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On the semiclassical stability of Minkowski spacetime

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Introduction

The coexistence of General Relativity and Quantum Mechanics is one of the biggest open problems in modern Physics. Both of them give high precision prediction at different scales, but when one tries to quantize GR, the resulting theory is not renormalizable and not well defined. There have been many attempts to overcome these difficulties, such as String theory, the still unknown M-theory, Loop Quantum Gravity, non-commutative general relativity and many others. However some of these approaches, at the moment, have difficulties in predicting observable phenomena that could be used to validate or falsify them.
In this context, it could be useful to study the semiclassical limit of Quantum Gravity, in order to obtain at least some qualitative expectation about the main features of the effective field theory that one could find from the low energy limit of a more complete approach to the quantization of gravity. While it has been proven that the flat Minkowski spacetime is an absolute minimum of the classical gravitational action in asymptotically flat spacetimes[1], it is not sure if it could be also a good ground state for the quantum theory. John Wheeler in 1957[2] observed that, at Planck scale, the spacetime manifold could be affected by geometrical and topological fluctuations. Such a spacetime, continuously changing its metric and topology, is known as "*spacetime foam*" and is expected to be a better ground state for quantum gravity.
At quantum level, it has been shown that the flat spacetime is unstable at first loop approximation respect to black holes nucleation when a nonzero temperature is introduced in the euclidean pure gravity path integral[3]. If $T \neq 0$, the euclidean Schwarzschild solution is an extremal point of the action end brings an imaginary part to the free energy of the functional integral. Moreover, if gravity is coupled with N conformally invariant scalar fields, the statement that the ground state of the theory is obtained in correspondence with the flat spacetime is false at leading order in $1/N$[4].
It could be interesting to see whether something similar happens also in the Hamiltonian approach to semiclassical Quantum Gravity, where no temperature is defined.
The main problem in this treatment is that general relativity is a covariant theory, that means it is invariant under spatial and temporal reparametriza-

tions by construction. Thus, the Hamiltonian, which generates time translations, is identically null for each possible spacetime, except for eventual boundary contributions. The constraint that imposes a null Hamiltonian density is known as Wheeler-DeWitt equation and it is the most important relation in the study of General Relativity as an Hamiltonian field theory. It is possible to treat this problem introducing a vacuum energy density as a quantum correction of the cosmological constant and study the WdW equation as a Sturm-Liouville eigenvalue problem[5].
While considering spacetimes with spatial topology different from R^3, R. Garattini studied the case of a Schwarzschild wormhole between two different asymptotically flat spacetimes[6]. If the two sides of the wormhole are symmetric, the black hole in the universe where the asymptotic observer who executes the measurements lives, will have a positive mass M, while the other one will have mass $-M$, forming a sort of black hole anti-black hole dipole. The result is a manifold with zero boundary energy like a couple of Minkowski spacetimes. That means the semiclassical computation of the spacetime energy has a crucial role in determining whether a flat metric could decay in a Schwarzschild wormhole, or in a foam composed by a large number of wormholes homogeneously distributed in spacetime.

Conventions:
in this thesis we use the signature (-,+,+,+) for the Lorentzian metric and we take the constants $\hbar$ and c equal to one.

Chapter 1

Field theory formulation and quantization of gravity

1.1 The Lagrangian formulation of gravity

In this section we are going to show that the pure-gravitational Hilbert-Einstein action for the field $g_{\mu\nu}$ gives exactly the Einstein vacuum equation, following the method used in [7].
We will begin considering the problem without boundary contributions. The Einstein-Hilbert action is

$$S = \frac{1}{16\pi G}\int_M d^4x\sqrt{-g}(R - 2\Lambda) \tag{1.1.1}$$

where R is the curvature scalar and Λ the cosmological constant. An important aspect of variational calculus respect to the metric $g_{\mu\nu}$ is that one have to pay great attention when uses the metric itself to rise and lower the indices of perturbations. In fact, the relation $(g+\delta g)^{\mu\rho}(g+\delta g)_{\rho\nu} = \delta^\mu_\nu$ must be respected, then we have to distinguish the variation of the inverse metric $\delta g^{\mu\nu}$ from the perturbation of $g_{\mu\nu}$, that we will call $\delta g_{\mu\nu}$. The requirement of a Kronecker delta implies the transformation $\delta g^{\mu\nu} = -g^{\rho\mu}g^{\sigma\nu}\delta g_{\rho\sigma}$.
We immediately have

$$\delta S = \frac{1}{16\pi G}\int_M d^4x \left[\delta(\sqrt{-g})(R - 2\Lambda) + \sqrt{-g}\delta R^{\mu\nu}g_{\mu\nu} + \sqrt{-g}R^{\mu\nu}\delta g_{\mu\nu}\right] \tag{1.1.2}$$

Let's consider a generic real symmetric matrix M, it can be put in a diagonal form and written as $M = e^m$ where m is a complex matrix. naturally we also have the inverse relation $m = \ln M$ and it is true the relation $\det M = e^{\mathrm{Tr}\, m}$. If we need the variation of the determinant of M, we can write it as

$$\delta \det M = e^{\mathrm{Tr}\{m\}}\,\mathrm{Tr}\{\delta m\} = \det M\ \ \mathrm{Tr}\big(M^{-1}\delta M\big) \tag{1.1.3}$$

Taking $M = g_{\mu\nu}$ and $M^{-1} = g^{\mu\nu}$, we find

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} \tag{1.1.4}$$

On the other side it is possible to compute the variation of the Ricci tensor. The Riemann tensor can be written as

$$R^{\rho}{}_{\mu\lambda\nu} = 2\partial_{[\lambda}\Gamma^{\rho}_{\nu]\mu} + 2\Gamma^{\rho}_{\sigma[\lambda}\Gamma^{\sigma}_{\nu]\mu} \tag{1.1.5}$$

Obviously a variation of the Riemann tensor, and consequently of the Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$, is strongly connected with the variation of the Christoffel symbols. Indeed it is true

$$\delta R^{\rho}{}_{\mu\lambda\nu} = 2\nabla_{[\lambda}\delta\Gamma^{\rho}_{\nu]\mu} \tag{1.1.6}$$

The Christoffel symbols do not transform as tensors, since a change of coordinates in the manifold modifies the partial derivatives and we want ∇_{μ} to be covariant. Anyway, if we consider the difference of two Christoffel symbols, the effects of the change of coordinates over partial derivatives should be nullified in the subtraction. So we expect $\delta\Gamma^{\rho}_{\mu\nu}$ to be a tensor and taking its covariant derivative as in (1.1.6) makes sense.
It can be shown that, for each metric, there is a unique torsionless connection that gives null covariant derivatives of the metric itself. A set of connection symbols $C^{\rho}_{\mu\nu}(g)$ can be seen as the corrections needed in order to transform a particular set of derivatives ∇_{μ} in the covariant derivatives naturally associated to $g_{\mu\nu}$ through the procedure $\nabla_{\mu}v^{\rho} \to \nabla_{\mu}v^{\rho} + C^{\rho}_{\mu\nu}(g)v^{\nu}$. The Christoffel symbols are those obtained when $\nabla_{\mu} = \partial_{\mu}$ and are given by

$$\Gamma^{\rho}_{\mu\nu}(g_{\mu\nu}) = \frac{1}{2}g^{\rho\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}) \tag{1.1.7}$$

Anyway this expression is valid also respect any other set of derivatives different from the usual partial derivatives ∂_{μ}. Now we can observe that $\delta\Gamma^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu}(g + \delta g) - \Gamma^{\rho}_{\mu\nu}(g)$ is exactly the connection symbol $C^{\rho}_{\mu\nu}(g + \delta g)$ needed by the covariant derivative associated to $g_{\mu\nu}$ to become the one that is null when applied to $g_{\mu\nu} + \delta g_{\mu\nu}$. That means, at first order in $\delta g_{\mu\nu}$,

$$\begin{aligned} C^{\rho}_{\mu\nu}(g + \delta g) &= \frac{1}{2}g^{\rho\sigma}[\nabla_{\mu}(g_{\nu\sigma} + \delta g_{\nu\sigma}) + \nabla_{\nu}(g_{\mu\sigma} + \delta g_{\mu\sigma}) - \nabla_{\sigma}(g_{\mu\nu} + \delta g_{\mu\nu})] \\ &= \frac{1}{2}g^{\rho\sigma}[\nabla_{\mu}\delta g_{\nu\sigma} + \nabla_{\nu}\delta g_{\mu\sigma} - \nabla_{\sigma}\delta g_{\mu\nu}] = \delta\Gamma^{\rho}_{\mu\nu} \end{aligned} \tag{1.1.8}$$

where ∇_{μ} represents the covariant derivative associated to the unperturbed metric $g_{\mu\nu}$. Thus,we have an expression for the variation of the Christoffel symbols.

A substitution of the last equation in the variation of the Riemann tensor (1.1) gives

$$g_{\mu\nu}\delta R^{\mu\nu} = \nabla_\rho(\nabla_\sigma \delta g^{\rho\sigma} - g_{\sigma\alpha}\nabla^\rho \delta g^{\sigma\alpha}) \tag{1.1.9}$$

that is a total derivative and so gives only boundary contributions. Hence, the result of the functional derivative of the action respect to the field $g_{\mu\nu}$ is

$$\frac{\delta S}{\delta g_{\mu\nu}} = \frac{1}{16\pi G}\sqrt{-g}[R^{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g^{\mu\nu}] \tag{1.1.10}$$

When we set this expression to zero in order to find the solution of motion, we obtain exactly the Einstein equation.
At this point we still have to treat the boundary contributions. If we consider also perturbations of the metric which are null on the boundary of the manifold, but don't have null derivatives, we can write, using Stokes theorem,

$$\int_M \nabla_\rho(\nabla_\sigma \delta g^{\rho\sigma} - g_{\sigma\alpha}\nabla^\rho \delta g^{\sigma\alpha}) = \int_{\partial M} n_\rho(\nabla_\sigma \delta g^{\rho\sigma} - g_{\sigma\alpha}\nabla^\rho \delta g^{\sigma\alpha}) \tag{1.1.11}$$

where n_ρ is the unitary vector field normal to the boundary ∂M.
The last integrand is equivalent to

$$n_\rho \sigma_{\mu\beta}(\nabla^\beta \delta g^{\rho\mu} - \nabla^\rho \delta g^{\mu\beta}) = -n_\rho \sigma_{\mu\beta}\nabla^\rho \delta g^{\mu\beta} \tag{1.1.12}$$

with $\sigma_{\mu\nu} = g_{\mu\nu} \pm n_\mu n_\nu$ representing the induced metric of the submanifold ∂M. This happens because $\delta g^{\mu\nu}$ is null in the integration region, and consequently $\sigma_{\mu\beta}\nabla^\beta \delta g^{\rho\mu} = 0$. The sign in the definition of $\sigma_{\mu\nu}$ depends on the boundary being a timelike or spacelike manifold.
This last expression is really strongly related to the variation of the trace of the extrinsic curvature $K = K^\mu_\mu = \sigma^\mu_\nu \nabla_\mu n^\nu$of the boundary manifold:

$$\delta K = \sigma^\mu_\nu \delta(\Gamma^\nu_{\mu\rho})n^\rho = \frac{1}{2}n^\rho h^{\mu\nu}\nabla_\rho \delta g^{\mu\nu} \tag{1.1.13}$$

then, if we want to maintain the Einstein equation as equation of the motion, we just have to add a contribution

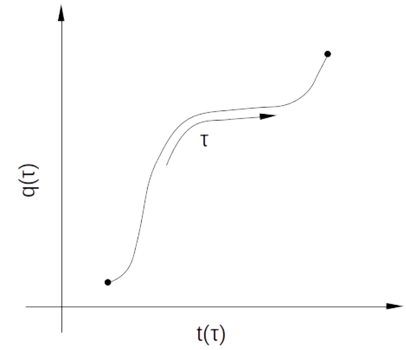
$$\frac{1}{8\pi G}\int_{\partial M} K d^3x\sqrt{\sigma} \tag{1.1.14}$$

where σ is the determinant of the metric of the boundary submanifold.

1.2 Dynamics of generally covariant systems

Before jumping in the Hamiltonian formulation of general relativity, it is important to understand which are the main features of generally covariant theories like GR [8].

One can transform any non covariant system in a covariant one simply introducing an arbitrary parametrization. Let's consider the motion of a point particle with position q and classical time t. What we are interested in is the trajectory $q(t)$, but, if we want to consider in the same way both time and space, we can introduce a parameter τ and express the information contained in $q(t)$ via the couple of functions $q(\tau)$ and $t(\tau)$.
It is always possible to construct two functions $q(\tau)$ and $t(\tau)$ which define



implicitly the relation between physical quantities $q(t)$, but it's not always true the opposite statement. That means the covariant formalism can represent a wider set of relations between t and q respect to the canonical $q(t)$ representation.
Obviously the total amount of information is the same, but now we are considering two functions in spite of one. This means the arbitrariness of the parametrization τ introduces a gauge degree of freedom in the theory. But how do the action and the Hamiltonian change in this process?
The action becomes a functional of two functions q and t thanks to a change of variable

$$S[q] = \int_{t_i}^{t_f} dt L(q(t), \dot{q}(t)) \to S[q,t] = \int_{\tau_i}^{\tau_f} d\tau \frac{dt(\tau)}{d\tau} L\left(q(\tau), \frac{dq(\tau)}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1}\right) \tag{1.2.1}$$

In the case of the point particle the typical action

$$L(q(t), \dot{q}(t)) = \frac{1}{2} m \dot{q}^2(t) - V(q) \tag{1.2.2}$$

becomes

$$L(q(\tau), t(\tau), \dot{q}(\tau), \dot{t}(\tau)) = \frac{1}{2} m \frac{\dot{q}^2}{\dot{t}} - \dot{t} V(q) \tag{1.2.3}$$

where now the dot stands for τ derivative. The resulting Euler-Lagrange equations are

$$q : \frac{d}{d\tau} m \frac{\dot{q}}{\dot{t}} + \dot{t} \nabla_q V(q) = 0 \tag{1.2.4}$$

$$t : \frac{d}{d\tau} \left(-\frac{1}{2} m \left(\frac{\dot{q}}{\dot{t}} \right)^2 - V(q) \right) = 0 \tag{1.2.5}$$

The first expression is exactly the Newton law, while the second one is equivalent to the conservation of energy and then it is a consequence of the former equation. As expected, the relevant information is held by (1.2.4), as happens in the non-parameterised version, and the equation of motion respect to the new variable $t(\tau)$ has just a trivial contribution.
If we want to study the Hamiltonian of the parameterised system, we need to define the conjugate momenta p_q and p_t.

$$p_t = \frac{\partial L}{\partial \dot{t}} = -\frac{1}{2} m \left(\frac{\dot{q}}{\dot{t}} \right)^2 - V(q) \tag{1.2.6}$$

$$p_q = \frac{\partial L}{\partial \dot{q}} = m \frac{\dot{q}}{\dot{t}} \tag{1.2.7}$$

It's easy to state the canonical variables found in this way are not independent from each other: we can substitute the expression for p_q in the equation (1.2.6) and write a constraint over canonical variables

$$C(t, q, p_t, p_q) = p_t + \frac{p_q^2}{2m} + V(q) = 0 \tag{1.2.8}$$

This happens because only the rate $\frac{\dot{q}}{\dot{t}}$ appears in the definition of the momenta, thus the relations between momenta and velocities are not invertible. The constraint we wrote substantially corresponds to the natural request that the momentum conjugated to the time, that we expect to generate translations respect to this variable, is exactly the Hamiltonian of the unparameterised system, which generates time translations in that context. On the other hand, the Hamiltonian H, given by the Legendre transform of the Lagrangian,

$$H = \dot{q} p_q + \dot{t} p_t - L \tag{1.2.9}$$

is proportional to the constraint and then identically null on the space of physical states.
the result just obtained is not surprising, since this Hamiltonian generates the evolution of the system respect to the parameter τ, but the system itself is invariant under τ reparametrization. The real dynamics of the system is held by the relations between the variables q, t and their momenta, which is expressed by the constraint $C(t, q, p_t, p_q)$.
This procedure is very similar to what we do when we study the motion

of a particle in special relativity, since we define the proper time τ and we write the 4-vectors of position, velocity and momentum where the time t is treated as one extra coordinate. In this case the constraint is the well known relation $p^2 - m^2 = 0$ and the action is

$$S = m \int d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu} \tag{1.2.10}$$

In general relativity the variables we want to study is the metric field $g_{\mu\nu}$ that is "parameterised" by a particular choice of a set of coordinates, or map, on the manifold. Einstein general relativity is generally covariant, that means the physical results it predicts do not depend on these coordinates and then $g_{\mu\nu}(x^\rho)$ behaves in a way quite similar to $q(\tau)$ and $t(\tau)$ in this example.
It is also possible to study this problem from a more formal point of view, with the theory of Hamiltonian systems with constraints[9].
A Lagrangian $L(q_i, \dot{q}_i)$ is singular if

$$\det \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = 0 \tag{1.2.11}$$

The momenta conjugated to q_i are defined as $p_i(q_i, \dot{q}_i) = \frac{\partial L}{\partial \dot{q}_i}$ and this expression, together with the identity over the position space, can be seen as an operator that permits to pass from the Lagrangian coordinates $(q_i, \dot{q}_i)$ to the canonical set (q_i, p_i). An operator between two manifolds written in terms of local coordinates is locally invertible if the determinant of its Jacobian is not null. Hence, the direct consequence of a zero Hessian determinant of the Lagrangian is that this operator is not invertible and there is not a set of unique formulas $\dot{q}_i(q, p)$ which expresses the velocities in terms of canonical variables. The canonical variables are not independent of each others and there exists a set of primary constraints

$$C_m(q, p) = 0 \tag{1.2.12}$$

The Hamiltonian H given by $H = p \cdot \dot{q} - L$ is also not unique and can be replaced by any alternative Hamiltonian $\tilde{H}$

$$\tilde{H} = H + N_m C_m \tag{1.2.13}$$

The new Hamiltonian gives a new set of equations of motion

$$\dot{q}_i = \{q_i, \tilde{H}\} = \frac{\partial H}{\partial p_i} + N_m \frac{\partial C_m}{\partial p_i} \tag{1.2.14}$$

$$\dot{p}_i = \{p_i, \tilde{H}\} = -\frac{\partial H}{\partial q_i} + N_m \frac{\partial C_m}{\partial q_i} \tag{1.2.15}$$

which preserves the constraints. Obviously we expect the constraints to hold over time, hence we can impose

$$\dot{C}_m = \{C_m, \tilde{H}\} = 0 \tag{1.2.16}$$

Sometimes it is automatically true, other times this procedure permits to find some new secondary constraints to be added to the others.
All these considerations can be extended to field theories. Given a field ϕ_a, the Lagrangian density is $\mathcal{L}\left(\phi_a(x^\mu), \frac{\partial\phi}{\partial x^\mu}\right)$ and the action is the integral over the whole spacetime of this density:

$$S[\phi_a(x)] = \int d^4x\mathcal{L}\left(\phi_a(x), \frac{\partial\phi}{\partial x^\mu}\right) \tag{1.2.17}$$

The conjugated momenta respect to the fields ϕ_a are

$$\pi^a = \frac{\delta\mathcal{L}}{\delta(\partial_0\phi_a)} \tag{1.2.18}$$

while the total Hamiltonian is

$$H = \int d^3x\mathcal{H} = \int d^3x(\pi^a\partial_0\phi_a - \mathcal{L}) \tag{1.2.19}$$

Now the constraints will have the form

$$C_m = (\phi_a, \pi_a, \partial_i\phi_a, \partial_i\pi_a) \tag{1.2.20}$$

where Latin indices run over spatial dimensions, and the arbitrariness of the Hamiltonian density $\mathcal{H}$ can be written as

$$\tilde{\mathcal{H}} = \mathcal{H} + N(x)_m C_m \tag{1.2.21}$$

In the example used above and in GR, the theory is respectively invariant respect to τ reparametrization and diffeomorphisms, so the Hamiltonian is composed only by constraints:

$$H = N_m C_m = 0 \tag{1.2.22}$$

In a covariant theory where a particular momentum appears linearly, one can solve the constraint and obtain a non covariant canonical system[10]: if we have a constraint

$$C : p^i(\tau) = -\mathfrak{H}(p^{j\neq i}(\tau), q_k(\tau)) \tag{1.2.23}$$

we can transform the Lagrangian

$$L = p^j\dot{q}_j + NC = p^j\dot{q}_j \tag{1.2.24}$$

in the form

$$L = p^{j\neq i}\dot{q}_{j\neq i} - \mathfrak{H}\dot{q}_i \tag{1.2.25}$$

and change the integration variable of the action

$$S = \int d\tau p^j\dot{q}_j + NC = \int dq_i p^{j\neq i}\frac{dq_{j\neq i}}{dq_i} - \mathfrak{H} \tag{1.2.26}$$

In this way one can consider the variable q_i as the non covariant time t and $\mathfrak{H}$ becomes the new non null Hamiltonian which generates translations along t.
A similar procedure can be accomplished with field theories with linear constraints and covariant respect a set of coordinates χ^α. In this case we expect to find four constraints (one for each dimension of the spacetime)

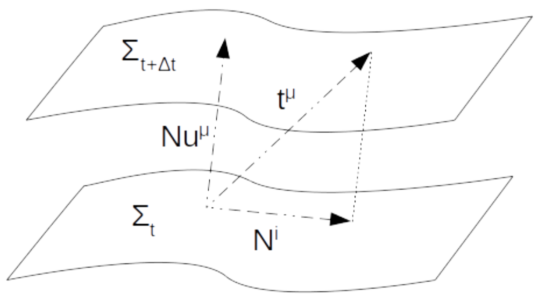
$$\pi^\mu(\chi^\alpha) = -\mathfrak{T}^{0\mu}(\pi^{j\neq\mu}(\chi^\alpha), \phi_i(\chi^\alpha)) \tag{1.2.27}$$

and we can write the action as

$$S = \int d^4\chi \pi^i \dot{\phi}_i + N^\mu C_\mu = \int d^4\phi \pi^{i\neq 0} \frac{d\phi_{i\neq 0}}{d\phi_0} - \mathfrak{T}^{00} \tag{1.2.28}$$

where the new set of proper coordinates x^μ is composed by the field components ϕ_μ and their associated components of the energy momentum tensor are $\mathfrak{T}^{0\mu}$. In particular $\mathfrak{T}^{00} = \mathfrak{H}$ is the non null Hamiltonian density.
However, as we will see in short time, in the GR case the constraint is of second order in all fields and momenta, then it can't be solved and it isn't possible to exclude all gauge degrees of freedom from the phase space in a trivial way.

