 + 2AB - 2A)u_j u_k - Bz_j z_k \tag{4.18}$$

The pressure and energy density terms are:

$$p_\perp = A^2 + A + B \tag{4.19}$$

$$\mu = 3A^2 + 2AB - 3A - B \tag{4.20}$$

It is very interesting to compare this solution with Stephani's solution (2.19): not only do the pressure and energy density terms change, as in the previous case of perfect fluid, but the nature of the field source itself is different. In fact, in Stephani the field source is a perfect fluid, while with Harada's solutions, the field source has an anisotropy term $-Bz_j z_k$. Therefore we can conclude that in the two cases the geometry of the universe itself will change: in one case it will be isotropic, in the other it will not.

Appendix A

Relativistic fluid tensors

Fluids are frequently used models in relativity and cosmology because they allow the radiation-matter contribution in the universe to be described. Assuming the universe to be homogeneous and isotropic, the energy-momentum tensor is described as a perfect fluid, i.e. a continuous distribution of matter of the form:

$$T_{ij} = (p + \mu)u_i u_j + p g_{ij} \tag{A.1}$$

where μ is the mass-energy density, p is the pressure. These two physical quantities are often related to each other by an equation of state (the most well-known is $p(\mu) = \frac{\mu}{3}$).The fluid is called perfect because is viscosity-free.

When we add a term of viscosity, we obtain the so-called imperfect fluid. it is written in the form:

$$T_{ij} = (\mu + p_\perp)u_i u_j + p_\perp g_{ij} + u_i q_j + u_j q_i + (p_r - p_\perp)\chi_j \chi_i \tag{A.2}$$

The different terms express:

- μ is the mass-energy density
- $p_\perp$ is the pressure perpendicular to the direction of fluid propagation
- p_r is the pressure along the radial direction
- χ_i is a space-like vector, perpendicular to u
- $u_i z_j + u_j z_i$ expresses the current flow of the fluid in motion.

If T_{jk} is the field source, then since it has anistotropy terms, the assumption of isotropy of the FW universe will fall.

Appendix B

Expansion of unit time-like vector

In differential geometry and General relativity, a *congruence* is a family of integral curves, defined by a vector field. A congruence can be time-like, null-like or space-like: it depends on whether the tangent vectors to such curves are everywhere time-type, light-type or space-type.

We focus ourselves on time-like unit vector field u_i, $u^i u_i = -1$; it can be physically interpreted as a family of particles having a time-like velocity field u.

On a Lorentzian manifold, given the vector field u, we can define the orthogonal projector $h_{ij} = g_{ij} + u_i u_j$, or rather the metric tensor of the hypersurface spanned by tangent vectors orthognonal to u. Thus, any vector can be separated into its parallel and orthogonal part to the vector u; this definition can also be applied to the covariant derivative, decomposing in irreducible parts:

$$\nabla_i u_j = \phi(g_{ij} + u_i u_j) + \omega_{ij} + \sigma_{ij} - u_i \dot{u}_j \tag{B.1}$$

Now we explain each component:

- $\dot{u}_j = u^m \nabla_m u_j$ is the acceleration: it represents the rate of variation of velocity in unit of time.
- ϕ is the expansion parameter: it represents the rate of variation of volume on the hypersurface, for unit volume.

$$\phi = (n-1)\frac{1}{\delta V}\frac{d(\delta V)}{dt}$$

- $\omega_{ij} = u_{[i;j]} + \dot{u}_{[i} u_{j]}$ is the vorticity: it is the relativistic extension of classical vorticity, which express the rotation of fluid streamlines.
- $\sigma_{ij} = u_{(i;j)} + \dot{u}_{(i} u_{j)} - \phi(g_{ij} + u_i u_j)$ is the shear It represents the deformation of spatial hypersurfaces, or rather the tendency of a sphere to become an ellipsoid of the same measure.

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