1.3 The Hamiltonian formulation of gravity (ADM)



A rigorous characterization of energy in a canonical formulation of gravity was given by Arnowitt, Deser and Misner in [11].
The Lagrangian formulation of gravity is spacetime covariant but, in order to apply a canonical quantization, we have to use the Hamiltonian formalism, that requires a breakup of spacetime in space and time. Indeed, as a first step, we choose a time function t and a vector field t^ν such that $t^\nu \nabla_\nu t = -1$. Now it is possible to define a foliation of spatial surfaces Σ_t of constant t with a unit normal vector $u^\nu \propto \nabla^\nu t$. We also define the lapse function

$$N = -g_{\mu\nu} t^\mu u^\nu = (u^\nu \nabla_\nu t)^{-1} \tag{1.3.1}$$

that represents the flow of proper time τ (defined by $\nabla^\nu \tau = u^\nu$) respect to the coordinate t, and the shift vector

$$N^\nu = h^\nu{}_\mu t^\mu \tag{1.3.2}$$

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ is the spatial metric induced in Σ_t. N^ν represent the component of t^ν tangent to the spatial hypersurface. One can write the normal vector in term of N, N^ν and t^ν in the following way

$$u^\nu = \frac{1}{N}(t^\nu - N^\nu) \tag{1.3.3}$$

Hence,the inverse spacetime metric can be substituted with

$$g^{\mu\nu} = h^{\mu\nu} - u^\mu u^\nu = h^{\mu\nu} - N^{-2}(t^\mu - N^\mu)(t^\nu - N^\nu) \tag{1.3.4}$$

In the Hamiltonian theory we will take as field variables the spatial metric h_{ij}, the scalar lapse function N and the covariant vector field $N_i = h_{ij}N^j$. From the last equation we see that, once chosen t and t^ν, the information in $g^{\mu\nu}$, and consequently in $g_{\mu\nu}$, is totally contained in the set (h_{ij},N,N_i). If we choose a good basis of coordinates which has a temporal generator normal to the spatial surface, we will have the relation $\sqrt{-g} = N\sqrt{h}$, where g and h are the determinants of the respective metric tensors.
From the results obtained in the preceding section, we can take as the pure-gravitational Hilbert-Einstein action in a manifold M with boundary ∂M the expression [12]

$$S = \frac{1}{16\pi G}\left[\int_M d^4x\sqrt{-g}(R - 2\Lambda) + 2\int_{\partial M} d^3x\sqrt{\sigma}(K - K^0)\right] \tag{1.3.5}$$

where K_{ij} the extrinsic curvature of the hypersurface with $K = K^i{}_i$ and K^0 the extrinsic curvature of ∂M embedded in a flat space, that has been added in order to give zero action to the Minkowski spacetime for each boundary, when $\Lambda = 0$.
Now we have to express it in term of (h_{ij},N,N_i) and their space derivatives. Two key elements to complete this task are the extrinsic curvature of Σ_t, $K_{ij} = h_i{}^k \nabla_k u_j$ and the Riemann tensor of the 3-dimensional submanifold ${}^3R^k{}_{ilj}$ [7]. Let's consider a one form over Σ_t ω_i, the 3d Riemann tensor is defined as

$${}^3R^k{}_{ilj}\omega_k = D_l D_j \omega_i - D_j D_l \omega_i \tag{1.3.6}$$

where D_i represents the covariant derivative in the spatial manifold and is equivalent to apply the 4-dimensional covariant derivative and later contract each index with the spatial metric h_{ij}. for example the 3d derivative of a vector field is $D_i v^j = h_{i\mu} h_k^j \nabla^\mu v^k$. We can expand D in the left term and find

$$D_l D_j \omega_i = D_l(h_j{}^k h_i{}^n \nabla_k \omega_n) = h_j{}^m h_i{}^p h_l{}^r \nabla_r (h_m^k h_p{}^n \nabla_k \omega_n) \tag{1.3.7}$$

but, from the definition of the spatial induced metric,

$$h_j{}^m h_l{}^r \nabla_r h_m^k = h_j{}^m h_l{}^r \nabla_r (g_m^k + u_m u^k) = K_{lm} n^k \tag{1.3.8}$$

that means

$$D_l D_j \omega_i = h_j{}^k h_i{}^n h_l{}^r \nabla_r \nabla_k \omega_n + h_i{}^n K_{lj} u^k \nabla_k \omega_n + h_j{}^k u^n \nabla_k \omega_n \tag{1.3.9}$$

and

$${}^3R^k{}_{ilj} \omega_k = h_j{}^k h_i{}^n h_l{}^r R^m{}_{rkn} \omega_m - K_{ij} K_l{}^m \omega_m + K_{lj} K_i{}^m \omega_m \tag{1.3.10}$$

for each one form ω_i. So we can consider the last equation as a relation between operators

$${}^3R^k{}_{ilj} = h_j{}^k h_i{}^n h_l{}^r h_p{}^m R^p{}_{rkn} - K_{ij} K_l{}^m + K_{lj} K_i{}^m \tag{1.3.11}$$

It's easy to observe that the 4d Riemann tensor totally contracted with the spatial metric is

$$R_{\mu\nu\rho\sigma} h^{\mu\rho} h^{\nu\sigma} = R_{\mu\nu\rho\sigma} (g^{\mu\rho} + u^\mu u^\rho)(g^{\nu\sigma} + u^\nu u^\sigma) = 2R_{\mu\rho} u^\mu u^\rho + R = 2G_{\mu\rho} u^\mu u^\rho \tag{1.3.12}$$

Now, if we contract the expression for the three dimensional Riemann tensor (1.3.11) with two induced metrics, we have

$$G_{\mu\nu} u^\mu u^\nu = \frac{1}{2}[{}^3R - K_{ij} K^{ij} + K^2] \tag{1.3.13}$$

In order to write R with the Hamiltonian variables, we still need a formula for $R_{\mu\rho} u^\mu u^\rho$, that can be found in the following way:

$$\begin{aligned} R^\nu{}_{\mu\nu\rho} u^\mu u^\rho &= -u^\mu (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla \mu) u^\nu \\ &= (\nabla_\mu u^\mu)(\nabla_\nu u^\nu) - (\nabla_\mu u^\nu)(\nabla_\nu u^\mu) - \nabla_\mu (u^\mu \nabla_\nu u^\nu) + \nabla_\nu (u^\mu \nabla_\mu u^\nu) \end{aligned} \tag{1.3.14}$$

thanks to the Leibniz rule and applying the definition of the extrinsic curvature one obtains

$$R_{\mu\nu} u^\mu u^\nu = K^2 - K_{ij} K^{ij} - \nabla_\mu (u^\mu \nabla_\nu u^\nu) + \nabla_\nu (u^\mu \nabla_\mu u^\nu) \tag{1.3.15}$$

The last two terms on the right side are pure divergences that we will consider later in boundary terms. Then curvature scalar is

$$R = 2G_{\mu\rho} u^\mu u^\rho - 2R_{\mu\rho} u^\mu u^\rho \tag{1.3.16}$$

and the Lagrangian density is

$$\mathcal{L} = \frac{1}{16\pi G} \sqrt{h} N [{}^3R + K_{ij} K^{ij} - K^2 - 2\Lambda] \tag{1.3.17}$$

We can define a time derivative of h_{ij} as $\dot{h}_{\mu\nu} \equiv h_\mu{}^\rho h_\nu{}^\sigma \pounds_t h_{\rho\sigma}$, where $\pounds_t$ represents the Lie derivative along t^ν, and we can compute the momentum canonically conjugate to h_{ij}

$$\pi^{ij} = \frac{\delta\mathcal{L}}{\delta\dot{h}_{ij}} = \frac{1}{16\pi G}\sqrt{h}(K^{ij} - Kh^{ij}) \tag{1.3.18}$$

We immediately see $\mathcal{L}$ does not contain temporal derivatives of N and N^i, then their conjugate momenta are identically null and N and N^i are not dynamical variable, but parameters which do not influence the evolution of Σ_t. Thus,we finally write the Hamiltonian density

$$\mathcal{H} = \pi^{ij}\dot{h}_{ij} - \mathcal{L} = (NH - H_i N^i) \tag{1.3.19}$$

with

$$H = -\frac{1}{16\pi G}\sqrt{h}({}^3R - 2\Lambda) + 16\pi G(h^{-1/2}\pi_{ij}\pi^{ij} - 1/2h^{-1/2}\pi^2) \tag{1.3.20}$$

$$H^i = D_j\pi^{ij} \tag{1.3.21}$$

and $\pi = \pi_i{}^i$.
The configuration space has been reduced to the space of all Riemannian metrics on Σ_t. It can be useful at this point to define a metric on such space, usually called superspace, the Wheeler-DeWitt metric

$$G_{ijkl} = \frac{1}{2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \tag{1.3.22}$$

It has the relevant property to have an hyperbolic signature (1,5) at every point in the three-surface independently of the signature of spacetime. We can also define the inverse of the supermetric G^{ijkl}, which acts on the momenta, with

$$G^{ijnm}G_{nmkl} = \frac{1}{2}(\delta^i_k\delta^j_l + \delta^i_l\delta^j_k) \tag{1.3.23}$$

The last relation implies

$$G^{ijkl} = (h^{ik}h^{jl} + h^{il}h^{jk} - 2h^{ij}h^{kl}) \tag{1.3.24}$$

The absence of an evolution of N and N^i means they behave as a set of Lagrange multiplier and with the Hamilton equations they give the Hamilton constraint and the momentum constraint

$$H = -\frac{1}{16\pi G}\sqrt{h}({}^3R - 2\Lambda) + 16\pi G h^{-1/2}G_{ijkl}\pi^{ij}\pi^{kl} = 0 \tag{1.3.25}$$

$$H_i = 0 \tag{1.3.26}$$

On the other hand the Hamilton equations applied to h_{ij} and π^{ij} give

$$\dot{h}_{ij} = \frac{\delta \mathcal{H}}{\delta \pi^{ij}} = 2h^{-1/2}N\left(\pi_{ij} - \frac{1}{2}h_{ij}\pi\right) + 2D_{(i}N_{j)} \tag{1.3.27}$$

$$\begin{aligned}\dot{\pi}^{ij} = \frac{\delta \mathcal{H}}{\delta h_{ij}} &= -N\sqrt{h}\left({}^3R^{ij} - \frac{1}{2}{}^3Rh^{ij}\right) + \frac{1}{2}Nh^{-1/2}h^{ij}G_{ijkl}\pi^{ij}\pi^{kl} \\ &-2Nh^{-1/2}\left(\pi^{ik}\pi_k^{\ j} - \frac{1}{2}\pi\pi^{ij}\right) + \sqrt{h}(D^iD^jN - h^{ij}D^kD_kN) \\ &+\sqrt{h}D_k(h^{-1/2}N^k\pi^{ij} - 2\pi^{k(i}D_kN^{j)}\end{aligned} \tag{1.3.28}$$

One can verify equations from (1.3.25) to (1.3.28) are equivalent to the Einstein equation, thus we have an Hamiltonian formulation of Einstein's gravity. The presence of two constrains means there are still some gauge degrees of freedom in h_{ij}.
The momentum constraint is the easiest to manage, since it is linear in the momenta and belongs to the general case treated in the previous section. We can act in a similar way to what is usually done in Maxwell theory of electromagnetism. In that field theory the interesting field is the four-potential A_μ and the action is

$$S_{em}[A] = \frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu} \tag{1.3.29}$$

where $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ is the Maxwell field. One can immediately see from the last expression that, thanks to the antisymmetry of $F_{\mu\nu}$, the time derivative of A_0 does not appear in the action and then the zero component of the vector potential, usually called scalar potential V, is not a dynamical quantity, exactly as N and N^i in GR. So V can be arbitrarily fixed and it defines a constraint on the canonical variables of electromagnetism. After the usual decomposition of the Maxwell field in electric field $E_i = F_{0i}$ and magnetic field $B_i = -\frac{1}{2}\epsilon_{ijk}F_{jk}$, the action is

$$S_{em}[A_i, V, E_i] = -\int dt\int d^3x(E^i\dot{A}_i - (E^2 + B^2) + V\partial_iE^i) \tag{1.3.30}$$

and the momentum π^i associated to the vector potential A_i is

$$\pi^i = \frac{\delta L}{\delta \dot{A}_i} = -E^i \tag{1.3.31}$$

The constraint given by a variation respect to V is $\nabla\cdot E = 0$, the Gauss equation in vacuum, and is quite similar to the momentum constraint (1.3.26). In electromagnetism the constraint is strongly connected with the residual gauge freedom of the theory: We all know that the theory is invariant under transformations $A_i \to A_i + \partial_i\lambda$ where lambda is a scalar function. Such a transformation does not have any effects over the fields E^i and B_i, then the only contribution to the action that could change with this substitution is

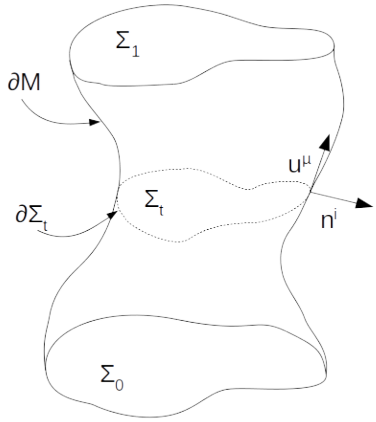
$$\int dt \int d^3x E^i \dot{A}_i \rightarrow \int dt \int d^3x E^i (\dot{A}_i + \partial_i \dot{\lambda}) \tag{1.3.32}$$

but we can integrate by part the spatial integral and obtain a total divergence null at infinity and the term $\dot{\lambda}\partial_i E^i$ that is equal to zero if the Gauss constraint is respected. The constraint permits us to restrict the phase space to the equivalence class of the vector field A_i configurations where two vector fields are equivalent if they differ only by a gauge transformation, i.e. by the gradient of a scalar function. In GR the physical system represented by a 3-metric is invariant under diffeomorphism and then $\mathcal{H}$ have to be invariant too. A more appropriate configuration space is the space of equivalence classes of metrics respect to diffeomorphic equivalence and the relative cotangent space π^{ij} is the space of linear functionals which act on metrics variations. Let's consider a transformation $x^i \rightarrow x^i + v^i$, the variations of the metric tensor change as $\delta h_{ij} + D_{(i}v_{j)}$ and, if π^{ij} acts on the space of equivalence classes, we must have

$$\int_M \pi^{ij}(\delta h_{ij} + D_{(i}v_{j)}) = \int_M \pi^{ij}\delta h_{ij} \tag{1.3.33}$$

that implies, in order to have a divergence on the left side, the momentum constraint (1.3.26).
In a similar way the Hamilton constraint represents the gauge arbitrariness in the choice of the function t and the corresponding spatial foliation Σ_t, but now the momenta π^{ij} appear quadratically in the constraint, so it isn't possible to deparametrize the theory by solving the constraint and restricting the phase space.
Thanks to the two constraints, it does not exist a nontrivial definition of energy in any manifold without boundary, since $\mathcal{H} = 0$ for each spacetime which is solution of the Einstein equation. We expected a similar behaviour from the example exposed above, because $\mathcal{H}$ generates t translations, but the theory is invariant under t reparametrization by construction.
It doesn't mean there is no dynamics in general relativity or time evolution is frozen: the dynamics of the system is given by the Wheeler-DeWitt equation and the evolution is held by the relative changes in the different components of h_{ij} and π^{ij} in the same way as, in the example of the previous section, the time evolution wasn't in the null Hamiltonian written for the arbitrary evolution parameter τ, but rather in the constraint C and in the relation between the physical variables q and t.

However boundary terms can give a meaningful energy in some particular cases, then now we will consider their contribution to the Hamiltonian. With a compact spacetime without spatial boundaries, the manifold can be represented by $M = I \times \Sigma_t$ and its boundaries are Σ_0 and Σ_1, if we take $I = [0, 1]$.



The boundary contributions in the action (1.3.5) are exactly nullified by the term $\nabla_\mu(u^\mu\nabla_\nu u^\nu)$ in eq. (1.3.15), since $g_\mu^{\ \nu}\nabla_\nu u^\mu = (h_\mu^{\ \nu} - u_\mu u^\nu)\nabla_\nu u^\mu = h_\mu^{\ \nu}\nabla_\nu u^\mu = K$, and, thanks to Stokes theorem, $\int_M \nabla_\mu(u^\mu K) = \int_{\partial M} u^\mu u_\mu K$. The argument of the divergence in the fourth term in the equation (1.3.15) is normal to u^ν, because $2u_\nu u^\mu\nabla_\mu u^\nu = u^\mu\nabla_\mu(u^\nu u_\nu) = 0$, so temporal boundaries do not give any contribution to the energy. But, if we have a spatial boundary ∂M, the foliation Σ_t can be naturally induced on it with $\partial M \cap \Sigma_t = \partial\Sigma_t$ and the boundary contribution to the Hamiltonian is

$$\int_{\partial\Sigma_t} d^2xN\sqrt{\sigma}\left[-2(\Theta - \Theta_0) + n_\mu(-u^\mu\nabla_\nu u^\nu + u^\nu\nabla_\nu u^\mu)\right] \tag{1.3.34}$$

where σ is the determinant of the induced metric of $\partial\Sigma_t$, Θ is the trace of its extrinsic curvature and n^μ is the unitary normal vector of the surface $\partial\Sigma_t$[13][14]. This expression can't be taken "as it is" to define an energy of the spacetime, since it directly depends on N, that is an arbitrary parameter. We define the quasi-local energy, or Brown-York energy [15], of the foliation the value of the Hamiltonian that generates unit time translations orthogonal to the spatial boundary, that means the Hamiltonian obtained with the parametrization that has $n \cdot u = 0$ and $|N| = 1$ on $\partial\Sigma_t$.
With this definition the boundary energy is reduced to

$$\int_{\partial\Sigma_t} d^2xN\sqrt{\sigma}\left[-2(\Theta - \Theta_0) + n_\mu u^\nu\nabla_\nu u^\mu\right] \tag{1.3.35}$$

If the spatial boundary do not changes over time, we can apply the Leibniz rule and write $n_\mu u^\nu\nabla_\nu u^\mu = u^\nu\nabla_\nu(n_\mu u^\mu) - u_\mu u^\nu\nabla_\nu n^\mu = 0$ and obtain an easier formula for the quasi-local energy

$$E_{ql} = -\frac{1}{8\pi G}\int_{\partial\Sigma_t} d^2x\sqrt{\sigma}(\Theta - \Theta_0) \tag{1.3.36}$$

A particular case is the asymptotically flat spacetime, where the boundary is taken in the limit $r \to \infty$, with r representing the radial coordinate. Near this boundary the metric is almost flat, then there is a parametrization t such that $\nabla_\mu u^\nu = 0$, $N = 1$ and $N^i = 0$. The result is

$$\begin{aligned} E_{ADM} &= - \lim_{r\to\infty} \frac{1}{8\pi G} \int_{\partial\Sigma_t} d^2x \sqrt{\sigma}(\Theta - \Theta_0) \\ &= \lim_{r\to\infty} \frac{1}{8\pi G} \int_{\partial\Sigma_t} d^2x r^i \left(\frac{\partial h^j{}_i}{\partial x^j} - \frac{\partial h^j{}_j}{\partial x^i} \right) \end{aligned} \tag{1.3.37}$$

and the total energy is $H_{tot} = \int_M \mathcal{H} + E_{ADM} = E_{ADM}$. This expression for energy in asymptotically flat spacetimes is usually called ADM energy. This means classical general relativity is a field theory where energy can be expressed as a surface term if a spatial boundary exists, otherwise energy is trivially zero.

1.4 Quantization

Canonical quantization

Now we want to proceed with the canonical quantization and introduce a wave functional defined on the superspace $\Psi[h_{ij}]$. In the canonical approach we will follow the technique developed by Wheeler and DeWitt [16]. The wave functional hasn't got a direct dependence on time t, since the intrinsic geometry of a particular submanifold Σ_t is enough to know its relative location in the spacetime. The usual substitutions in Dirac quantization give

$$h_{ij} |\Psi\rangle = h_{ij} \Psi[h_{ij}] \tag{1.4.1}$$

$$\pi^{ij} |\Psi\rangle = -i \frac{\delta}{\delta h_{ij}} \Psi[h_{ij}] \tag{1.4.2}$$

while the inner Hilbert product on Schrdinger functionals is

$$\langle \Psi | \Phi \rangle = \int \mathcal{D}[h_{ij}] \Psi^*[h_{ij}] \Phi[h_{ij}] \tag{1.4.3}$$

Considering (1.3.25) and (1.3.26) as operator equations would yield to no dynamics at all, so the classical constraints must be taken as requests of annihilation of the wave functional:

$$H_i |\Psi\rangle = 2iD_j \frac{\delta}{\delta h_{ij}} \Psi[h_{ij}] = 0 \tag{1.4.4}$$

$$H |\Psi\rangle = \left[-16\pi G h^{-1/2} G_{ijkl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} - \frac{1}{16\pi G} \sqrt{h} ({}^3R - 2\Lambda) \right] \Psi[h_{mn}] = 0 \tag{1.4.5}$$

Obviously the substitution of momenta with functional derivatives gives some ordering problem, since the supermetric depends on the field h_{ij}, but we will discuss them in a second moment. The Wheeler-DeWitt (WDW) equation (1.4.5) is a second order hyperbolic functional differential equation which describe the dynamical evolution of the wave functional. In general, there are many solution of the Wheeler-DeWitt equation, then only boundary conditions on the superspace permit to have some predictive power in cosmological models.
The invariance under 4-diffeomorphism of classical general relativity has as a result the constraints (1.3.25) (1.3.26) showed above. After quantization, the role of those equations is taken by the Wheeler-DeWitt equation. One of the consequences of the WDW equation is the so called "problem of time": while in other field theories, where time is an external parameter, one obtains a Schrdinger equation $\frac{\partial \Psi}{\partial t} = H\Psi$ that determines the time evolution of the wave functional, in the canonical quantization of general relativity time is just an arbitrary parametrization. How to obtain a well defined concept of time emerging from the classical limit is one of the great problems of quantum gravity theories.

Path integral quantization

An alternative quantization procedure is the path integral method and we will deal with it following the Hawking approach to the problem [17]. In the path integral approach to the quantization of a field ϕ, the amplitude to go from a field configuration ϕ_1 at time t_1 to a configuration ϕ_2 at time t_2 is expressed by

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \int D[\phi] e^{iS[\phi]} \tag{1.4.6}$$

where the integration is over all paths that begins in ϕ_1 at time t_1 and finish in the configuration ϕ_2 at time t_2. On the other hand, in Schrdinger representation,

$$\langle \phi_2, t_2 | \phi_1, t_1 \rangle = \langle \phi_2, t_2 | \, e^{-iH(t_2 - t_1)} \, | \phi_1, t_1 \rangle \tag{1.4.7}$$

After a Wick rotation with the substitution $t_1 - t_2 = i\beta$ we can compare the two expressions for the probability amplitude. If we take $\phi_2 = \phi_1$, i. e. we consider periodic paths with period β in imaginary time, and sum these amplitudes over a complete base of the Hilbert space $\{|\phi\rangle\}$, we obtain

$$Z = \operatorname{Tr} e^{-\beta H} = \int D[\phi] e^{-I[\phi]} \tag{1.4.8}$$

where I is the euclidean form of the action, which means the result of the Wick rotation of S. Thus,one has a partition function Z for the field ϕ in a similar way respect to a statistical ensemble. It is possible to evaluate

the moments of the distribution from it, which are exactly the n points propagators of the field.
Our aim is to apply this procedure to the 3 metric field h_{ij}, since, as showed above, it contains all the physical information of $g^{\mu\nu}$, but there are some difficulties. A path for the spatial submanifold is a family of surfaces Σ_t to varying of t, hence it corresponds to a configuration of $g^{\mu\nu}$. In order to fix the extremes of integration we have to consider only metrics of the spacetime which have as time boundaries the requested three surfaces and three metrics, but are free to vary off these surfaces. Actually, the only physical freedom is the specification of h_{ij} in the Σ_t surfaces which do not have extremal positions, then we will have to manage some gauge freedom in the integration. Consequently the quantum mechanical superposition of two configurations h_{ij} and h'_{ij} at the extrema of the paths appears as

$$\langle h'_{ij}|h_{ij}\rangle = \int D[g]e^{iS[g]+gauge\ fixing\ terms} \tag{1.4.9}$$

Now we would like to apply again a Wick rotation and define a partition function

$$Z = \int D[g]e^{-I[g]+gauge\ fixing\ terms} \tag{1.4.10}$$

Written with canonical variables, the Minkowskian gravitational action (1.3.5) is

$$S = \left[\int_M \pi^{ij}\dot{h}_{ij} - (NH - H_iN^i) + \frac{1}{8\pi G}\int_{\partial M} d^3x\sqrt{\sigma}(\Theta - \Theta^0)\right] \tag{1.4.11}$$

while the euclidean partition function takes the form

$$Z = \int D[N^i]D[N]D[h_{ij}]D[\pi_{ij}]\exp\left\{-\int_0^\beta dt\left[\int_{\Sigma_t} d^3x(i\pi^{ij}\dot{h}_{ij} + NH - H_iN^i)\right.\right.$$
$$\left.\left.-\frac{1}{8\pi G}\int_{\partial\Sigma_t} d^2x\sqrt{\sigma}(\Theta - \Theta^0)\right] + gauge\ fixing\ terms\right\} \tag{1.4.12}$$

where the integration over π_{ij} can be done explicitly, since it appears only quadratically in H. However the euclidean action is not positive definite as in the scalar field, or at least semi-definite, as in usual gauge theories (electromagnetism and Yang-Mills). Indeed, since the supermetric (1.3.22) that appears in H is hyperbolic (it has a negative signature respect to the trace component $\sqrt{h}$), the euclidean action is not bounded from below for each direction chosen in the Wick rotation. Hence,the path integral will not converge with paths over real euclidean metrics. Convergence is achieved only carrying out integrations over complex contours in the space of complex four-metrics. Moreover the integration result to be dependent of the chosen

contour and is not clear how to return to a Minkowski framework after the Wick rotation.
It is possible to define a wave functional of the spacetime with the help of functional integration [18]: once chosen an initial state $|\Psi_0\rangle = |\bar{h}_{ij}\rangle$, thanks to the transition amplitude (1.4.9), the functional $\Psi_0[h_{ij}]$ is given by

$$\Psi_0[h_{ij}] = \langle \bar{h}_{ij} | h_{ij} \rangle = \mathcal{N} \int D[g] e^{-I[g] + gauge\ fixing\ terms} \tag{1.4.13}$$

where $\mathcal{N}$ is a normalization and the integration is done over all 4-geometries with a final spacelike boundary with induced metric h_{ij} and an initial boundary with induced metric $\bar{h}_{ij}$. Obviously from this formula one can obtain also state functional with more complex initial condition, for example with a superposition of eigenstates of the operator $\hat{h}_{ij}$.
The wave functional obtained in this way is independent of the time parametrization t and then of the scalar field N. Thus,the variation respect to N at the ending boundary of the path integral must be null:

$$0 = \frac{\delta \Psi}{\delta N} = \mathcal{N} \int D[g] \frac{\delta I}{\delta N(t_f)} e^{-I[g] + gauge\ fixing\ terms} \tag{1.4.14}$$

The derivative of the action $\frac{\delta I}{\delta N(t_f)} = -H$ can be brought out of the integral after the canonical substitution $\pi^{ij} \to -i \frac{\delta}{\delta h_{ij}}$, since it depends only from the fixed boundary conditions. The result is exactly the WDW equation (1.4.5) $H\Psi[h_{ij}] = 0$ and an analogue argument can be applied to 3 dimensional diffeomorphisms on the spatial manifold, giving the momentum constraints $H^i \Psi[h_{ij}] = 0$. A rigorous demonstration of the consistence of canonical quantum constraints with the path integral wave function, which take account of possible variations of the integration measures and ghosts, can be found in [19].
The choice of an integration path and an initial state, i. e. the the choice of class of paths over which the integration is done, has a role comparable to the choice of boundary condition in the research of solutions of the WDW equation.

Chapter 2

Approximations and perturbative methods

2.1 Minisuperspace

The superspace, that represents the configuration space of Hamiltonian gravity, is infinite dimensional, then practical calculations on it result very difficult. A common method to resolve this problem is the minisuperspace[12]: in minisuperspace framework almost all the degrees of freedom of superspace are frozen, except of one or two parameters. This machinery permits to represents spacetimes with strong symmetries (for example homogeneous or spherical symmetric manifolds) and to reduce the field theory of gravity to a simpler problem of quantum mechanics.
Clearly the minisuperspace is not a rigorous approximation of the full theory since many criticalities emerge in the freezing process: the request to set identically to zero most of the field modes and its conjugated momenta violates the uncertainty principle and the interferences of modes in the full theory could bring to results far from the minisuperspace model predictions. Anyway minisuperspace can be seen as a toy model useful in order to predict some general behaviour and can be used as a base for a more complete perturbative theory. In this Chapter we will momentarily ignore the $16\pi G$ factors in order to simplify the expressions. we will gradually reintroduce them in the second part, with semiclassical approximations.
In a minisuperspace model one chooses a finite set of variables $\{q^\alpha\}$, which completely fix an highly symmetric spacetime, and evaluates the metric f^{ab} induced on the minisuperspace by the supermetric G^{ijkl}. For example, in a spatially homogeneous spacetime

$$ds^2 = -N^2(t)dt^2 + a^2(t)d\Omega_3^2 \tag{2.1.1}$$

a good choice of variable could be the conformal factor $a(t)$, to be accompanied by $N(t)$ and $N^i = 0$. It is important to notice that the metric f^{ab},

exactly as the supermetric, can be not positive semi-definite (this surely happens in this example, since the variable a of the minisuperspace is the conformal factor of the spacelike hypersurface, that correspond to the negative mode of G^{ijkl}).
One way or another, one can always obtain an action of the form

$$S[q^\alpha(t), N(t)] = \int_0^1 dtN\left[\frac{1}{2N^2}f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta - U(q)\right] = \int Ldt \qquad (2.1.2)$$

Such an action has the same form of the action of a relativistic particle in a curved spacetime and then, with a variation respect of q^α, gives an equation of motion that describes a geodesic time evolution with a forcing term

$$\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{q}^\alpha}{N}\right) + \frac{1}{N^2}\Gamma^\alpha{}_{\beta\gamma}\dot{q}^\beta\dot{q}^\gamma + f^{\alpha\beta}\frac{\partial U}{\partial q^\beta} = 0 \qquad (2.1.3)$$

where $\Gamma^\alpha{}_{\beta\gamma}$ is the connection respect to the metric $f_{\alpha\beta}$. A variation respect to the parameter N permits to write the constraint

$$\frac{1}{2N^2}f_{\alpha\beta}(q)\dot{q}^\alpha\dot{q}^\beta + U(q) = 0 \qquad (2.1.4)$$

These equations clearly have to be consistent with the Einstein equation, that means a solution (q^α, N), once inserted in the original metric, must give

$$R_{\mu\nu}(q^\alpha, N) + \left[\Lambda - \frac{1}{2}R(q^\alpha, N)\right]g_{\mu\nu}(q^\alpha, N) = 0 \qquad (2.1.5)$$

This is not always guaranteed, hence only the cases when the equivalence is true can be treated with minisuperspace.
Using canonical variables, can find as usual the momenta $p_\alpha = f_{\alpha\beta}\frac{\dot{q}^\beta}{N}$ and the Hamiltonian

$$\mathcal{H} = N\left[\frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q)\right] = NH \qquad (2.1.6)$$

The equations of motion are

$$\dot{q}^\alpha = N\{q^\alpha, H\} = \frac{\partial H}{\partial p_\alpha},\ \dot{p}_\alpha = N\{p_\alpha, H\} = -\frac{\partial H}{\partial q^\alpha} \qquad (2.1.7)$$

and the Hamiltonian constraint given by variations respect to N

$$H = \frac{1}{2}f^{\alpha\beta}p_\alpha p_\beta + U(q) = 0 \qquad (2.1.8)$$

grants again the invariance of the theory under time reparametrizations. In fact, given a transformation $t \to t + \epsilon(t)$, we have

$$\delta q^\alpha = \epsilon(t)\{q^\alpha, H\},\ \delta p_\alpha = \epsilon(t)\{p_\alpha, H\},\ \delta N = \dot{\epsilon}(t) \qquad (2.1.9)$$

and the action variation

$$\delta S = \left[\epsilon(t)\left(p_\alpha \frac{\partial H}{\partial p_\alpha} - H\right)\right]_0^1 \tag{2.1.10}$$

vanishes if and only if $\epsilon(0) = \epsilon(1) = 0$, since the constraint is quadratic in momenta.
In the case of the example (2.1.1), the minisuperspace action is

$$\begin{aligned} S[a(t), N(t)] &= \int_0^1 dtN\left[-\frac{6a}{N^2}\dot{a} + (6k - a^2 2\Lambda)a\right] = \\ &= \int_0^1 dtN\left[-\frac{1}{24a}p^2 + (6k - a^2 2\Lambda)a\right] \end{aligned} \tag{2.1.11}$$

where the momentum p is $p = -12\frac{a\dot{a}}{N}$ and k is respectively equal to 1, 0 or -1 if the spatially homogeneous manifold is spherical, flat or hyperbolic. The Hamiltonian constraint descending from this action is

$$\frac{1}{24a}p^2 + (6k - a^2 2\Lambda)a = 0 \tag{2.1.12}$$

By means of canonical quantization, it is possible to define a time independent wave function $\Psi(q^\alpha)$ and write a minisuperspace Wheeler-DeWitt equation

$$H(q^\alpha, -i\frac{\partial}{\partial q^\alpha})\Psi(q^\alpha) = 0 \tag{2.1.13}$$

Since the metric $f^{\alpha\beta}$ depends on q^α, there can be some ordering issues in the WDW equation.
Bringing on the spatially homogeneous example, one finds a continuous family of Wheeler-DeWitt equations [20]

$$\left[\frac{1}{24}\frac{1}{a^i}\frac{\partial}{\partial a}\frac{1}{a^j}\frac{\partial}{\partial a}\frac{1}{a^k} - (6k - a^2 2\Lambda)a\right]\Psi(a) = 0 \tag{2.1.14}$$

depending from the real parameters i, j, k which respects the constraint $i + j + k = 1$. They represent the arbitrariness in the ordering of the operator $\frac{1}{a}$ and the derivatives respect to a. If we want to exclude part of the ordering dependence from the differential problem, we can redefine the wave function as $\Psi(a) = a^{1+\frac{k-i}{2}}\Phi(b)$ with $b = 6a^2$, thus the equation to solve is

$$\frac{d^2\Phi}{db^2} + \frac{1}{b}\frac{d\Phi}{db} - \left(k - \frac{b}{18}\Lambda + \frac{1}{16b^2}(j+1)^2\right)\Phi = 0 \tag{2.1.15}$$

where now only the parameter j appears. However, as we will see in the next section, the effect of ordering is impossible to resolve in a semiclassical limit.

The minisuperspace can be quantized also with the path integral formalism. We still have a residual invariance under reparametrizations of $N(t)$, so we have to impose a gauge fixing condition that satisfies the following restrictions[21]:

- it must completely fix the gauge arbitrariness, that means there must be only one point of intersection between the surface identified in the phase space by the constraint and the submanifold characterised by the gauge fixing condition. This condition can be represented in some cases by the relation $\delta\chi = N\{\chi, H\} \neq 0$, since it means that a transformation along the constraint surface does not preserve the gauge fixing condition. However, if the hypersurface determined by the constraint is topologically non trivial, it can be more difficult to find a suitable gauge fixing that respect this request. It is called the Gribov problem.

- the gauge condition must be reachable by any path via some gauge transformations which do not change the action

In this case a good gauge-fixing has the form

$$\chi = \dot{N} - \chi^*(p, q, N) = 0 \tag{2.1.16}$$

It can be shown with the Fradkin-Vilkovisky theorem[22][23] that the path integral with such a gauge fixing is independent of χ^* and its total action is

$$S = \int_0^1 dt[p_\alpha q^\alpha - NH + \Pi(\dot{N} - \chi^*)] + S_{gh} \tag{2.1.17}$$

where $\Pi = \Pi(t)$ is a Lagrange multiplier associated to the gauge fixing and S_{gh} is the ghosts' contribution to the action. If we chose the gauge $\dot{N} = 0$, the ghost fields decouple from the physical quantities, then we obtain only a constant Faddev Popov determinant, and we find again the equations of motion of the theory without gauge fixing (2.1.7). Thus, the path integral formula for the wave function

$$\Psi(\bar{q}^\alpha) = \int Dp_\alpha Dq^\alpha DN e^{iS[p,q,N]} \tag{2.1.18}$$

where $\bar{q}^\alpha$ sets the boundary condition $q^\alpha(1) = \bar{q}^\alpha$ on the possible paths, is reduced to

$$\Psi(\bar{q}^\alpha) = \int dN \int Dp_\alpha Dq^\alpha e^{iS[p,q,N]} \tag{2.1.19}$$

This happens because with $\chi^* = 0$ the functional integration over Π forces N to be constant along the path integral and its functional integral is reduced to a Riemannian integration. A relevant question about this equation is the interval of integration of N. If we rename

$$\psi(\bar{q}^\alpha, N) = \int Dp_\alpha Dq^\alpha e^{iS[p,q,N]} \tag{2.1.20}$$

the wave function ψ is solution of a Schrdinger equation respect to N and the Hamiltonian H: with change of variable $dtN = dN$ the wave functional is

$$\psi(\bar{q}^\alpha, N) = \int Dp_\alpha Dq^\alpha \exp\left[i\int_0^N dN'\,(p_\alpha \dot{q}^\alpha - H)\right] \tag{2.1.21}$$

and it automatically solves the equation

$$i\frac{\partial\psi}{\partial N} = H\psi \tag{2.1.22}$$

Hence,the Hamiltonian constraint over the total wave function (2.1.13) takes the form

$$H\Psi = \int dN H\psi = \int dN i\frac{\partial\psi}{\partial N} = i\psi|_{N_2}^{N_2} \tag{2.1.23}$$

with N_1 and N_2 representing the extrema of integration. Clearly the WDW constraint is respected only if the right hand side of the last equation is null and the most used solutions are to take the N integration over a complex closed contour or over a path from $-\infty$ to $+\infty$ with $\psi \to 0$ when $|N| \to \infty$. Also in this case one can apply a Wick rotation and use the euclidean time $\beta = it$. After integrating out the momenta, the Euclidean functional integral is

$$\Psi(\bar{q}^\alpha) = \int dN \int Dq^\alpha e^{-I[q^\alpha(\beta),N]} \tag{2.1.24}$$

with

$$I[q^\alpha(\beta), N] = \int_0^1 d\beta N\left[\frac{1}{2N^2} f_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta + U(q)\right] \tag{2.1.25}$$

We can define an Euclidean partition function for the minisuperspace

$$Z = \int dN \int Dq^\alpha e^{-I[q^\alpha(\beta),N]} \tag{2.1.26}$$

where the integration is carried over periodic paths.
Again the indefiniteness of the metric used in the action forces us to use complex integration contours in order to obtain a meaningful path integral.

2.2 Semiclassical approximations

In order to make real computations it is common to use semiclassical limits as the WKB approximation or the equivalent steepest descent method in the path integral formulation.
In the WKB approximation we suppose to have a wave function with the form $\Psi = Ce^{iS}$, where S, at the moment, is a generic function of the variables q^α and C variate slowly respect to S with q^α. If we want to make a

semiclassical approximation, we need a scale parameter that permits to separate classical from quantum phenomena. In this case we choose $m_p = G^{-1/2}$ as a large parameter and we reinsert it in the WDW equation, obtaining

$$\left[-\frac{1}{2m_p^2}\nabla^2 + m_p^2 U(q)\right] C(q)e^{iS} = 0 \tag{2.2.1}$$

where ∇ represent the covariant derivation respect to q^α with the metric $f^{\alpha\beta}$. Then we expand the function S as $S = S_0 m_p^2 + S_1 + ...$ and we equate different orders of m_p in eq (2.2.1). The leading one is

$$\frac{1}{2}(\nabla S_0)^2 + U(q) = 0 \tag{2.2.2}$$

that is the Lorentzian Hamilton-Jacobi equation and can be used in order to find in another way the classical equations of motion, as we will show below.
A canonical transformation $(p, q) \to (P, Q)$ with a second type generating function $G_0(q, P)$ fulfils the classical relations

$$P = \frac{\partial G_0}{\partial q},\ Q = \frac{\partial G_0}{\partial P} \tag{2.2.3}$$

In quantum mechanics, the equivalent transformation of the wave function $\psi(q)$, is given by

$$\Psi(P) = \int dq e^{-iG(q,P)}\psi(q) \tag{2.2.4}$$

where $G = G_0$ in a first order approximation in Plank's constant. If we take a WKB wave function and we look for a transformation

$$P = p - \frac{\partial S}{\partial q},\ Q = q \tag{2.2.5}$$

a good generator is $G_0(q, P) = qP + S(q)$. The transformed wave function result to be $\Psi(P) = \delta(P)$ to leading order, thus we can consider the equation

$$p = m_p^2 \frac{\partial S_0}{\partial q} \tag{2.2.6}$$

as a strong correlation between momenta and coordinates respected by wave functions generated by S_0. Moreover, if one consider the ordering arbitrariness in eq (2.2.1), will immediately notice that all the emerging therms will be of order $O(m_p^{-1})$ or lower, because the only source of m_p factors is the derivative of the wave function Ψ. That means at leading order the derivatives have to act on Ψ and the possible q-depending terms can be set without problems to the left side of derivatives. However, if one desires to make a more accurate computation, must consider also higher orders where ordering choices have tangible effects.

At the moment, we do not have a good definition of classical time in our quantum theory, however we can take the affine parameter of the evolution of the function S_0 as a proper time emerging in the semiclassical limit. This is a meaningful choice, since if we take

$$\frac{d}{ds} = f^{\alpha\beta}\frac{\partial S_0}{\partial q^\alpha}\frac{\partial}{\partial q^\beta} \tag{2.2.7}$$

where s is the proper time defined by $ds = Ndt$, we find the canonical relation between momenta and velocities $p_\alpha = f_{\alpha\beta}\frac{dq^\beta}{ds}$.
After a differentiation respect to q^γ of (2.2.2), one obtains

$$\frac{1}{2}\frac{\partial f^{\alpha\beta}}{\partial q_\gamma}\frac{\partial S_0}{\partial q^\alpha}\frac{\partial S_0}{\partial q^\beta} + f^{\alpha\beta}\frac{\partial S_0}{\partial q^\alpha}\frac{\partial^2 S_0}{\partial q^\beta \partial q^\gamma} + \frac{\partial U}{\partial q^\gamma} = 0 \tag{2.2.8}$$

and, by plugging in p_α and ds,

$$\frac{dp_\gamma}{ds} + \frac{1}{2m_p^2}\frac{\partial f^{\alpha\beta}}{\partial q_\gamma}p^\alpha p^\beta + m_p^2\frac{\partial U}{\partial q^\gamma} = 0 \tag{2.2.9}$$

that is exactly eq (2.1.3), while (2.1.4) is already implied by the Hamilton-Jacobi equation.
Hence,S_0 is the classical Hamilton's principal function, since it gives the classical equations of motion, and e^{iS_0} is a superposition of a set of classical solutions of the spacetime evolution around which the WKB wave function is peaked. Moreover S_0 itself permits to define a semiclassical notion of time.
We want to apply the WKB formalism to the minisuperspace example considered previously (2.1.1), so we obtain the equation

$$\frac{1}{24a}\left(\frac{\partial S_0}{\partial a}\right)^2 + (6k - a^2 2\Lambda)a = 0 \tag{2.2.10}$$

In the regions classically allowed with $aU(a) = (6k - a^2 2\Lambda)a^2 < 0$ we find two oscillatory solutions

$$\Psi_\pm^{(1)}(a) = \exp\left[\pm i\int_{a_0}^{a} da' a'\sqrt{-(6k - a'^2 2\Lambda)} \mp \frac{i\pi}{4}\right] \tag{2.2.11}$$

where a_0 is the point where the allowed region begins, i.e. $6k - a^2 2\Lambda = 0$. On the other hand, when $aU(a) > 0$, we have quantum tunnelling and an exponential behaviour

$$\Psi_\pm^{(2)}(a) = \exp\left(\pm\int_{a}^{a_0} da' a'\sqrt{6k - a'^2 2\Lambda}\right) \tag{2.2.12}$$

At this point boundary conditions are fundamental in choosing the right combination of solutions of the WKB form of the WDW equation that grants

continuity of the wave function in a_0. The semiclassical time evolution of the wave function can be computed with the derivative just defined and, for example, one can find that the solution $\Psi_+^{(1)}(a)$ describes a contracting universe, while $\Psi_-^{(1)}(a)$ is an expanding spacetime[24].
The method equivalent to the WKB approximation in the path integral quantization is the steepest descent technique. In this case one considers the path integral wave function (2.1.24) and notes that classical solutions of Euclidean gravity are minima of the action I, then they have more relevance in the integration. The result, also in this case, is a wave function peaked near to classical solutions of Euclidean gravity. We can expand the wave function near a solution I_{cl} and write

$$\Psi(\bar{q}^\alpha) = e^{-I_{cl}} \int dN \int D\tilde{q}^\alpha e^{-I_2[\tilde{q}^\alpha, N]} \tag{2.2.13}$$

where $\tilde{q}^\alpha$ is the difference $\tilde{q}^\alpha = q^\alpha - q^\alpha_{cl}$ and I_2 is the second variation of the action. In the same way the leading order of the partition function is

$$Z = e^{-I_{cl}} \int dN \int D\tilde{q}^\alpha e^{-I_2[\tilde{q}^\alpha, N]} \tag{2.2.14}$$

However the indefiniteness of the metric introduces some trouble, since these solutions are just relative minima and not absolute. We will study the effects of this problem later.

2.3 Inhomogeneous perturbations of the metric

Our aim is to treat a complete theory of quantum gravity, then, once found a semiclassical limit in the minisuperspace toy model, we try to return to the full theory in a perturbative way [12][25]. We take a WKB state $e^{i(m_p^2 S_0 + S_1)}$ and we call $\bar{g}_{ij}(q^\alpha)$ the classical solution of the equation of motion of the action S_0, while the field $g_{ij} = \bar{g}_{ij} + h_{ij}$ will be the complete spatial metric. We consider some Inhomogeneous perturbations h_{ij} of the three metric near to the minisuperspace model, hence the variations of action and Hamiltonian are

$$S[g_{ij}, N] = S_0[\bar{g}_{ij}(q^\alpha), N] + S_2[q^\alpha, h_{ij}, N] \tag{2.3.1}$$

$$\mathcal{H} = N\left(H_0 + H_2\right) + N^i H_i \tag{2.3.2}$$

The resulting Wheeler-DeWitt equation is

$$\left[-\frac{1}{2m_p^2}\nabla^2 + m_p^2 U(q) + \int d^3x H_2\right]\Psi[q^\alpha, h_{ij}] = 0 \tag{2.3.3}$$

where the operator ∇ acts only on q^α, not on perturbations. If we consider solution of the form

$$\Psi[q^\alpha, h_{ij}] = e^{i(m_p^2 S_0 + S_1)}\psi[q^\alpha, h_{ij}] \tag{2.3.4}$$

At the leading order we clearly obtain again the Hamilton-Jacobi equation (2.2.2) for S_0 and we can define the semiclassical time $\frac{\partial}{\partial t} = \nabla S_0 \cdot \nabla$. At the next to leading order we have

$$\psi \left[\nabla S_0 \cdot \nabla (S_1 + ordering) - \frac{i}{2} \nabla^2 S_0 \right] = i \nabla S_0 \cdot \nabla \psi - H_2 \psi \qquad (2.3.5)$$

where we have included also the possible contributions from the ordering arbitrariness. Given an Hilbert product (1.4.3) on the space of wave functionals ψ, we set $\langle \psi[h_{ij}] | \psi[h_{ij}] \rangle = 1$ and consequently $\frac{d}{dt} \langle \psi | \psi \rangle = 0$. That means $\left\langle i \frac{\partial \psi}{\partial t} \middle| \psi \right\rangle = \left\langle \psi \middle| i \frac{\partial \psi}{\partial t} \right\rangle$, hence the last product is real. Thus,an inner product with $|\psi\rangle$ of (2.3.5), shows that the left hand side of the WDW equation have to be real, since H_2 is Hermitian. The result is the condition

$$\nabla S_0 \cdot \nabla [\mathrm{Im}(S_1 + ordering)] - \frac{1}{2} \nabla^2 S_0 = 0 \qquad (2.3.6)$$

because S_0 is the classical Hilbert-Einstein action and then it is real. Finally the WDW equation becomes, after a redefinition $\psi = e^{i\,\mathrm{Re}(S_1 + ordering)} \psi$,

$$i \frac{\partial \psi}{\partial t} = \int d^3 x H_2 \psi \qquad (2.3.7)$$

that is an emerging nontrivial Schrdinger equation. Moreover the wave functional has taken the form

$$\Psi[q^\alpha, h_{ij}] = C(q) e^{i S_0 m_p^2} \psi[q^\alpha, h_{ij}] \qquad (2.3.8)$$

where $C(q)$ is the usual WKB prefactor that now contains the term $e^{i\,\mathrm{Re}\,S_1 - \mathrm{Im}\, ordering}$ which varies slowly respect to $e^{i S_0 m_p^2}$ with q. That means the Wheeler-DeWitt equation reduces in a semiclassical limit to a quantum field theory for the fluctuations h_{ij} in a classical fixed background $\bar{g}_{ij}$.
In the path integral computation, the same perturbations h_{ij} give a partition function

$$Z = e^{-I_{cl}[\bar{g}_{ij}, N]} \int dN \int D h_{ij} e^{-I_2[h_{ij}, N]} \qquad (2.3.9)$$

Actually there can be more than one classical solution of the equation of motion, given a set of boundary conditions. In this case the complete partition function will be the sum of the contribution given by the saddle point expansions made near to different classical spacetimes (or instantons). Asking which of these instantons is more relevant corresponds to searching the best ground state for a quantum theory of gravity.

Variations respect to the metric

To make something a bit more practical, we have to compute the second order variations of booth the Hamiltonian and the action. They are really

strongly correlated, then we will find the second order variation of the action in a covariant spacetime, and only in a second time we will introduce the foliation and obtain the variation of the Hamiltonian function. As stated in the first section, the relation $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$ brings some complications in variational calculus respect to the metric tensor. In this section we will distinguish the variation of the inverse metric $\delta g^{\mu\nu} = \tilde{h}^{\mu\nu}$ from the perturbation of $g_{\mu\nu}$, that we will call $h_{\mu\nu}$, with indices risen by $\bar{g}^{\mu\nu}$, $h^{\mu\nu}$. The requirement of a Kronecker delta implies the transformation $\tilde{h}^{\mu\nu} = -\bar{g}^{\rho\mu}\bar{g}^{\sigma\nu}h_{\rho\sigma} = -h^{\mu\nu}$. In a similar way $h = \bar{g}^{\mu\nu}h_{\mu\nu} = -\bar{g}_{\mu\nu}\tilde{h}^{\mu\nu}$.
Moreover we will consider only variations of the metric $h_{\mu\nu}$ which are null on the boundary and also have null normal derivatives there. This will permit us to ignore the boundary terms. Thus, in spite of the complete action (1.3.5), we will variate only the euclidean Lagrangian density

$$-\frac{1}{16\pi G}\sqrt{g}(R - 2\Lambda) \tag{2.3.10}$$

Where the prefactor $16\pi G$ has been reintroduced. At the first order we obviously obtain an expression similar to the Einstein equation

$$I_1 = \frac{1}{16\pi G}\int_M d^4x\sqrt{\bar{g}}[R_{\rho\sigma}\bar{g}^{\mu\rho}\bar{g}^{\sigma\nu} - \frac{1}{2}\bar{g}^{\mu\nu}R + \Lambda\bar{g}^{\mu\nu}]h_{\mu\nu} \tag{2.3.11}$$

The second variation will be

$$I_2 = \frac{1}{2}\frac{\delta I_1}{\delta\bar{g}_{\rho\sigma}}h_{\rho\sigma} \tag{2.3.12}$$

From [7] we know

$$\delta\sqrt{g} = \frac{1}{2}\sqrt{g}g^{\mu\nu}h_{\mu\nu} \tag{2.3.13}$$

and

$$\delta R_{\mu\nu} = -\frac{1}{2}\nabla_\mu\nabla_\nu h - \frac{1}{2}\nabla^\rho\nabla_\rho h_{\mu\nu} + \nabla^\rho\nabla_{(\mu}h_{\nu)\rho} \tag{2.3.14}$$

hence

$$\begin{aligned} I_2 = \frac{1}{32\pi G}\int_M d^4x\sqrt{g}\Big[&\frac{1}{2}(hR^{\mu\nu}h_{\mu\nu} - \frac{1}{2}Rh^2 + \Lambda h^2) - 2R^{\mu}{}_{\sigma}h^{\sigma\nu}h_{\mu\nu} + \\ &+h^{\mu\nu}(-\frac{1}{2}\nabla_\mu\nabla_\nu h - \frac{1}{2}\nabla^\rho\nabla_\rho h_{\mu\nu} + \nabla^\rho\nabla_\mu h_{\nu\rho}) + \frac{1}{2}Rh_{\mu\nu}h^{\mu\nu} + \frac{1}{2}hR^{\mu\nu}h_{\mu\nu} \\ &+\frac{1}{2}h\nabla^\rho\nabla_\rho h - \frac{1}{2}h\nabla^\mu\nabla^\nu h_{\mu\nu} - \Lambda h^{\mu\nu}h_{\mu\nu}\Big] \end{aligned} \tag{2.3.15}$$

It can be put in a more familiar form with the definition of the Riemann tensor

$$(\nabla_\mu\nabla_\gamma - \nabla_\gamma\nabla_\mu)h_{\alpha\nu} = R_{\mu\gamma\alpha}{}^{\rho}h_{\rho\nu} + R_{\mu\gamma\nu}{}^{\rho}h_{\alpha\rho} \tag{2.3.16}$$

that implies, after a contraction with $\bar{g}^{\alpha\gamma}$,

$$\nabla^\alpha \nabla_\mu h_{\alpha\nu} = \nabla_\mu \nabla^\alpha h_{\alpha\nu} + R_\mu{}^\rho h_{\rho\nu} - R_\mu{}^\alpha{}_\nu{}^\rho h_{\alpha\rho} \tag{2.3.17}$$

Consequently we can define the Lichnerowicz operator

$$\Delta_L^{\mu\rho\nu\sigma} = -\nabla^\gamma \nabla_\gamma \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} - 2R^{\mu\rho\nu\sigma} + R^{\mu\rho} \bar{g}^{\nu\sigma} + R^{\mu\sigma} \bar{g}^{\nu\rho} \tag{2.3.18}$$

and obtain

$$I_2 = \frac{1}{16\pi G} \int_M d^4x \sqrt{\bar{g}} \left[\frac{1}{4} h_{\mu\nu} \Delta_L^{\mu\rho\nu\sigma} h_{\rho\sigma} - R^\mu{}_\sigma h^{\sigma\nu} h_{\mu\nu} + \frac{1}{2} h^{\mu\nu} \nabla_\mu \nabla^\rho h_{\nu\rho} + \right. \\ + \frac{1}{2} h R^{\mu\nu} h_{\mu\nu} - \frac{1}{8} R h^2 + \frac{1}{4} \Lambda h^2 - \frac{1}{2} h \nabla^\mu \nabla^\nu h_{\mu\nu} - \frac{1}{2} \Lambda h^{\mu\nu} h_{\mu\nu} + \\ \left. + \frac{1}{4} R h_{\mu\nu} h^{\mu\nu} + \frac{1}{4} h \nabla^\rho \nabla_\rho h \right] \tag{2.3.19}$$

On the spatial manifold

In a 3+1 dimensional spacetime the Lagrangian density has the form (1.3.17), where we can substitute

$$K_{ij} = \frac{1}{2N} (i\dot{g}_{ij} - D_i N_j - D_j N_i) \tag{2.3.20}$$

where the imaginary unit i comes from the Wick rotation. A good choice is taking $N^i = 0$, that gives

$$\mathcal{L}_E = \frac{1}{16\pi G} \sqrt{{}^3g} N [-{}^3R + \frac{1}{4N^2} \dot{g}_{ij} \dot{g}^{ij} - \frac{1}{4} \dot{g}^2 + 2\Lambda] \tag{2.3.21}$$

with $\dot{g}$ representing the trace $\dot{g}_{ij} g^{ij}$ and ${}^3g$ being the determinant of the spatial metric g_{ij}. The second variation of the new part containing time derivatives of the metric is

$$\frac{\sqrt{{}^3\bar{g}}}{N} \left[\frac{1}{4} h^{ij} \frac{\partial^2 h_{ij}}{\partial t^2} - \frac{1}{4} h \frac{\partial^2 h}{\partial t^2} \right] \tag{2.3.22}$$

if the background metric is static $\dot{\bar{g}}_{ij} = 0$. Hence,the variation of the action after a space-time separation is

$$I_2 = \frac{1}{16\pi G} \int_t dt \int_{\Sigma_t} d^3x \sqrt{{}^3\bar{g}} N \left[\frac{1}{4} h_{ij} \Delta_L^{ikjl} h_{kl} - R^j{}_k h^{ki} h_{ij} + \frac{1}{2} h^{ij} \nabla_i \nabla^k h_{jk} + \right. \\ + \frac{1}{2} h R^{ij} h_{ij} - \frac{1}{8} {}^3R h^2 + \frac{1}{4} \Lambda h^2 - \frac{1}{2} h \nabla^i \nabla^j h_{ij} - \frac{1}{2} \Lambda h^{ij} h_{ij} + \frac{1}{4} {}^3R h_{ij} h^{ij} + \\ \left. + \frac{1}{4} h \nabla^k \nabla_k h - \frac{1}{4N^2} h^{ij} \frac{\partial^2 h_{ij}}{\partial t^2} + \frac{1}{4N^2} h \frac{\partial^2 h}{\partial t^2} \right] \tag{2.3.23}$$

In case one prefers to use the canonical quantization, one will obtain $\mathcal{H} = NH$ and, since π^{ij} is already quadratic in H, the only part to consider is $-\sqrt{{}^3g}({}^3R-2\Lambda)$. This is substantially the 3-dimensional case of the variation of the action found above, then

$$
\begin{aligned}
H_2 = 16\pi G h^{-1/2} G_{ijkl}\pi^{ij}\pi^{kl} + \frac{1}{16\pi G}\sqrt{{}^3g}\Bigg[\frac{1}{4}h_{ij}\Delta_L^{ikjl}h_{kl} - R^j{}_k h^{ki}h_{ij} + \\
+\frac{1}{2}h^{ij}\nabla_i\nabla^k h_{jk} + \frac{1}{2}hR^{ij}h_{ij} - \frac{1}{8}{}^3Rh^2 + \frac{1}{4}\Lambda h^2 - \frac{1}{2}h\nabla^i\nabla^j h_{ij} - \frac{1}{2}\Lambda h^{ij}h_{ij} + \\
+\frac{1}{4}{}^3Rh_{ij}h^{ij} + \frac{1}{4}h\nabla^k\nabla_k h\Bigg]
\end{aligned}
\tag{2.3.24}
$$

where covariant derivatives and the Riemann tensor are computed respect to the background metric $\bar{g}_{ij}$.

Chapter 3

Zero point energy

In this chapter we will explain how to compute a semiclassical approximation of the zero point energy of a spacetime of pure gravity. We will stress and try to manage the most relevant criticalities in a one loop computation in quantum gravity.

WDW equation as a Sturm-Liouville problem

First of all, we have to put the WDW equation in a more transparent form[5]. The Einstein equation, that describes the dynamics of a classical spacetime, is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_c g_{\mu\nu} = 0 \tag{3.0.1}$$

when we exclude any matter field from the system. However, in a quantum theory, a vacuum energy will emerge, so we can add an energy momentum tensor to the right side $T_{\mu\nu} = -\langle\rho\rangle g_{\mu\nu}$. This contribution can be interpreted as an induced cosmological constant and moved again to the left side

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda_{eff} g_{\mu\nu} = 0 \tag{3.0.2}$$

There Λ_{eff} is the effective cosmological constant, equal to the classical term Λ_c summed with the quantum effects represented by Λ_q. Clearly the energy density, given by $T_{\mu\nu}u^\mu u^\nu$, is $\langle\rho\rangle = \frac{\Lambda_q}{8\pi G}$. With the Hamiltonian formalism described above, we have

$$[-\sqrt{^3g}\,^3R + (16\pi G)^2 g^{-1/2} G_{ijkl}\pi^{ij}\pi^{kl}]\Psi[g_{ij}] = -2\sqrt{^3g}\Lambda_{eff}\Psi[g_{ij}] \tag{3.0.3}$$

If we rename

$$\frac{1}{2}[-\sqrt{^3g}\,^3R + (16\pi G)^2 g^{-1/2} G_{ijkl}\pi^{ij}\pi^{kl}] = \hat{\Lambda}_\Sigma \tag{3.0.4}$$

we obtain

$$\hat{\Lambda}_\Sigma \Psi[g_{ij}] = -\Lambda_{eff}\Psi[g_{ij}] \tag{3.0.5}$$

Let's consider again the minisuperspace spatially homogeneous model, the WDW equation with all the $16\pi G$ factors is

$$\begin{aligned}
&\left[-a^{-q}\frac{\partial}{\partial a}a^{q}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}(ka^2-\frac{\Lambda_{eff}}{3}a^4)\right]\Psi(a)=\\
&=\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}(ka^2-\frac{\Lambda_{eff}}{3}a^4)\right]\Psi(a)=0
\end{aligned}\tag{3.0.6}$$

where has been chosen an ordering convention $i = 1 - q$, $j = q$ and $k = 0$. This expression has the form of the Sturm-Liouville differential equation[26]

$$\frac{d}{dx}\left(p(x)\frac{dy(x)}{dx}\right)+q(x)y(x)+\lambda w(x)y(x)=0 \tag{3.0.7}$$

with a normalization of the function $y(x)$ defined by

$$\int dxw(x)y^*(x)y(x) \tag{3.0.8}$$

Such a differential equation is usually treated as a variational problem trough the functional

$$-\frac{\int dxy^*(x)\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y(x)}{\int dxw(x)y^*(x)y(x)}=\mathcal{F}[y(x)] \tag{3.0.9}$$

If $y(x)$ is an eigenfunction of the differential equation with eigenvalue λ, the functional $\mathcal{F}$ assume the value λ, while the search of a minimum for $\mathcal{F}[y(x)]$ respect to y or y^* variations gives the following Euler-Lagrange equation

$$\frac{\left[\frac{d}{dx}\left(p(x)\frac{d}{dx}\right)+q(x)\right]y(x)}{\int dxw(x)y^*(x)y(x)}+\lambda_m\frac{w(x)y(x)}{\int dxw(x)y^*(x)y(x)}=0 \tag{3.0.10}$$

where λ_m is the minimum of $\mathcal{F}$. It is equivalent to the Sturm-Liouville problem (3.0.7), hence a local minimum to $\mathcal{F}$ is an eigenfunction of the differential equation and the global minimum corresponds with the lowest eigenvalue.
With the substitutions

$$x \to a \tag{3.0.11}$$
$$p(x) \to a^q \tag{3.0.12}$$
$$q(x) \to \left(\frac{3\pi}{2G}\right)^2 a^{q+2} \tag{3.0.13}$$
$$w(x) \to a^{q+4} \tag{3.0.14}$$
$$y \to \Psi(a) \tag{3.0.15}$$
$$\lambda \to \frac{\Lambda}{3}\left(\frac{3\pi}{2G}\right)^2 \tag{3.0.16}$$

the equation we have to consider is

$$\frac{\int daa^q\Psi^*(a)\left[-\frac{\partial^2}{\partial a^2}-\frac{q}{a}\frac{\partial}{\partial a}+\frac{9\pi^2}{4G^2}(ka^2-\frac{\Lambda_{eff}}{3}a^4)\right]\Psi(a)}{\int daa^{q+4}\Psi^*(a)\Psi(a)}=\frac{3\Lambda\pi^2}{4G^2} \tag{3.0.17}$$

The generalisation to the complete quantum theory permits to hold the formal structure of the functional $\mathcal{F}$ derived from the Sturm-Liouville problem. By multiplying the expression (3.0.5) by $\Psi^*[g_{ij}]$ and integrating it on g_{ij}, we can rewrite it as an expectation value of the operator $\hat{\Lambda}_\Sigma$

$$\frac{1}{V}\frac{\int\mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi[g_{ij}]}{\int\mathcal{D}[g_{ij}]\Psi^*[g_{ij}]\Psi[g_{ij}]}=-\Lambda_{eff} \tag{3.0.18}$$

with the normalization condition given by the Hilbert product (1.4.3). At this point it could be useful to exclude the classical effects, then we decompose again the spacelike metric g_{ij} in a stationary background part $\bar{g}_{ij}$, which is solution of the classical Einstein equation, and a perturbation h_{ij}

$$g_{ij}=\bar{g}_{ij}+h_{ij} \tag{3.0.19}$$

and we expand near $\bar{g}_{ij}$ the operator $\hat{\Lambda}_\Sigma$. Thus,eq (3.0.18) becomes

$$\frac{1}{V}\frac{\langle\Psi|\int_\Sigma d^3x\left[\hat{\Lambda}_\Sigma^{(0)}+\hat{\Lambda}_\Sigma^{(1)}+\hat{\Lambda}_\Sigma^{(2)}+...\right]|\Psi\rangle}{\langle\Psi|\Psi\rangle}=-\Lambda_{eff} \tag{3.0.20}$$

where $\hat{\Lambda}_\Sigma^0$ is independent of h_{ij} and equal to Λ_c, since $\bar{g}_{ij}$ is a classical solution. In (3.0.4) the kinetic term is already quadratic in momenta, so we only have to expand $\sqrt{^3g}\ {}^3R$ to the second order. It is important to notice that, thanks to definition (3.0.4), $\hat{\Lambda}_\Sigma^{(2)}$ is substantially proportional to the variation of H_2 found in section 2.3, except for the absence of Λ, that has been moved to the right side before the functional derivative.

Energy from euclidean path integral

A similar expression can be reached also from the partition function Z[27]. If we define as $F=-\ln Z$ the free energy of the statistical system associated to the Euclidean section, it is useful to consider the functional derivative

$$\frac{2}{\sqrt{-g}}\frac{\delta F}{\delta g_{\mu\nu}}=\frac{\int Dg_{\mu\nu}\frac{\delta I[g_{\mu\nu}]}{\delta g_{\mu\nu}}e^{-I[g_{\mu\nu}]}}{Z} \tag{3.0.21}$$

Now, the functional derivative of the action gives the left side of the Einstein equation (3.0.1), so we have again an expression for $\langle T_{\mu\nu}\rangle$. With a use of the cosmological constant similar to what has been made for the WDW equation, we obtain

$$\frac{2}{\sqrt{-g}}\frac{\delta F}{\delta g_{\mu\nu}} = -\Lambda_{eff} g^{\mu\nu} \tag{3.0.22}$$

where now F do not contain anymore Λ_{eff} in the action. Moreover it is possible to apply a saddle point approximation, that gives $F = F_0 + F_{1loop} + \ldots$. Clearly the classical part gives $\frac{2}{\sqrt{-g}}\frac{\delta F_0}{\delta g_{\mu\nu}} = \Lambda_c g^{\mu\nu}$, since there is no functional integration, while the derivative of the one loop expansion of the free energy is an estimation of the contribution of quantum fluctuations Λ_q. With N^i=0, the cosmological energy density is

$$\frac{2}{\sqrt{-g}}\frac{\delta F}{\delta N} = -\Lambda_{eff} \tag{3.0.23}$$

3.1 Disentangling the gauge modes

If we want to make some real calculation with booth quantization methods, we have to identify the meaningful degrees of freedom in h_{ij} and separate them from the pure gauge variables. With this aim, first of all we decompose the tensor h_{ij}.

Orthogonal decomposition

We have defined G_{ijkl} and chosen a background metric $\bar{g}_{ij}$, so we can introduce an inner product on the tangent space of the superspace, i. e. the space of metric variations,

$$\langle h, k\rangle := \int_\Sigma \sqrt{^3\bar{g}} G^{ijkl} h_{ij}(x) k_{kl}(x) d^3x \tag{3.1.1}$$

and an analogue product in the cotangent space

$$\langle p, q\rangle := \int_\Sigma \sqrt{^3\bar{g}} G_{ijkl} p^{ij}(x) q^{kl}(x) d^3x \tag{3.1.2}$$

A good decomposition, in order to make computations, is the following[28]:

$$h_{ij} = h_{ij}^{TT} + (L\xi)_{ij} + \frac{1}{3}(\sigma + 2\nabla\cdot\xi)\bar{g}_{ij} \tag{3.1.3}$$

where the longitudinal and traceless part $(L\xi)_{ij}$ is

$$h_{ij}^L = (L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}\bar{g}_{ij}\nabla\cdot\xi \tag{3.1.4}$$

with ξ_i representing a covariant field, the trace part is

$$(\sigma + 2\nabla\cdot\xi) = h = h^i{}_i \tag{3.1.5}$$

while h_{ij}^{TT} is, as we will see below, the transverse and traceless part of the symmetric field h_{ij}, that means

$$h^{i}{}_{i}^{TT} = 0, \quad \nabla^i h_{ij}^{TT} = 0 \tag{3.1.6}$$

In fact the traceless condition is trivially satisfied by construction, since h_{ij}^{TT} is defined by

$$h_{ij}^{TT} = h_{ij} - \frac{1}{3} h \bar{g}_{ij} - (L\xi)_{ij} \tag{3.1.7}$$

while the transversality request is equivalent to the equation

$$\nabla^i (L\xi)_{ij} = \nabla^i \left(h_{ij} - \frac{1}{3} h \bar{g}_{ij} \right) \tag{3.1.8}$$

It can be shown that always exists a unique solution ξ^i of the last equation, modulo conformal killing vectors. A conformal killing vector is a vector field such that the transformation of coordinates it generates preserves the conformal structure of the metric, then $g_{ij} \to \Omega g_{ij}$. Given a vector field v^i, this property is equivalent to

$$\nabla_i v_j + \nabla_j v_i = \frac{2}{3} \bar{g}_{ij} \nabla \cdot v \tag{3.1.9}$$

and $(Lv)_{ij} = 0$. Thus,the arbitrariness in the choice of ξ^i respect to conformal killing vectors can't influence h_{ij}^{TT} because of definition (3.1.7) and the decomposition is unique. Such decomposition is also orthogonal, since $\frac{1}{3} h \bar{g}_{ij}$ is poinwise orthogonal to h_{ij}^{TT} and $(L\xi)_{ij}$ as they are traceless, and the product $\langle (L\xi)_{ij}, h_{ij}^{TT} \rangle$ is clearly null if we integrate by parts: it can be easily seen that

$$\langle (L\xi)_{ij}, h_{ij}^{TT} \rangle = -2 \langle \xi_i, \nabla^i h_{ij}^{TT} \rangle = 0 \tag{3.1.10}$$

if the tensor field h_{ij} is null on the boundary of the three-surface.
Now we will study the behaviour of different components under conformal transformations and diffeomorphisms. Let's consider a conformal transformation with conformal factor ϕ: it acts as $\bar{g}_{ij} \to \tilde{g}_{ij} = \phi^{-4} \bar{g}_{ij}$ on the background and the variation of the metric will clearly scale in the same way. The transverse traceless decomposition (3.1.7) can be seen as

$$h_{ij}^{TT} = \phi^4 \left(\tilde{h}_{ij} - \frac{1}{3} \tilde{h} \tilde{g}_{ij} \right) - (L\xi)_{ij} \tag{3.1.11}$$

Since the connections coefficients, under conformal maps, transform as

$$\tilde{\Gamma}^i{}_{jk} = \Gamma^i{}_{jk} + 2(\delta^i_j \nabla_k \ln \phi + \delta^i_k \nabla_j \ln \phi - \bar{g}_{jk} \nabla^i \ln \phi) \tag{3.1.12}$$

the following equations are true:

$$(L\xi)_{ij} = \phi^4 \left(\tilde{\nabla}_i \xi_j + \tilde{\nabla}_j \xi_i - \frac{2}{3} \tilde{g}_{ij} \tilde{\nabla}^k \xi_k \right) = \phi^4 (\tilde{L}\xi)_{ij} \tag{3.1.13}$$

$$\phi^{-4} \nabla^i \left(h_{ij} - \frac{1}{3} h \bar{g}_{ij} \right) = \tilde{\nabla}^i \left(\tilde{h}_{ij} - \frac{1}{3} \tilde{h} \tilde{g}_{ij} \right) \tag{3.1.14}$$

and we can define

$$\tilde{h}^{TT}_{ij} = \phi^{-4} h^{TT}_{ij} = \left(\tilde{h}_{ij} - \frac{1}{3}\tilde{h}\tilde{g}_{ij} \right) - (\tilde{L}\xi)_{ij} \tag{3.1.15}$$

The component $\tilde{h}^{TT}_{ij}$ is obviously traceless and also transverse, because the condition

$$\tilde{\nabla}^i (\tilde{L}\xi)_{ij} = \tilde{\nabla}^i \left(\tilde{h}_{ij} - \frac{1}{3}\tilde{h}\tilde{g}_{ij} \right) \tag{3.1.16}$$

is completely equivalent to (3.1.8), as can be shown with the help of (3.1.12), (3.1.13) and (3.1.14), that means they have the same solution ξ_i. Thus,a conformal transformation do non change the TT transformation, except for a multiplicative factor ϕ^{-4}.
If we consider a three dimensional diffeomorphism generated by an infinitesimal coordinate shift $x^i \to x^i + v^i$, the change in the metric is $\delta g_{ij} = \nabla_i v_j + \nabla_j v_i$, while the metric variation respect to $\bar{g}_{ij}$ becomes $h_{ij} \to h_{ij} + \nabla_i v_j + \nabla_j v_i$. The new longitudinal part must solve the new form of equation (3.1.8)

$$\nabla^i (L\xi')_{ij} = \nabla^i \left[h_{ij} + \nabla_i v_j + \nabla_j v_i - \frac{1}{3}(h + 2\nabla \cdot v)\bar{g}_{ij} \right] \tag{3.1.17}$$

which imply $\xi'_i = \xi_i + v_i$, since (3.1.8) is linear, and

$$h'^{TT}_{ij} = h^{TT}_{ij} \tag{3.1.18}$$

Hence,the longitudinal part is a pure gauge contribution, because the physical system do not change under diffeomorphisms and the transformation generated by $-\xi_i$ nullifies it. The only therm corresponding to a variation of the intrinsic conformal geometry is h^{TT}_{ij}, while the trace component $h\bar{g}_{ij}$ generates conformal transformations.

Decomposition of $\hat{\Lambda}^{(2)}_{\Sigma}$

The part containing momenta in the operator $\hat{\Lambda}^{(2)}_{\Sigma}$ is an inner product of those defined in the precedent paragraph, then we can decompose the momenta and write

$$G_{ijkl}\pi^{ij}\pi^{kl} = \pi^{(TT)ij}\pi^{TT}_{ij} - \frac{1}{6}\pi^2 + \pi^{(L)ij}\pi^{L}_{ij} \tag{3.1.19}$$

In general, as explained in the first chapter, to different boundary conditions correspond different wave functionals solving the WDW equation. In this work we will consider only Gaussian wave functionals of the type

$$\begin{aligned}\Psi[h_{ij}] = \mathcal{N} \exp \Big\{ -\frac{1}{4G} \big[&\langle h, K^{TT\ -1} h \rangle^{TT} + \langle (L\xi), K^{L\ -1} (L\xi) \rangle + \\ &+ \langle h, K^{Tr\ -1} h \rangle^{Tr} \big] \Big\}\end{aligned} \tag{3.1.20}$$

where K are a set of propagators we will determinate with a variational method. We are looking for a ground state of quantum gravity, that we expect to be located in a minimum of the potential. A minimum can always be approximated by an harmonic oscillator, so a Gaussian structure of the wave functional seems to be a reasonable assumption. Moreover, this choice has been made in order to reproduce the separation of momentum components observed in $\hat{\Lambda}^{(2)}_{\Sigma}$ also in the part containing the field h_{ij}. Such a wave functional can be factorised in

$$\Psi[h_{ij}] = \mathcal{N}\Psi[h^{TT}_{ij}]\Psi[\xi_i]\Psi[\sigma] \tag{3.1.21}$$

and permits to neglect $\hat{\Lambda}^{(1)}_{\Sigma}$ and cross terms between different components in the decomposition of $\hat{\Lambda}^{(2)}_{\Sigma}$, since the first moment of a Gaussian integration is always null. The traceless-transverse decomposition also involves a redefinition in the measure of functional integration, in fact we have $Dh_{ij} \to Dh^{TT}_{ij} D\sigma D\xi J$, where J is the Jacobian determinant induced by the new variable ξ [29]

$$J = \left[\det\left(\Delta g^{ij} + \frac{1}{3}\nabla^i\nabla^j - R^{ij}\right)\right]^{1/2} \tag{3.1.22}$$

We have

$$\frac{1}{V}\frac{\int Dh^{TT}_{ij} D\sigma D\xi J \Psi^*[h_{ij}] \int_\Sigma d^3x \left[\hat{\Lambda}^{TT}_{\Sigma} + \hat{\Lambda}^{\xi}_{\Sigma} + \hat{\Lambda}^{\sigma}_{\Sigma}\right]\Psi[h_{ij}]}{\langle\Psi|\Psi\rangle} = -\Lambda_q \tag{3.1.23}$$

where $\hat{\Lambda}^{TT}_{\Sigma}$ $\hat{\Lambda}^{\xi}_{\Sigma}$ and $\hat{\Lambda}^{\sigma}_{\Sigma}$ are the components of the variation $\hat{\Lambda}^{(2)}_{\Sigma}$.
The factorised structure of the wave functional permit us to treat separately different parts of the perturbation and, since we have shown ξ is pure gauge, we can fix $\xi = 0$. The contribution of $\hat{\Lambda}^{\xi}_{\Sigma}$ will be null and the determinant J will be cancelled by an equal term at the denominator.
Once treated 3-diffeomorphisms, there is still one exceeding degree of freedom in the wave functional: the Hamiltonian constraint hides another non physical component of h_{ij}, but it can't be solved trivially. Despite this obstacle, following the work of ADM[10], we can observe that, at the first order in the perturbation of the Hamiltonian constraint, we have

$$\left(-\Delta + \frac{1}{2}R\right)\sigma = -\mathfrak{H} = -R^{ij}h^{TT}_{i}j - (16\pi G)^2 G_{ijkl}\bar{\pi}^{ij}\pi^{kl} \tag{3.1.24}$$

That means it is possible, in linear gravity, to take σ as the momentum associated to semiclassical time $t = -1/2\Delta\pi$. The choice to consider the trace component as the degree of freedom associated to classical time seems natural since, as we observed in the first chapter, the trace has negative signature respect to the superspace metric given by eq. (1.3.22).

In the full theory, as already widely discussed, the constraint isn't such simple, but in general it is possible to write something with the same form, where the right side is now a nonlinear function of h_{ij} and π^{ij}. Thus, one can solve this equation, at least by a perturbation-iteration expansion, for h.
The final result is that the only physical degrees of freedom are held by h_{ij}^{TT}, so we have

$$\begin{aligned}\hat{\Lambda}_{\Sigma}^{TT} = \frac{(16\pi G)^2}{2} g^{-1/2} G_{ijkl} \pi^{(TT)ij} \pi^{(TT)kl} + \\ + \frac{1}{2}\sqrt{{}^3\bar{g}} \left[\frac{1}{4} h_{ij} \Delta_L^{ikjl} h_{kl} - R^j{}_k h^{ki} h_{ij} + \frac{1}{4} {}^3R h_{ij} h^{ij} \right]\end{aligned} \tag{3.1.25}$$

In a 3-dimensional torsion free manifold the Weyl tensor, i. e. the traceless part of the Riemann tensor, is null, that means all the information in R^{ijkl} is already present in R^{ij}. Then it is true the equivalence

$$R_{ikjl} = g_{ij} R_{kl} - g_{il} R_{kj} - g_{kj} R_{il} + g_{kl} R_{ij} - \frac{R}{2}(g_{ij} g_{kl} - g_{il} g_{kj}) \tag{3.1.26}$$

and the operator $\hat{\Lambda}_{\Sigma}^{TT}$ can be rewritten as

$$\hat{\Lambda}_{\Sigma}^{TT} = \frac{(16\pi G)^2}{2} g^{-1/2} G_{ijkl} \pi^{ij} \pi^{kl} + \frac{1}{8}\sqrt{{}^3\bar{g}} \left[-h^{ij} \nabla^k \nabla_k h_{ij} + 2 h^{ij} R_{il} h^l{}_j \right] \tag{3.1.27}$$

that is the energy density operator for the graviton.
The next step will be to calculate the contribution to the one loop cosmological constant brought by the traceless transverse sector of the operator $\hat{\Lambda}_{\Sigma}^{(2)}$

$$\langle \hat{\Lambda}_{\Sigma}^{TT} \rangle = \frac{1}{V} \frac{\langle \Psi | \int_{\Sigma} d^3x \hat{\Lambda}_{\Sigma}^{TT} | \Psi \rangle}{\langle \Psi | \Psi \rangle} \tag{3.1.28}$$

Gauge fixing in path integral

If one prefer to work with the path integral from expression (3.0.22), one have to put more attention on determinants coming from changes of variables and gauge fixing. We will mainly work with the totally covariant form of path integral, as happens in [30] and [3]. Obviously the decomposition showed in 3.1 is valid for each dimension of the manifold where the tensor $h_{\mu\nu}$ lives, at the cost of substituting 3 with the dimension of the manifold d in all denominators, since $g_{\mu\nu} g^{\mu\nu} = d$. In this situation, if we consider only spacetimes with null classical cosmological constant Λ_c, $\bar{g}_{\mu\nu}$ describes an Einstein spacetime with $R_{\mu\nu} = 0$.
The integration measure is $Dh_{\mu\nu} = Dh_{\mu\nu}^{TT} DhD(L\xi)_{\mu\nu}$ where the space of 2-tensors $(L\xi)_{\mu\nu}$ is substantially isomorphic to the space of vector fields ξ after a quotient over conformal killing vectors. Now we are capable to manage

the gauge freedom brought by diffeomorphisms of the type $x^\mu \to x^\mu + v^\mu$, which induces the change $L\xi \to L(\xi+v) =: L\xi^{(v)}$ and $h \to h + 2\nabla_\mu v^\mu$ in the metric variations. We want to treat these nonphysical fields in the usual way in path integral quantum field theory, introducing a Dirac delta in order to fix gauge freedom with the following identity

$$1 = \int Dv\delta(G(L\xi^{(v)}))\det\left(\frac{\delta(G(L\xi^{(v)}))}{\delta v}\right) \tag{3.1.29}$$

The most convenient gauge fixing function G would be $L\xi^{(v)}$ itself, but the operator L has not a well defined determinant, since it is an operator that goes from vector fields to 2-tensors. To avoid this complication, we consider the module of the longitudinal mode $\langle L(\xi+v), L(\xi+v)\rangle$. The product is evaluated respect to the supermetric, which is positive definite on the longitudinal sector, since $L\xi$ has null trace. In this case it reduces to

$$\langle L(\xi+v), L(\xi+v)\rangle = \int_M d^4x\sqrt{-g}L(\xi+v)^{\mu\nu}L(\xi+v)_{\mu\nu} \tag{3.1.30}$$

With an integration by part, we obtain $\xi L^\dagger L\xi$ in the integrand, thus setting $L\xi^{(v)}$ to zero is equivalent to set $\xi^{(v)}L^\dagger L\xi^{(v)} = 0$ and consequently $L^\dagger L\xi^{(v)} = 0$. Then we will use the square root of the determinant of the quadratic operator $L^\dagger L$.

This determinant is independent of $L\xi$ and the action I_2 is gauge invariant, so we can shift the variable $h_{\mu\nu}$ to $h_{\mu\nu} - \nabla_\mu v_\nu - \nabla_\nu v_\mu$ and obtain

$$\begin{aligned}\int Dh^{TT}DhD(L\xi)e^{-I_2[h^{TT},L\xi,h]} &= \\ = \int Dv\int Dh^{TT}DhD(L\xi)e^{-I_2[h^{TT},h-2\nabla_\mu v^\mu]}\delta(L\xi)\det\Big(L^\dagger L\Big)^{1/2}\end{aligned} \tag{3.1.31}$$

After a last change of variable $\sigma = h - 2\nabla_\mu v^\mu$, the functional integral is

$$\int Dv\int Dh^{TT}D\sigma e^{-I_2[h^{TT},\sigma]}\det\Big(L^\dagger L\Big)^{1/2} \tag{3.1.32}$$

where the integration over v is an infinite multiplicative factor that can be cancelled by a normalization. The quadratic part of the action is

$$I_2 = \frac{1}{16\pi G}\int_M d^4x\sqrt{g}\left[\frac{1}{4}h_{\mu\nu}\Delta_{TT}^{\mu\rho\nu\sigma}h_{\rho\sigma} + \frac{3}{32}\sigma\nabla^\mu\nabla_\mu\sigma\right] \tag{3.1.33}$$

thus we can reduce the partition function to

$$Z = \det_{h^{TT}}(\Delta_{TT})^{-1/2}\det_{\xi}(L^\dagger L)^{1/2}\det_{\sigma}(-\Delta)^{-1/2} \tag{3.1.34}$$

with $\Delta = -\nabla^\mu \nabla_\mu$. The next step is computing the factor $\det\left(L^\dagger L\right)^{1/2}$ in order to obtain a more manageable expression.
We know from Gaussian integration that

$$\det\left(L^\dagger L\right)^{-1/2} = \int D\xi e^{-\frac{1}{2}\int_M d^4x\sqrt{\bar{g}}\xi L^\dagger L\xi} \tag{3.1.35}$$

were conformal killing vectors, which compose the kernel of L, are excluded from the integration over ξ. The quadratic operator has the form

$$(L^\dagger L)_\mu{}^\nu = -2\left(-\Delta\delta_\mu{}^\nu + \left(1-\frac{2}{d}\right)\nabla_\mu\nabla^\nu + R_\mu{}^\nu\right) \tag{3.1.36}$$

but it can be simplified even more with the assumption $R^{\mu\nu} = 0$ and $d = 4$. Moreover, with the help of Hodge decomposition, we can write

$$\xi = d\psi + \xi^H + \delta\omega = \nabla\psi + \xi^H + \xi^T \tag{3.1.37}$$

where d now stands for the external derivative, and δ is the adjoint operator respect to d. In this decomposition ψ is a scalar field, ξ^H is an harmonic vector field ($\Delta\xi^H = 0$) and ω is a 2-form that generate the transverse part of the vector ($\nabla\cdot\xi = 0$). In fact the harmonic part often can be neglected, since the dimension of the space of harmonic n-forms is equal to the dimension of the nth cohomology group of the manifold associated with $\bar{g}_{\mu\nu}$[3]. The expression $\xi^\mu(L^\dagger L)_\mu{}^\nu\xi_\nu$ becomes

$$2\xi^{T\mu}\Delta\xi^T_\mu - 3\psi\Delta^2\psi - \xi^{H\mu}\nabla_\mu\nabla^\nu\xi^H_\nu \tag{3.1.38}$$

The change of variable $D\xi \to D\xi^T D\xi^H D\psi$ brings a Jacobian factor with the form $\det(\Delta)^{1/2}$ and therefore

$$\det\left(L^\dagger L\right)^{1/2} = \det_\psi(-\Delta)^{1/2}\det_{\xi^T}(\Delta)^{1/2}\det_{\xi^H}(\nabla_\mu\nabla^\nu)^{1/2} \tag{3.1.39}$$

Hence,the effective partition function is

$$Z = \det_{h^{TT}}(\Delta_L)^{-1/2}\det_\sigma(-\Delta)^{-1/2}\det_\psi(-\Delta)^{1/2}\det_{\xi^T}(\Delta)^{1/2}\det_{\xi^H}(\nabla_\mu\nabla^\nu)^{1/2} \tag{3.1.40}$$

The field σ and ψ are both scalar, then it would be interesting to know whether the two determinants cancel each other. With this aim, it is important to consider over which field configuration we are effectively integrating. We have excluded all configurations of ξ corresponding to conformal killing vectors and the conformal killing condition (3.1.9) applied to the longitudinal sector of ξ becomes

$$\nabla_\mu\nabla_\nu\psi = \frac{1}{d}g_{\mu\nu}\Delta\psi \tag{3.1.41}$$

Yano and Nagano have shown that, if a connected Einstein space of dimension $d > 2$ admits a non-trivial conformal killing vector field which do not generates an homothetic transformation, then it has constant curvature and is homeomorphic to the sphere S^d[31]. In this paper we will mainly discuss Schwarzschild and Minkowski spacetimes, that means we can reduce the excluded conformal killing vectors to true killing vectors and homotheties. In particular the Schwarzschild spacetime do not admit conformal killing vectors at all, thus only killing vectors remain out of functional integration and correspond to the configurations of ψ whose second derivatives are null. Anyway some of the zero modes of the Laplacian is not comprehended in the integral over ψ, while they are considered in the determinant given by the field σ. So it is not possible to simplify expression (3.1.40) without taking the decision to exclude conformal modes of perturbation.

3.2 Expectation value of $\hat{\Lambda}_\Sigma^{(2)}$

The wave functional form (3.1.20) is convenient in computing expectation values, in fact we have

$$\frac{\langle\Psi|\, h_{ij}^{TT}(x) h_{jk}^{TT}(y)\,|\Psi\rangle}{\langle\Psi|\Psi\rangle} = K_{ijkl}^{TT}(x,y) \tag{3.2.1}$$

Moreover, with the canonical substitution of momenta in Dirac quantization we can easily compute also $\langle\pi^{ij}\pi^{kl}\rangle$[6]. We have

$$\pi^{ij}(x)\pi^{kl}(y)\,|\Psi\rangle = -\frac{\delta^2\Psi[h]}{\delta h_{ij}(x)\delta h_{kl}(y)} \tag{3.2.2}$$

and then

$$\begin{aligned}\pi^{ij}(x)\pi^{kl}(y)\,|\Psi\rangle = {}& \frac{1}{2}K^{-1\ klij}(x,y)\sqrt{g(x)}\sqrt{g(y)}\Psi[h]- \\ & -\frac{1}{4}\int d^3z d^3z' \sqrt{g(x)}\sqrt{g(y)}\sqrt{g(z)}\sqrt{g(z')}K^{-1\ mnij}(x,z')h_{mn}(z)\times \\ & \times K^{-1\ klpq}(y,z')h_{pq}(z')\Psi[h]\end{aligned} \tag{3.2.3}$$

After a contraction with $\langle\Psi|$ it becomes

$$\frac{\langle\Psi|\,\pi^{ij}(x)\pi^{kl}(y)\,|\Psi\rangle}{\langle\Psi|\Psi\rangle} = \frac{1}{4}K^{-1\ ijkl}(x,y)\sqrt{g(x)}\sqrt{g(y)} \tag{3.2.4}$$

These results permit us to rewrite the expectation value of $\hat{\Lambda}_\Sigma^{TT}$ in the form

$$\begin{aligned}\langle\hat{\Lambda}_\Sigma^{TT}\rangle = \frac{1}{8V}\int_\Sigma d^3x\sqrt{{}^3\bar{g}}G_{ijkl}\Big[&(16\pi G)^2 K^{TT\ -1\ ijkl}(x,x)+ \\ &+\Delta_{TT\ m}^{j} K^{TT\ imkl}(x,x)\Big]\end{aligned} \tag{3.2.5}$$

where the transverse Laplacian operator $\Delta_{TT\ m}^{j}$ is

$$\Delta_{TT\ m}^{j} = \Delta\delta^{j}{}_{m} + 2R^{j}{}_{m} = \Delta_{L\ m}^{j} - 4R^{j}{}_{m} + R\delta^{j}{}_{m} \tag{3.2.6}$$

We do not have any expressions for the propagator K^{TT} , but we can write it as a combination of eigenfunctions of the potential energy operator, which weights will be set by the minimisation of the energy of the spacetime. Since we are interested only in some expectation values, we can consider only diagonal terms in this representation, hence

$$K_{ijkl}^{TT}(x,y) = \sum_{\tau} \frac{h_{ij}^{(\tau)TT}(x) h_{kl}^{(\tau)TT}(y)}{2\lambda^{TT}(\tau)} \tag{3.2.7}$$

where $h_{ij}^{(\tau)TT}$ is a complete set of eigenfunctions of Δ_{TT} normalized respect to the inner product $\langle \cdot, \cdot \rangle$, while $\lambda^{TT}(\tau)$ is a set of variational parameters will be fixed by means of minimisation of the zero point energy.
Thus,the expectation value is

$$\langle \hat{\Lambda}_{\Sigma}^{TT} \rangle = \sum_{\tau} \frac{1}{8V} \left[(16\pi G)^2 \lambda^{TT}(\tau) + \frac{\omega^{TT\ 2}(\tau)}{\lambda^{TT}(\tau)} \right] \tag{3.2.8}$$

with $\omega^2(\tau)$ representing the eigenvalues of the eigenfunctions $h(\tau)$. A quick differentiation respect to the parameters $\lambda^{TT}(\tau)$ shows that the energy density has a stationary point for $\lambda^{TT}(\tau) = \frac{\sqrt{\omega^{TT\ 2}}}{16\pi G}$. The following expression can be taken as good approximation of the cosmological energy density:

$$\langle \hat{\Lambda}_{\Sigma}^{TT} \rangle = \frac{1}{V} \sum_{\tau} 2\pi G \sqrt{\omega^{TT\ 2}(\tau)} \tag{3.2.9}$$

The last expression make sense only if $\omega^2 > 0$, then eventual negative modes must be considered separately.
At this point the main task in order to have an estimation of energy is to find the eigenvalues and eigenfunctions of the differential operator Δ_{TT}. It can be quite hard, then an alternative way is the WKB method used by 'tHooft in the brick wall problem[32], that consist in counting the density of modes as a function of the energy eigenvalue ω^2.
However, also with this procedure there is a tricky aspect: In general the operator Δ_{TT} we want to study is not an endomorphism of the traceless transverse subspace. the transverse Laplacian operator $\Delta_{TT\ a}^{j} = \Delta\delta_{a}^{j} + 2R_{a}^{j}$ appears in the expectation value of the cosmological constant in a sandwich product between two time the traceless-transverse component of the metric variation $h_{j}^{TT\ i} \Delta_{TT\ a}^{j} h_{i}^{TT\ a}$. We can immediately see the Laplacian $\Delta = -\nabla^k \nabla_k$ is a scalar operator, then it conserves the traceless property and symmetry of $h_i^{TT} j$, while R_a^i, seen as an operator acting on the superspace, is

not in general an endomorphism of the traceless-transverse sector, moreover it isn't an endomorphism of symmetric matrices.
The notion of eigenvalue and eigenvector clearly has sense only when we consider endomorphisms over a certain space, so we have to reduce the operator $\Delta^{j}_{TT\ a}$ to an endomorphism of the traceless-tansverse sector of metric variations. This is possible because, as we stated above, Δ_{TT} is totally contracted in the inner product and in this product the different subspaces we are going to consider are orthogonal to each other.
first of all we substitute the Ricci tensor acting on the metric variation $2R^i_a h^{TT\ a}_j$ with a symmetrized form $R^i_a h^{TT\ a}_j + R_{ja} h^{TT\ ai}$ and we subtract the trace in order to have a traceless tensor

$$2R^i_a h^{TT\ a}_j \rightarrow R^i_a h^{TT\ a}_j + R_{ja} h^{TT\ ai} - \frac{2}{3}\delta^i_j R^k_a h^{TT\ a}_k \tag{3.2.10}$$

In this way we have obtained a traceless symmetric tensor from $\Delta^{j}_{TT\ a} h^{TT\ a}_i$ without modifying the inner product. The following step should consist in removing an eventual emerging longitudinal part. We can separate the modified transverse Laplacian operator in the Laplacian and a second part containing the Ricci tensor.

$$\begin{aligned}\nabla_i \nabla_k \nabla^k h^{TT\ i}_j = \nabla_k \nabla_i \nabla^k h^{TT\ i}_j - R_{ikl}{}^k \nabla^l h^{TT\ i}_j - R_{ikl}{}^i \nabla^k h^{TT\ l}_j + \\ + R_{ikj}{}^l \nabla^k h^{TT\ i}_l = \nabla_k \nabla_i \nabla^k h^{TT\ i}_j + R_{ikj}{}^l \nabla^k h^{TT\ i}_l\end{aligned} \tag{3.2.11}$$

After another commutation between derivatives and using the null divergence of $h^{TT\ i}_j$

$$\nabla_i \nabla_k \nabla^k h^{TT\ i}_j = -\nabla^k R_{ikl}{}^i h^{TT\ l}_j + \nabla^k R_{ikj}{}^l h^{TT\ i}_l + R_{ikj}{}^l \nabla^k h^{TT\ i}_l \tag{3.2.12}$$

In a three-dimensional manifold the Riemann tensor can be written in terms of the Ricci tensor, so

$$\begin{aligned}\nabla_i \nabla_k \nabla^k h^{TT\ i}_j = \nabla^k R_{kl} h^{TT\ l}_j + \nabla^k (R_{kl} h^{TT\ l}_j - g_{kj} R^l_i h^{TT\ i}_l + R_{ij} h^{TT\ i}_k) + \\ + R_{kl} \nabla^k h^{TT\ l}_j - R^l_i \nabla_j h^{TT\ i}_l\end{aligned} \tag{3.2.13}$$

The contracted Bianchi identities grant null divergence of the Ricci tensor, hence we can write

$$\nabla_i \nabla_k \nabla^k h^{TT\ i}_j = 3R_{kl} \nabla^k h^{TT\ l}_j - \nabla_j R^l_i h^{TT\ i}_l + h^{TT\ i}_k \nabla^k R_{ij} - R^l_i \nabla_j h^{TT\ i}_l \tag{3.2.14}$$

The divergence of the second part of the modified transverse Laplacian operator applied to h^{TT} is

$$\nabla_i R^i_k h^{TT\ k}_j + \nabla_i R_{jk} h^{TT\ ki} - \frac{2}{3} \nabla_j R^k_l h^{TT\ l}_k \tag{3.2.15}$$

and, summing the two contributions, the total divergence is

$$\nabla_j \Delta^j_{TT\ a} h_i^{TT\ a} = \nabla_j(\Delta h_a^{TT\ j} + R^j_a h_i^{TT\ a} + R_{ia} h^{TT\ aj} - \frac{2}{3}\delta^j_i R^k_a h_k^{TT\ a})$$
$$= -2R_{kl}\nabla^k h_i^{TT\ l} + R^j_k \nabla_i h_j^{TT\ k} + \frac{1}{3}\nabla_i (R^j_k h_j^{TT\ k}) \tag{3.2.16}$$

We have just shown that in general the transverse Laplacian operator does not preserve the transversality of h^{TT}_{ij}. Removing the emergent longitudinal part is not trivial, because one have to find a vector η such that the divergence $\nabla^i(L\eta)_{ij}$ is equal to the latter expression.

$$\nabla^j (L\eta)_{ij} = \nabla_j \Delta^j_{TT\ a} h_i^{TT\ a} \tag{3.2.17}$$

The eigenvalue equation to solve should have the form

$$\Delta h_j^{TT\ i} + R^i_a h_j^{TT\ a} + R_{ja} h^{TT\ ai} - \frac{2}{3}\delta^i_j R^k_a h_k^{TT\ a} - (L\eta)^i_j = \lambda h_j^{TT\ i} \tag{3.2.18}$$

All these problems are far more easy to manage in the particular case of spatial manifold solving the Einstein equation: in this situation the 3d Ricci tensor has the form

$$R_{ij} = \Lambda_c g_{ij} \tag{3.2.19}$$

and trivially commutes with covariant derivatives. Moreover also the divergence of the Laplacian operator (3.2.14) is null, since, with this form of the Ricci tensor, the first term is equal to the divergence of h^{TT}, the derivatives of R_{ij} are null and $R^{ij} h^{TT}_{ij}$ is proportional to the trace $h_i^{TT\ i} = 0$.
In this case, if one is capable to reduce the problem to an eigenvalue equation of the type

$$\frac{d^2 f(x)}{dx^2} + (\omega^2 - V(x))f(x) = 0 \tag{3.2.20}$$

as happen for example with a separation of radial modes and spherical harmonics in a spherically symmetric spacetime, it is possible to apply the WKB approximation to the wave function $f(x)$ and define the wave number $k^2 = \omega^2 - V$. In such a framework one can obtain the number ν of modes of the function f with energy less or equal than a certain level ω^2 with the relation

$$\pi\nu = \int dx \sqrt{k^2} \tag{3.2.21}$$

where the domain of integration corresponds with the region where $k^2 > 0$. Once found the mode density $\frac{dg}{d\omega}$, where $g(\omega)$ is the number of modes with energy less than ω of the metric perturbation h, the sum over the complete set of eigenvalues can be substituted with the integration

$$\int_0^\infty d\omega\omega \frac{dg}{d\omega} \tag{3.2.22}$$

Thus,we have

$$- \Lambda_q = \frac{G\pi}{V} \int_0^\infty d\omega\omega \frac{2dg^{TT}(\omega)}{d\omega} \tag{3.2.23}$$

This integral usually gives divergent results, which can be regularized and renormalized at first loop. However it isn't a definitive solution, since the quantum field theory of the spin 2 graviton is not renormalizable at each perturbative order with a finite set of running parameters.

Spherically symmetric spaces

Let's consider a spherically symmetric spatial manifold. In a wide set of cases, its background metric can be written as

$$\bar{g}_{ij} dx^i dx^j = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2^2 \tag{3.2.24}$$

where $b(r)$ is an arbitrary positive function and $d\Omega_2$ is the measure of the 2-sphere $d\theta^2 + \sin^2\theta d\phi^2$. The Ricci tensor of these spaces is

$$R^i_j = diag\left[\frac{b'(r)}{r^2} - \frac{b(r)}{r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}, \frac{b'(r)}{2r^2} + \frac{b(r)}{2r^3}\right] \tag{3.2.25}$$

while the curvature scalar is ${}^3R = 2\frac{b'(r)}{r^2}$.
In such a class of manifolds we can follow the Regge-Wheeler decomposition [33] when dealing with the perturbations h_{ij}. We consider modes with fixed parity, angular momentum l and m, and frequency ω^2. Since the system is spherically symmetric, we can set the angular momentum component along the z axis m as we prefer in the interval $(-l, -l+1, ..., l)$ and we will choose $m = 0$.
In the best case, that means it is possible to write a meaningful eigenvalue equation for the modified operator Δ_L, we expect to find something like

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m^2(r)\right] f(x) = \omega^2 f(x) \tag{3.2.26}$$

where $m(r)$ is an r dependent effective mass. Hence,the eigenvalue problem has been reduced to the desired form (3.2.20), where $V = \frac{l(l+1)}{r^2} + m^2(r)$. Following the path indicated above, we define

$$k^2(\omega, r, l) = \omega^2 - \frac{l(l+1)}{r^2} - m^2(r) \tag{3.2.27}$$

and

$$\nu(\omega, l) = \frac{1}{\pi} \int dx \sqrt{k^2(\omega, r, l)} \tag{3.2.28}$$

and then we obtain the estimation of the number of states with energy less than ω with

$$g(\omega) = \int_0^{l_{max}} dl\nu(\omega, l)(2l+1) \tag{3.2.29}$$

where the discrete sum over l has been transformed in an integral and l_{max} is the maximum value of the angular momentum that permits to hold $k^2 \geq 0$. The factor $(2l+1)$ obviously comes from the m degeneracy of eigenstates. The expression we actually need is the derivative of $g(\omega)$, that result to be

$$\frac{dg(\omega)}{d\omega} = \frac{1}{\pi}\int dx \int_0^{l_{max}} dl \frac{2l+1}{2\sqrt{k_i^2}}\frac{dk^2}{d\omega} = \frac{1}{\pi}\int dx\omega \int_0^{l_{max}} dl \frac{2l+1}{\sqrt{k^2}} \tag{3.2.30}$$

The integration over l can be immediately solved, since $\frac{2l+1}{r^2}$ is exactly the l derivative of k^2 and $k^2(\omega, r, l_{max}) = 0$ by definition of l_{max}, then

$$\frac{dg(\omega)}{d\omega} = \frac{2}{\pi}\int dxr^2\omega\sqrt{\omega^2 - m^2(r)} \tag{3.2.31}$$

The contribution of the traceless transverse component will respect the following equation

$$-\Lambda_q^{TT} = \frac{4G}{V}\int dxr^2 \int d\omega\omega^2\sqrt{\omega^2 - m^2(r)} \tag{3.2.32}$$

The domains of integration of x and ω are not trivial, since the approximation of ν from WKB theory has sense only if $k^2 \geq 0$. We want to have an integration over the whole space, so we can fix as lower extreme of the energy integration a certain value $\omega^*(r) > 0$, that is the minimum energy which grants $k^2 \geq 0$ in a certain point $r(x)$. Now, with an integration over the entire space, we can write the latter formula as

$$\frac{4\pi}{V}\int dxr^2\left(\Lambda_q^{TT} + \frac{G}{\pi}\int_{\omega^*(r)}^{\infty} d\omega\omega^2\sqrt{\omega^2 - m^2(r)}\right) = 0 \tag{3.2.33}$$

that can be seen as a weak identity

$$-\Lambda_q^{TT}(r) = \frac{G}{\pi}\int_{\omega^*(r)}^{\infty} d\omega\omega^2\sqrt{\omega^2 - m^2(r)} \tag{3.2.34}$$

The contribution to the cosmological constant is UV divergent in the ω integration, then a regularization is needed.

3.3 Regularization and renormalization

In order to regularize this one loop diverging result, we use the zeta function regularization scheme, that consists in changing the exponent of the wave number in the integrand and introduce a mass parameter in order to restore the correct dimension in regularized quantities.

$$-\Lambda_q^{TT} = \mu^{2\epsilon}\frac{G}{\pi}\int_{\omega^*(r)}^{\infty} d\omega\omega^2(\omega^2 - m^2(r))^{1/2-\epsilon} \tag{3.3.1}$$

where ϵ is taken small. We can reduce this computation to two relevant cases: when $m(r) > 0$ and when $m(r) < 0$.
If $m(r) > 0$ the integral is

$$\int_{\omega^*(r)}^{\infty} d\omega\omega^2(\omega^2 - m^2(r))^{1/2-\epsilon} \tag{3.3.2}$$

and, after a change of variable $t = \omega/\sqrt{|m^2(r)|}$, it becomes

$$m(r)^{4-2\epsilon}\int_1^{\infty} dtt^2(t^2 - 1)^{1/2-\epsilon} \tag{3.3.3}$$

We can use the integral relation 3.251.3 in [34] that states

$$\int_1^{\infty} dtt^{\mu-1}(t^p - 1)^{\nu-1} = \frac{1}{p}B\left(1 - \nu - \frac{\mu}{p}, \nu\right) \tag{3.3.4}$$

where B is the beta function related to the gamma functions via

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)} \tag{3.3.5}$$

and find

$$m(r)^{4-2\epsilon}B\left(\frac{3}{2}, \epsilon - 2\right) \tag{3.3.6}$$

If $m(r) < 0$ the integral is

$$\int_0^{\infty} d\omega\omega^2(\omega^2 + |m^2(r)|)^{1/2-\epsilon} \tag{3.3.7}$$

and with the same substitution it can be reduced to

$$m(r)^{4-2\epsilon}\int_1^{\infty} dtt^2(t^2 + 1)^{1/2-\epsilon} \tag{3.3.8}$$

In this case the most useful formula is the 3.251.2 of [34]

$$\int_0^{\infty} dtt^{\mu-1}(t^2 + 1)^{\nu-1} = \frac{1}{2}B\left(\frac{\mu}{2}, 1 - \nu - \frac{\mu}{2}\right) \tag{3.3.9}$$

Applied to our integral the result is

$$m(r)^{4-2\epsilon}B\left(\epsilon - 2, \frac{3}{2} - \epsilon\right) \tag{3.3.10}$$

The Gamma function has poles in negative integers, but we can use the recursive relation $\Gamma(n-1) = \Gamma(n)/(n-1)$ and the first order expansions

$$\Gamma(1+\epsilon) = 1 - \gamma_E + O(\epsilon^2) \tag{3.3.11}$$

$$\Gamma\left(\epsilon + \frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right)[1 - \epsilon(\gamma_E + 2\ln 2)] + O(\epsilon^2) \tag{3.3.12}$$

where γ_E is the Euler constant, in order to study its behaviour near the singularities for $\epsilon \to 0$.
In both cases described above, the $O(1)$ approximation of the cosmological energy density is [35]

$$\Lambda_q^{TT} = \frac{G}{16\pi} m^4(r) \left[\frac{1}{\epsilon} + \ln\left(\frac{\mu^2}{|m^2(r)|}\right) + 2\ln 2 - \frac{1}{2}\right] \tag{3.3.13}$$

Hence, in the rest of the chapter, we will use $m(r)^2$ to indicate its absolute value.
The next step is to separate the divergent part for $\epsilon \to 0$ and define

$$\Lambda_q^{TT,div} = \frac{G}{16\pi\epsilon} m^4(r) \tag{3.3.14}$$

Consequently we redefine in the cosmological constant $\Lambda_{eff}^{TT} = \Lambda_0^{TT} + \Lambda_q^{TT}$ the quantities $\Lambda_0^{TT} \to \Lambda_c^{TT} + \Lambda_q^{TT,div}$ and $\Lambda_q^{TT} \to \Lambda_q^{TT} - \Lambda_q^{TT,div}$, absorbing the diverging quantities in the classical therm[36]. Hence,one obtains

$$\Lambda_q^{TT}(\mu, r) = \frac{G}{16\pi} m^4(r) \left[\ln\left(\frac{\mu^2}{m^2(r)}\right) + 2\ln 2 - \frac{1}{2}\right] \tag{3.3.15}$$

the quantity Λ_{eff}^{TT} is a physical observable, but, at the moment, it depends from an arbitrary mass μ, then, in order to cancel such a dependence, we request Λ_{eff}^{TT} to respect a renormalization group equation

$$\mu \frac{d\Lambda_{eff}^{TT}}{d\mu} = 0 \tag{3.3.16}$$

It means the bare cosmological constant now behave as a running parameter, which respect the evolution equation

$$\mu \frac{d\Lambda_0^{TT}(\mu)}{d\mu} = -\mu \frac{d\Lambda_q^{TT}(\mu)}{d\mu} = -\frac{G}{8\pi} m^4(r) \tag{3.3.17}$$

and we can write

$$\Lambda_0^{TT}(\mu, r) = \Lambda_0^{TT}(\mu_0, r) + \frac{G}{8\pi} m^4(r) \ln\frac{\mu_0}{\mu} \tag{3.3.18}$$

At low energy we expect to find the classical cosmological constant, that is independent of r, then

$$\Lambda_0^{TT}(\mu, r) = \Lambda_c + \lim_{\mu_0 \to 0} \frac{G}{8\pi} m^4(r) \ln \frac{\mu_0}{\mu} \tag{3.3.19}$$

and the r dependence of $\Lambda_0^{TT}(\mu, r) - \Lambda_c$ comes only from the factor m^4. So we can express the effect of the starting point μ_0 in a mass parameter Ω

$$\Omega = \mu_0 e^{\frac{8\pi}{G m^4(r)}\left(\Lambda_0^{TT}(\mu,r) - \Lambda_c\right)} \tag{3.3.20}$$

that is homogeneous and gives

$$\Lambda_0^{TT}(\mu, r) = \Lambda_c + \frac{G}{8\pi} m^4(r) \ln \frac{\Omega^2}{\mu} \tag{3.3.21}$$

The parameter Ω plays the role of a fundamental energy scale, analogue to the Λ_{qcd} scale. Thus,the cosmological constant is

$$\Lambda_{eff}^{TT}(r) = \Lambda_c + \frac{G}{16\pi} m^4(r) \left[\ln \frac{\Omega^2}{m(r)^2} + 2 \ln 2 - \frac{1}{2}\right] \tag{3.3.22}$$

At this point we have a one loop regular theory that can be applied to some real computation in order to find the zero point energy of different spacetime background configurations.

Chapter 4

Testing quantum stability of the Minkowski spacetime

Gravity, already at classical level, is an universally attractive force that can't be screened. This feature, that permits us to detect such a weak coupling force, is source of a great number of instabilities. An homogeneous distribution of matter interacting through Newtonian gravity is unstable respect to long wavelength density perturbations (Jeans instability) and the attractiveness of gravity in general relativity provokes the collapse of matter in spacetime singularities. Given the presence of these phenomena, a logical argument of research is the existence and the stability of an eventual ground state of the quantum theory of gravity.
We have already outlined the fact that the gravitational action is not semi-positive definite and sometimes admits any negative modes and this feature will have great consequences in this chapter.

4.1 Using path integral

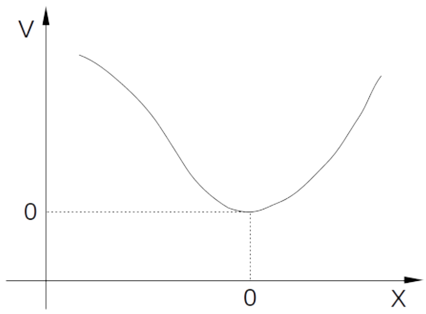
The first who showed an instability of the Minkowski spacetime in a thermal pure gravity quantum field theory were Gross, Perry and Yaffe [3], so in the rest of this section we well follow their discussion. Moreover, from now on, we will take spacetimes with null classical cosmological constant.

False vacuum states in a toy model

let's consider a toy model of a pointwise particle interacting with a potential $V(x)$. The transition amplitude with imaginary time β is

$$\langle x_i|x_f\rangle = \langle x_i| e^{-H\beta} |x_f\rangle = \int D[x]e^{-I[x]} \tag{4.1.1}$$

where $\langle x|$ are eigenstates of the position operator and the functional integral



is made over path that goes from x_i to x_f. We want to study the behaviour of a false ground state, i. e. a metastable equilibrium state, in order to reproduce the same procedure in the gravity field theory. The euclidean action clearly will be

$$I = \int_{-\beta/2}^{\beta/2} dt \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + V(x) \right] \tag{4.1.2}$$

and we can again make a saddle point expansion next to a classical solution $\bar{x}(t)$.

Let's consider a minimum in the potential $x = 0$, we can expand the action in an harmonic oscillator form and write

$$I_2 = \int_{-\beta/2}^{\beta/2} dt \frac{1}{2} \left[\left(\frac{dx}{dt} \right)^2 + \omega^2 x^2 \right] \tag{4.1.3}$$

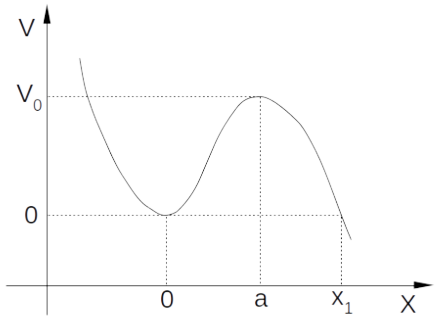
If it is a true ground state, the only classical solution of the equation of motion

$$\frac{d^2x}{dt^2} - \frac{dV(x)}{dx} = 0 \tag{4.1.4}$$

with boundary condition $x_i = x_f = 0$ will be $x(t) = 0$. Then the transition amplitude from the state $x = 0$ to itself will be

$$e^{-I_{cl}} \int D[x] e^{-I_2[x]} \approx \mathcal{N} \det \left[-\frac{d^2}{dt^2} + \omega^2 \right] \approx \left(\frac{\omega}{\pi} \right)^{1/2} e^{-\beta\omega/2} \tag{4.1.5}$$

in the big β limit. This expression corresponds with the ground state energy of the harmonic oscillator and, returning to the real time $it = \beta$, we have $| \langle x = 0, t_2 | n = 0, t_1 \rangle |^2 \approx \frac{\omega}{\pi}$, where $|n = 0\rangle$ is the ground state of the system, since the exponential is pure imaginary. That respects our expectation from the harmonic oscillator ground state.
But, if we consider a potential with a barrier of height $V(a) = V_0$ and where $x = 0$ is just a local minimum $V(0) = 0$, there will be other classical solutions



of the equation of motion around which make a saddle point expansion.
Equation (4.1.4) is clearly the usual equation of motion of a particle in a potential field $-V(x)$, then the new periodic classical solution (or instanton) will be the trajectory $x(t)$ where the particle goes down from the point $x = 0$ and return back after passing through the barrier and reaching a point x_1 with the same potential energy $V(x_1) = V(0)$. The euclidean energy $E = -\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + V(x) = 0$ is a constant of motion, hence we can compute the period needed to pass under the potential barrier and return to the point $x = 0$, that results to be $\tau = \int_0^{x_1} dx(2V)^{-1/2} = \infty$. So, with $\beta \to 0$, the instanton can be included in the path integral. We will call B the classical action of this solution, that can be simply computed with

$$B = \int_0^{x_1} dx\sqrt{2V(x)} = \int_{-\beta/2}^{\beta/2} dt \left(\frac{dx}{dt}\right)^2 \tag{4.1.6}$$

Moreover, for $|t|$ big, the solution is approximately $\bar{x}(t) = e^{-\omega|t|}$, if the instant $t = 0$ is set in correspondence with the centre of the bounce. Thus, the instanton is substantially localised in a region of order $\frac{1}{\omega}$, while in the rest of the time, the particle remains in the local minimum $x = 0$. this feature permits us to account in the integration every path with an arbitrary number of bounces in any possible instant, since it is a good approximation of an action's minimum. The resulting transition amplitude will be [37][38]

$$\Sigma_{n=0}^{\infty} e^{-\beta\omega/2} \frac{(Ke^{-B}\beta)^n}{n!} = \exp\bigl(-\beta\omega/2 + Ke^{-B}\beta\bigr) \tag{4.1.7}$$

where K is the result of the functional integration of variations near to a path with one bounce.
We would expect $K \propto \det\left[-\frac{d^2}{dt^2} + \frac{d^2V(\bar{x}(t))}{dx^2}\right]$ and if would be so the contribution of the bounce would be negligible in the high β limit, but the operator $-\frac{d^2}{dt^2} + \frac{d^2V(\bar{x}(t))}{dx^2}$ has a negative mode, then the Gaussian integral that gives the determinant is not well definite. In fact the path $\delta x(t) = \frac{d\bar{x}}{dt}$

is zero on the extrema and has the property

$$\left[-\frac{d^2}{dt^2}+\frac{d^2V(\bar{x}(t))}{dx^2}\right]\delta x(t)=\frac{d}{dt}\left[-\frac{d^2\bar{x}}{dt^2}+\frac{dV(\bar{x}(t))}{dx}\right]=0 \tag{4.1.8}$$

thanks to the equation of motion. The variation $\frac{d\bar{x}}{dt}$ is substantially the result of an infinitesimal time translation and changes sign during the bounce, then it has a zero. Hence,there must exist an eigenfunction of the operator $-\frac{d^2}{dt^2}+\frac{d^2V(\bar{x}(t))}{dx^2}$ that doesn't change sign and with an eigenvalue smaller than zero. The presence of a negative eigenvalue forces us to consider complex path in the functional integration and gives an imaginary part to K that has a fundamental role. After some calculation one obtains

$$\operatorname{Im}K=\frac{1}{2}\sqrt{B/2\pi}\left|\frac{\det'\left[-\frac{d^2}{dt^2}+\frac{d^2V(\bar{x}(t))}{dx^2}\right]}{\det\left[-\frac{d^2}{dt^2}+\omega^2\right]}\right|^{-1/2} \tag{4.1.9}$$

where $\det'$ means zero eigenvalues are excluded from the determinant. This reduced determinant appears because the summation over time translations and solutions with any number of bounces n is equivalent to accounting the translational zero mode of the operator.
An important aspect is that the treatment by GPY is true if there exists only one negative eigenvalue, since, if they are more, one have to consider their complex product in the determinant evaluation. If we consider again the real time transition amplitude, we have a factor $e^{-\beta\Gamma}$ with $\Gamma=e^{-B}\operatorname{Im}K$ that is the decay probability per unit time of the false ground state. After an explicit evaluation of determinants, one can find it is also equal to $-2\operatorname{Im}E_0$, the well known result of quantum tunnelling in quantum mechanics. Now, if we extend this result to the entire ensamble at low temperature $1/\beta$, the tunnelling probability will be proportional to the imaginary part of the average energy, so $\Gamma=-2\operatorname{Im}F$ where F is the free energy computed with the path integral.
If the temperature is not zero, the instanton individuated above doesn't respect the periodicity condition, since it need unlimited time to return to $x=0$, consequently quantum tunnelling from the point $x=0$ can't happen. Anyway quantum tunnelling is not the only possible phenomenon, given that at higher temperatures also thermal excitation permits to cross the barrier. In this case we can't restrict our research to the classical low energy solution $x(t)=0$ and we have to consider also solutions with energy $E>0$. When $\beta<\frac{2\pi}{-V''(a)}$, the period of the bounces with any energy in the interval $[0,V_0]$

$$\tau(E)=\int_{x_3}^{x_2}dx[2V(x)-E]^{-1/2} \tag{4.1.10}$$

is bigger than the period imposed by the imaginary time, and this type of solution degenerate to the static solution $\bar{x}(t)=a$. This means at high temperature quantum tunnelling is suppressed. The static instanton has a very simple classical action βV_0 and its contribution to the imaginary part of the free energy is

$$\mathrm{Im}\, F = \frac{1}{2} e^{-\beta V_0} \left| \frac{\det' \left[-\frac{d^2}{dt^2} + \frac{d^2 V(\bar{x}(t))}{dx^2} \right]}{\det \left[-\frac{d^2}{dt^2} + \omega^2 \right]} \right|^{-1/2} \tag{4.1.11}$$

because a is a local maximum of the potential and its second derivative is negative. Finally, in the thermal process, $\Gamma = \frac{V''(a)\beta}{\pi} \,\mathrm{Im}\, F$.
While the tunnelling process represents a particle that passes from one classically allowed region to the other, the thermal barrier crossing represents the probability to have a thermally excited particle on the top of the barrier, that immediately after will roll down one of the two sides of the wall.

The quantum gravity case

It's time to consider the gravitational case. The Minkowski spacetime is surely stable at classical level, since the ADM energy defined in the first chapter is always ≥ 0 in pure gravity and furthermore $E_{ADM} = 0$ only in the Minkowski space if we consider asymptotically flat spacetimes. At the quantum level the situation is more complicated and depends on the temperature $\frac{1}{\beta}$ of the euclidean path integral.
At zero temperature the boundary conditions of the euclidean path integral are called asymptotically Euclidean (AE). An AE spacetime is approximately flat outside a compact subset of the four dimensional manifold. The positive action theorem states that any AE spacetime which has also $R = 0$ must have euclidean action $I \geq 0$. In particular it has $I = 0$ if and only if the spacetime is flat. A consequence is that the only AE instanton of the euclidean gravitational action is the flat space. This is true since the action must be invariant under diffeomorphisms, while the transformation of coordinates $x \to \lambda x$ acts on the metric as $g_{\mu\nu} \to \lambda g_{\mu\nu}$. So the boundary term of the action, the only one remaining when $R = 0$, behaves in the same manner. The result is an action that scales with λ, but this is compatible with the diffeomorphism invariance only if $I = 0$, then the spacetime is flat. With a saddle point expansion near the flat space, The 4-dimensional Lichnerowicz operator Δ_L is reduced to the Laplacian and then is positive definite and the Gaussian functional integral can be carried out without the problem of negative modes.
The zero temperature flat spacetime is stable at first loop, however something different happens when one considers the finite temperature case. Now the boundary conditions are called asymptotically flat and they demand the spacetime to be periodic in time with period β and asymptotically flat in its spatial submanifold. the result is a spacetime that approaches the flat metric

with topology $S^1 \times R^3$ outside a compact subset. The semiclassical calculation in the saddle point approximation near the flat instanton is not much different from the zero temperature case, except for the periodicity condition over the eigenfunctions of quadratic operators. If we would consider higher orders in the expansion we should account the graviton-graviton interaction in the thermal gas of gravitons that fills the spacetime when $T > 0$, and this would produce an unrenormalizable ultraviolet divergence and an infrared divergence that is the quantum equivalent of Jeans instability.
Anyway, from our point of view, the most relevant aspect of the finite temperature perturbation theory is that in this case there are other instantons different from the trivial flat space, like the Kerr black hole and in particular the Schwarzschild solution. The classical action of the Schwarzschild black hole is, for an arbitrary mass M,

$$I = \frac{1}{2}M\beta + 2GM^2\left(\frac{\beta}{4MG} - 2\pi\right) \tag{4.1.12}$$

that is zero with null mass and has a maximum $I = 4\pi GM^2$ for $M = \frac{\beta}{8\pi G}$. After this value the action decreases to $-\infty$. The particular value of the black hole mass that maximise the action is also relevant since the request of periodicity of the spacetime over imaginary time brings to a conical singularity in $r = 2GM$ for any other mass. This behaviour of the action arouses the suspicion that the Schwarzschild solution of the equation of motion could behave in a way similar to the static bounce $x(t) = a$ in the latter example. If one studies the operator Δ_L with a decomposition in radial modes and spherical harmonics similar to what we have done with its 3-dimensional form in the preceding chapter, will discover all odd variations and also even ones with $l > 1$ have positive eigenvalues, while there is an even eigenvector of the radial operator with $l = 1$ which generates spatial translations of the black hole and has zero eigenvalue. It remains to consider the $l = 0$ even modes and, from a numerical computation, we know Δ_L has one negative mode with eigenvalue $\lambda \approx -0,19(GM)^{-2}$ in this sector. Thus,the free energy acquires an imaginary part as happened in the latter example. According to Coleman [39], this is a signal of a transition from a false vacuum to a true one.
In the toy model we had to consider as different paths all possible time translations of the bounce and also paths with multiple instantons, the same procedure with collective coordinates must be done with the Schwarzschild black hole, where now are taken in account all 3-dimensional translations. this process permits again to exclude zero modes from functional determinant.
The first loop evaluation in curved spacetime has an UV divergence that is proportional to the Euler character of the manifold. This topological property is null in the flat space but equal to 2 in the Schwarzschild instanton.

So it's necessary the introduction of a regulator mass μ. The result is a decay rate of the flat spacetime

$$\Gamma \approx \frac{0,87}{\beta}(\mu\beta)^{212/45}\frac{1}{64(G\pi^2)^{3/2}}e^{-\frac{\beta^2}{16\pi G}} \tag{4.1.13}$$

The instability is absent at $T = 0$, then it isn't a tunnelling process. the imaginary part of the free energy means there is a non null probability in the flat euclidean spacetime to nucleate a black hole with mass $M = \frac{\beta}{8\pi G}$. Once created, the singularity will grow or evaporate depending on the temperature of the surrounding matter and its thermal Hawking radiation.

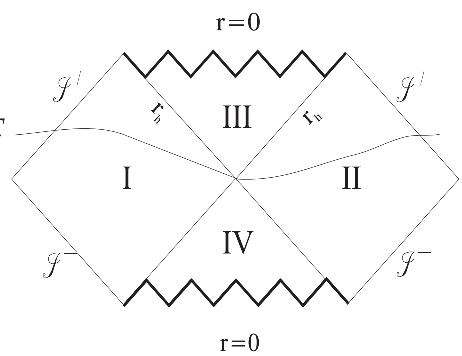
4.2 Hamiltonian research of a ground state

Until now, we have seen quantum instability just as a thermal phenomenon. However, if we consider also spacetimes with different topologies, something similar can happen also without considering temperature.
The ADM formalism is defined in such a way to give zero classical energy to the Minkowski spacetime, moreover, with a Minkowskian background, also the spatial submanifold identified by $t = const$ is an Einstein space and it has a spherical symmetry, so we can easily apply the formalism described above in order to make first order quantum calculations.
The effective potential defined in the previous chapter $m(r)$ is trivially null and the trace transverse energy density is just

$$-\Lambda_q^{TT}(r) = \frac{G}{\pi}\int_0^\infty d\omega\omega^3 \tag{4.2.1}$$

that gives a ω^4 divergence but, after the renormalization, the energy density (3.3.15) is null. So we can state that the Minkowski spacetime has exactly null energy in this framework.

The Schwarzschild wormhole



As we have learnt from the path integral, the Schwarzschild solution has a fundamental role in the stability of Minkowski spacetime is semiclassical gravity. In this section we will try to treat this problem with the canonical quantization and we will try to evaluate the ZPE of the extended Krustal-Schwarzschild solution and compare it with the one loop energy of the Minkowski spacetime.
With the Krustal null coordinates (U, V) substituting t and r, the line element has the form

$$ds^2 = -\frac{32(MG)^3}{r} \exp\left(-\frac{r}{2MG}\right) dU dV + r^2 d\Omega_2^2 \tag{4.2.2}$$

. The usual coordinates t and r can be evaluated from the relations

$$\left(1 - \frac{r}{2MG}\right) \exp\left(-\frac{r}{2MG}\right) = UV \tag{4.2.3}$$

$$t = 2MG \in \left|\frac{-V}{U}\right| \tag{4.2.4}$$

This spacetime is the union of four regions, or wedges, R_+, R_-, T_+ and T_-. The regions R_+ and R_- are asymptotically flat and we have $U < 0$, $V > 0$ in R_+ and $U > 0$ and $V < 0$ is R_-. On the other side the regions T_+ and T_- represent respectively the inner region of the black hole and the white hole.
The line element (4.2.2) is also invariant respect to the symmetries

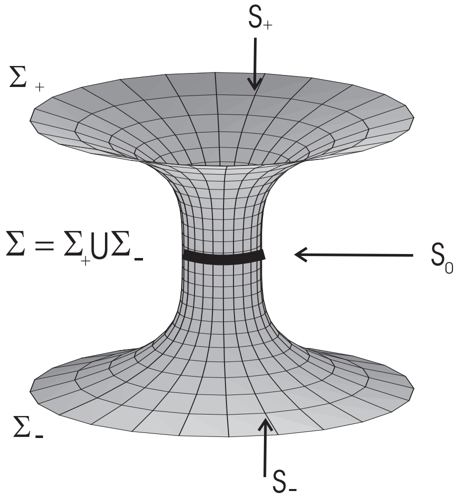
$$I : U \to -U,\ V \to -V;\ L : U \to -V, V \to -U \tag{4.2.5}$$

that means the spacetime is invariant respect to the bifurcation surface S_0, the intersection of the future and past horizons H_+ and H_-. We can construct a spatial foliation of the eternal Schwarzschild black hole, such that the manifold with constant t Σ_t has the topology $R^1 \times S^2$ of the Einstein-Rosen Bridge and the metric

$$ds^2 = dx^2 + r^2(x) d\Omega_2^2 \tag{4.2.6}$$

It has substantially the structure of the spatially symmetric spacetime (3.2.24) with $b = 2MG$, except of x that now runs from $-\infty$ to ∞. The regions with different sign correspond to submanifolds of the two asymptotically flat regions, which are in touch through the bifurcation surface S_0 in $x = 0$ or equivalently $r = 2MG$. This surface in the Einstein-Rosen topology is usually called the throat of the wormhole. We denote by $\Sigma_\pm$ the the part of Σ lying in $R_\pm$ respectively. Consequently the radial coordinates in the two different sectors are defined by

$$dx = \pm \frac{dr}{\sqrt{1 - 2MG/r}} \tag{4.2.7}$$



Moreover all the hypersurfaces Σ_t are invariant respect to transformation L. In this calculation we want to consider only a part of the total Krustal space-time, so we fix some spatial and temporal boundaries in the AF regions. We take a temporal interval $[t', t"]$ and we consider the foliation Σ_t of the region included between $\Sigma_{t'}$ and $\sigma_{t"}$. On the spatial side we fix a three dimensional boundary B composed by an hypersurface B_+ with constant $x_+ \geq 2MG$ in R_+ and an analogue manifold B_- characterised by $x = x_- \leq -2MG$ embedded in R_-. We will call the resulting region bounded by those hypersurfaces $\mathcal{M} = \mathcal{M}_+ \cup \mathcal{M}_-$ while the spatial boundary of a time slice is composed by $S^2_\pm = \Sigma_\pm \cap B_\pm$. Thus,we have a foliation of hypersurfaces with structure $I_\Sigma \times S^2$ where I_Σ is a spacelike interval, on the other hand the boundaries $B_\pm$ have topologies $I_t \times S^2_\pm$, where I_t is a finite timelike distance.
The classical energy is the integral of expression (1.3.36) evaluated over both the boundaries of the spatial manifold $S^2_\pm$. The timelike unitary vector normal to the spacelike foliation is $u^\mu = \frac{1}{N}\delta^\mu_t$, where the lapse function is $N = \sqrt{1 - \frac{2MG}{r(x)}}$ in $\mathcal{M}_+$ and $-N$ in $\mathcal{M}_-$. On the other side the outgoing spatial normal vector on the boundaries $S^2_\pm$ is equal to $n^\mu = \pm\delta^\mu_x$. That implies the two normal vectors are orthogonal ($n \cdot u = 0$) all over the boundaries and the quasi-local energy can be reduced to the expression

$$E_{ql} = E_+ - E_- = -\frac{1}{(8\pi G)}\int_{S^2_+} d^2x\sqrt{\sigma}(\Theta - \Theta_0)_+ + \frac{1}{(8\pi G)}\int_{S^2_-} d^2x\sqrt{\sigma}(\Theta - \Theta_0)_- \tag{4.2.8}$$

In the latter expression σ is the determinant of the submanifolds $S^2_\pm$, which inherit the induced metric

$$\sigma_{ij} = diag[r(x)^2, r^2(x)\sin^2\theta] \tag{4.2.9}$$

So $\sqrt{\sigma} = r^2(x)\sin\theta$, while the trace of the extrinsic curvature $\Theta_\pm$ has the form

$$\Theta_{\pm} = \sigma^{ij}\Theta_{\pm ij} = \nabla \cdot n = \frac{1}{\sqrt{\sigma}}\partial_i(\sqrt{\sigma}n^i) = \pm\frac{1}{r^2(x)}\partial_x(r^2(x)) = \frac{2\sqrt{1-2MG/r}}{r(x)} \tag{4.2.10}$$

A flat space with the same topology would have an equal boundary metric with $r(x) = |x|$, hence the extrinsic curvature subtracted is

$$\Theta_{0\pm} = \frac{2}{r} \tag{4.2.11}$$

We have found a quasi-local energy that receives contributions of opposite sign from the wedges $\mathcal{M}\pm$ and depends from the mass associated to the black hole and the position of the spatial boundaries

$$E_{ql}(r) = \frac{1}{G}\{[r(1-\sqrt{1-2MG/r})]_{x=x_+} - [r(1-\sqrt{1-2MG/r})]_{x=x_-}\} \tag{4.2.12}$$

Booth the terms $E_{\pm}$ approach to the ADM mass M for $x_{\pm} \to \pm\infty$, while they assume a value near to $2M$ when $x_{\pm} \to 0$. However if $x_{\pm} = 0$ the lapse function is null and the energy itself is zero, because doesn't exist a reparametrization with $|N| = 1$.
This is a quite interesting result: if one considers spacetimes with "doubly flat" asymptotic behaviour, that means there are two disconnected asymptotically flat regions, it is possible to find configurations with zero Brown-York energy different from a couple of separated flat Minkowski spacetimes.
In the next step we will consider quantum contributions. Thanks to the sign change in N, the total Hamiltonian can be written as

$$H_{tot} = E_{ql} + \int_{\mathcal{M}_+} d^3xN(r(x))H - \int_{\mathcal{M}_-} d^3xN(r(x))H \tag{4.2.13}$$

after a functional derivative respect to N we obtain the WDW equation

$$\begin{cases} H = 0 \; for \; x > 0 \\ -H = 0 \; for \; x < 0 \end{cases} \tag{4.2.14}$$

Obviously the two wedges behave in the same manner, hence if we consider the Sturm-Liouville problem (3.0.18) we obtain

$$\begin{aligned} -\Lambda_q V_{\mathcal{M}} &= 4\pi\frac{\langle\Psi|\left[\int_0^{x_+} dxr^2(x)\hat{\Lambda}_{\Sigma}^{(2)} + \int_{x_-}^0 dxr^2(x)\hat{\Lambda}_{\Sigma}^{(2)}\right]|\Psi\rangle}{\langle\Psi|\Psi\rangle} \\ &= 4\pi\frac{\langle\Psi|\left(\int_0^{x_+} + \int_0^{x_-}\right)dxr^2(x)\hat{\Lambda}_{\Sigma}^{(2)}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \end{aligned} \tag{4.2.15}$$

The equation for the quantum contributions to the cosmological constant has been reduced to the sum of two terms with a spherical symmetry, so we can try to apply the formalism explained above.

Even modes

With $b = 2MG$, according with the Regge-Wheeler representation, the spherical harmonics decomposition of the even perturbations of the metric has the form

$$h^i_j = \begin{pmatrix} H(r) & \frac{r-2MG}{r}h_1(r)\partial_\theta & \frac{r-2MG}{r}h_1(r)\partial_\phi \\ \frac{1}{r^2}h_1(r)\partial_\theta & K(r)+G(r)\partial^2_\theta & G(r)(\partial_\theta\partial_\phi - \cot\theta\partial_\phi) \\ \frac{1}{r^2\sin^2\theta}h_1(r)\partial_\phi & \frac{1}{\sin^2\theta}G(r)(\partial_\theta\partial_\phi - \cot\theta\partial_\phi) & K(r)+G(r)(\frac{1}{\sin^2\theta}\partial^2\phi + \cot\theta\partial_\theta) \end{pmatrix} Y_{l,m}(\theta,\phi) \tag{4.2.16}$$

In a spacetime with spherical symmetry, energy do not depends on angular components, thus we can fix $m = 0$ and all ϕ derivatives become null. the result is the mixed tensor

$$h^i_j = \begin{pmatrix} H(r) & \frac{r-2MG}{r}h_1(r)\partial_\theta & 0 \\ \frac{1}{r^2}h_1(r)\partial_\theta & K(r)+G(r)\partial^2_\theta & 0 \\ 0 & 0 & K(r)+G(r)\cot\theta\partial_\theta \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.17}$$

We have reduced the metric perturbations to four independent radial functions, and we can still impose the gauge choice and the traceless condition in order to reduce the object of our studies to the traceless-transverse sector. The null trace request and the gauge fixing condition $\nabla_i h^i_j = 0$ can be represented by three equations

$$H + 2K - l(l+1)G = 0 \tag{4.2.18}$$

$$-\left(\partial_r + \frac{3}{r}\right)H + \frac{1}{r^2}h_1 l(l+1) = 0 \tag{4.2.19}$$

$$\left(-\frac{3MG}{r^2} + \frac{2}{r}\right)h_1 + \left(1 - \frac{2MG}{r}\right)\partial_r h_1 + K + G[1 - l(l+1)] = 0 \tag{4.2.20}$$

Since we have four unknown functions and three constraints, we expect to reduce the radial dependence to only one function. The second degree of freedom expected to be found in the graviton representation is carried by the ϕ-dependent angular part, which do not influences the energy expectation value.
The eigenvalue problem we have to solve (3.2.18) is

$$\begin{aligned}\left[-\frac{r-2MG}{r}\partial^2_r - \frac{2r-3MG}{r^2}\partial_r + \frac{1}{r^2}l(l+1) + 4\frac{r-2MG}{r^3} - \frac{8MG}{3r^3}\right]H(r) - \\ -2\frac{r-\frac{4}{3}MG}{r^3}(2K - l(l+1)G) - 4\frac{r-2MG}{r^4}l(l+1)h_1 - (L\eta)^1_1 = \lambda H\end{aligned} \tag{4.2.21}$$

$$\left[-\frac{r-2MG}{r}\partial_r^2-\frac{2r-3MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)+2\frac{r-2MG}{r^3}+\right.$$
$$\left.+\frac{4MG}{3r^3}\right](K+G\partial_\theta^2)-4\frac{r-2MG}{r^4}l(l+1)h_1\partial_\theta^2-\frac{2}{r^2}G(\partial_\theta^2-\cot^2\theta\partial_\theta)+$$
$$+\left(\frac{4MG}{3r^3}-2\frac{r-2MG}{r^3}\right)H-\frac{2MG}{3r^3}(K+G\cot\theta\partial_\theta)-(L\eta)_2^2=\lambda(K+G\partial_\theta^2) \tag{4.2.22}$$

$$\left[-\frac{r-2MG}{r}\partial_r^2-\frac{2r-3MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)+2\frac{r-2MG}{r^3}+\right.$$
$$\left.+\frac{4MG}{3r^3}\right](K+G\cot\theta\partial_\theta)-4\frac{r-2MG}{r^4}l(l+1)h_1\cot\theta\partial_\theta+$$
$$+\frac{2}{r^2}G(\partial_\theta^2-\cot^2\theta\partial_\theta)+\left(\frac{4MG}{3r^3}-2\frac{r-2MG}{r^3}\right)H-\frac{2MG}{3r^3}(K+G\partial_\theta^2)-$$
$$-(L\eta)_3^3=\lambda(K+G\cot\theta\partial_\theta) \tag{4.2.23}$$

$$\left[-\frac{r-2MG}{r}\partial_r^2-\frac{4r-5MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)+\frac{2r-7MG}{r^3}\right]\frac{1}{r^2}h_1\partial_\theta+$$
$$+\frac{2}{r^3}H\partial_\theta+\frac{2}{r^3}G\cot\theta(\partial_\theta^2-\cot^2\theta\partial_\theta)+\frac{2}{r^3}(K\partial_\theta+G\partial_\theta^3)-(L\eta)_1^2=\lambda\frac{1}{r^2}h_1\partial_\theta \tag{4.2.24}$$

We can immediately observe that the null trace condition is compatible with the first three equations of the eigenvalue problem, in fact if we sum these expressions we obtain

$$\left[-\frac{r-2MG}{r}\partial_r^2-\frac{2r-3MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)\right](H+2K-l(l+1)G)$$
$$=\lambda(H+2K-l(l+1)G) \tag{4.2.25}$$

thanks to the null trace of $(L\eta)^i_j$. If also the two differential constraints coming from the request of transversality are compatible with the latter system of equations, we can reduce it to just one relevant equation. If we substitute (4.2.18) and (4.2.19) in (4.2.21), we have

$$\left[-\frac{r-2MG}{r}\partial_r^2-\frac{2r-3MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)+6\frac{r-2MG}{r^3}-\frac{2MG}{r^3}\right]H$$
$$-4\frac{r-2MG}{r^2}\left(\partial_r+\frac{3}{r}\right)H-(L\eta)_1^1=\lambda H \tag{4.2.26}$$

Except for the contribution of $(L\eta)_1^1$, which is an unknown tensor at the moment, we have reduced the problem to a differential equation respect to

only one radial function $H(r)$.
We need to write $(L\eta)_{ij}$ as a function of h_{ij}^{TT}, its derivatives and R_{ij}. The right side of equation (3.2.17) for the even mode is

$$\nabla_j \Delta^j_{TT\ a} h_i^{TT\ a} = \begin{pmatrix} \frac{MG}{r^3}\left(2\partial_r + \frac{3}{r}\right) H \\ -\frac{4MG}{r^3} H\partial_\theta + \frac{6MG}{r^4}\left[\frac{MG}{r} + (r-2MG)\partial_r\right] h_1 \partial_\theta \\ 0 \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.27}$$

Also the vector η_i can be decomposed in an even and an odd part, the even part with $m = 0$ can be represented with the form

$$\eta_i = \begin{pmatrix} U \\ V\partial_\theta \\ 0 \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.28}$$

Now we can work on the left side of (3.2.17).

$$\nabla^j (L\eta)_{ij} = -\Delta\eta_i + \nabla^j \nabla_i \eta_j - \frac{2}{3}\nabla_i \nabla_j \eta^j = -\Delta\eta_i + \frac{1}{3}\nabla_i \nabla_j \eta^j + R_i^j \eta_j \tag{4.2.29}$$

One can notice the second term of the last expression is a covariant derivative of a scalar, that can be substituted by a simple partial derivative

$$\nabla_i \nabla_j \eta^j = \partial_i \nabla_j \eta^j = \partial_i \left[\frac{2r-3MG}{r^2} U + \frac{r-2MG}{r}\partial_r U - \frac{l(l+1)}{r^2} V\right] \tag{4.2.30}$$

The three-dimensional Laplacian of η is

$$-\Delta\eta_i = \nabla_k \nabla^k \eta_i = \begin{pmatrix} \left[\frac{r-2MG}{r}\partial_r^2 + \frac{2r-MG}{r^2}\partial_r - \frac{1}{r^2} l(l+1) - 2\frac{r-2MG}{r^3}\right] U + \frac{2l(l+1)}{r^3} V \\ \left[\frac{r-2MG}{r}\partial_r^2 + \frac{MG}{r^2}\partial_r - \frac{1}{r^2} l(l+1) + \frac{MG}{r^3}\right] V\partial_\theta + 2\frac{r-2MG}{r^2} U\partial_\theta \\ 0 \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.31}$$

If we put together all these pieces we find two equations

$$\begin{aligned} &\left[\frac{4(r-2MG)}{3r}\partial_r^2 + \frac{8r-4MG}{3r^2}\partial_r - \frac{l(l+1)}{r^2} - \frac{8r-12MG}{3r^3}\right] U + \\ &\qquad + \frac{l(l+1)}{r^2}\left(\frac{4}{3r} - \frac{1}{3}\partial_r\right) V = \frac{MG}{r^3}\left(2\partial_r + \frac{3}{r}\right) H \end{aligned} \tag{4.2.32}$$

$$\begin{aligned} &\left[\frac{r-2MG}{r}\partial_r^2 + \frac{MG}{r^2}\partial_r - \frac{4}{3r^2} l(l+1) + \frac{2MG}{r^3}\right] V + \\ &\qquad + \left(\frac{8r-15MG}{3r^2} + \frac{r-2MG}{3r}\partial_r\right) U = \\ &= -\frac{4MG}{r^3} H + \frac{6MG}{r^4}\left[\frac{MG}{r} + (r-2MG)\partial_r\right] h_1 \end{aligned} \tag{4.2.33}$$

Odd modes

On the other hand, the odd modes with $m = 0$ have the form

$$h^i_j = \begin{pmatrix} 0 & 0 & \frac{r-2MG}{r} f_1(r) \sin\theta\partial_\theta \\ 0 & 0 & -\frac{1}{2r^2} f_2(r)\sin\theta l(l+1) \\ \frac{1}{r^2\sin\theta} f_1(r)\partial_\theta & -\frac{1}{2r^2\sin\theta} f_2(r) l(l+1) & 0 \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.34}$$

When we consider odd perturbations, the diagonal components in spherical coordinates are null, then the total contraction between the Ricci tensor and h^{TT}_{ij} is null and the third term in the divergence (3.2.16) is also null.
The request of null trace is automatically fulfilled, while transversality condition can be reduced to just one equation

$$\left[\left(\partial_r - \frac{MG}{r^(r-2MG)} + \frac{1}{r}\right)(r-2MG)r + (r-2MG)\right]\frac{1}{r^2\sin\theta} f_1(r)\partial_\theta - \\ -(\partial_\theta + 3\cot\theta)\frac{1}{2r^2\sin\theta} f_2(r) l(l+1) = 0 \tag{4.2.35}$$

the eigenvalue problem can be reduced again to only one equation, in particular we will consider the eigenvalue equation for the component $h_1^{TT\ 3}$

$$\left[-\frac{r-2MG}{r}\partial_r^2 - \frac{4r-5MG}{r^2}\partial_r + \frac{1}{r^2}l(l+1) + \frac{2r-10MG}{r^3}\right]\frac{1}{r^2\sin\theta} f_1(r)\partial_\theta + \\ + \frac{2}{r^3}(\partial_\theta + 3\cot\theta)\frac{1}{2r^2\sin\theta} f_2(r) l(l+1) - (L\eta)^3_1 = \lambda\frac{1}{r^2\sin\theta} f_1(r)\partial_\theta \tag{4.2.36}$$

An easy substitution of equation (4.2.35) in the last expression gives

$$\left[-\frac{r-2MG}{r}\partial_r^2 - \frac{2r-MG}{r^2}\partial_r + \frac{1}{r^2}l(l+1) + \frac{8r-20MG}{r^3}\right]\frac{1}{r^2\sin\theta} f_1(r)\partial_\theta - \\ -(L\eta)^3_1 = \lambda\frac{1}{r^2\sin\theta} f_1(r)\partial_\theta \tag{4.2.37}$$

The result is very similar to the even mode: we have an eigenvalue equation with the unknown contribution of $(L\eta)^3_1$.
In this case the odd part of the vector field is

$$\eta_i = \begin{pmatrix} 0 \\ 0 \\ D(r)\sin\theta\partial_\theta \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.38}$$

and $(L\eta)^3_1$ is equal to

$$(L\eta)^3_1 = (\partial_r - \frac{2}{r})D(r)\sin\theta\partial_\theta Y_{l,0}(\theta) \tag{4.2.39}$$

Also in this case we should find an expression for $D(r) = D(f_1(r))$ from the equation (3.2.17), but in the odd mode $\nabla \cdot \eta = 0$ for each function D, then the left side of the equation is reduced to

$$\nabla^j (L\eta)_{ij} = -\Delta\eta_i + R_i^j \eta_j \tag{4.2.40}$$

The Laplacian of the odd vector is

$$-\Delta\eta_i = \nabla_k \nabla^k \eta_i = \begin{pmatrix} 0 \\ 0 \\ \left[\frac{r-2MG}{r}\partial_r^2 - \frac{l(l+1)}{r^2} + \frac{MG}{r^2}\partial_r + \frac{MG}{r^3}\right] D(r) \sin\theta \partial_\theta \end{pmatrix} Y_{l,0}(\theta) \tag{4.2.41}$$

Regarding the right side of eq (3.2.17), the total contraction between the Ricci tensor and the odd mode of the metric $R_k^j h_j^{TT\ k}$ is zero, since R_{ij} is diagonal, while $h_j^{TT\ k}$ has only off diagonal terms. That means (3.2.16) receives only contributions from the first two terms. The result is

$$\begin{aligned} &\nabla_j \Delta^j_{TT\ a} h_i^{TT\ a} = \\ &= \begin{pmatrix} 0 \\ 0 \\ \frac{4MG}{r^3}\left[\partial_r \frac{r-2MG}{r} - \frac{MG}{r^2} - \frac{r-2MG}{r^2}\right] f_1(r) \sin\theta\partial_\theta + \frac{MG}{r^5}\sin^2\theta(\partial_\theta + 3\cot\theta)\frac{l(l+1)}{\sin\theta} f_2(r) \end{pmatrix} Y_{l,0}(\theta) \end{aligned} \tag{4.2.42}$$

We can immediately remove f_2 from the last equation thanks to the relation (4.2.35).
The eq (3.2.17) is finally reduced to

$$\begin{aligned} &\frac{MG}{r^3}\left[\partial_r \frac{r-2MG}{r} - \frac{MG}{r^2}\right] f_1(r) = \\ &\qquad = \left[\frac{r-2MG}{r}\partial_r^2 - \frac{l(l+1)}{r^2} + \frac{MG}{r^2}\partial_r + \frac{2MG}{r^3}\right] D(r) \end{aligned} \tag{4.2.43}$$

Near the throat

One could try to consider The operator Δ_{TT} as a perturbed version of the Lichnerowicz operator Δ_L and find an eigenfunction basis of the latter operator.
The original Lichnerowicz operator in $3Dim$ is

$$(\triangle_L h)_{ij} = -\nabla^a \nabla_a h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h_j^k + R_{jk}h_i^k. \tag{4.2.44}$$

This is traceless and for the divergence, one gets

$$\nabla^i (\triangle_L h)_{ij} = -\nabla^k \nabla_k \left(\nabla^i h_{ij}\right) + R_j^l \left(\nabla^i h_{il}\right) - \left(\nabla_j R_k^l\right) h_l^k. \tag{4.2.45}$$

While the first two terms vanish for the null divergence condition on h^{TT}, for the third one we get for j = 1

$$\frac{9h_1^1(r,\theta,\phi)MG}{r^4} \tag{4.2.46}$$

while for $j = 2$, we have

$$-\frac{3h_2^1\,(r,\theta,\phi)\,MG}{r^4} \tag{4.2.47}$$

and, finally for $j = 3$, we obtain

$$-\frac{3h_3^1\,(r,\theta,\phi)\,MG}{r^4} \tag{4.2.48}$$

The equations for the even mode are simpler, but they still need a vector field η_i.
We have are working in WKB approximation, under the hypothesis that the derivative of the potential is negligible. This is equivalent to state $\left(\nabla_j R_k^l\right) h_l^k \approx 0$. If we want to have a non null potential in this approximation, we need, on the diagonal components, $R_i^i \gg \partial_r R_i^i$, that means $M \gg 1$ in plank units. As expected, the semiclassical approach loses validity at Planck scale, since we would need a complete quantum theory, however we can hope to obtain some meaningful qualitative information in the quasi Planck region.
Following this procedure, we find the TT sector to be Δ_L invariant. Now we have an operator which can be studied with an eigenvalue equation in the TT space and we would like to introduce the difference between Δ_{TT} and Δ_L as a perturbation depending on the parameter γ.

$$\begin{aligned}\left[-\frac{r-2MG}{r}\partial_r^2 - \frac{2r-3MG}{r^2}\partial_r + \frac{1}{r^2}l(l+1) + 6\frac{r-2MG}{r^3} + \frac{2MG}{r^3}\right] H \\ -4\frac{r-2MG}{r^2}\left(\partial_r + \frac{3}{r}\right) H - \gamma\frac{4MG}{r^3}H = \lambda H\end{aligned} \tag{4.2.49}$$

Passing to the new variable $H = h/r$ we have

$$\begin{aligned}\left[-\frac{r-2MG}{r}\partial_r^2 - \left(2\frac{r-2MG}{r^2} + \frac{2r-3MG}{r^2}\right)\partial_r + \frac{1}{r^2}l(l+1)\right. \\ \left. -6\frac{r-2MG}{r^3} + \frac{3MG}{r^3}\right] h - \gamma\frac{4MG}{r^3}h = \lambda h\end{aligned} \tag{4.2.50}$$

and, respect to the proper distance x

$$\begin{aligned}\left[-\partial_x^2 - 4\sqrt{\frac{r-2MG}{r^2}}\partial_r + \frac{1}{r^2}l(l+1)\right. \\ \left. -6\frac{r-2MG}{r^3} + \frac{3MG}{r^3}\right] h - \gamma\frac{4MG}{r^3}h = \lambda h\end{aligned} \tag{4.2.51}$$

We expect to observe quantum effects mainly in the region near the throat, where gravity is stronger. Hence, for $r \approx 2MG$,

$$\left[-\partial_x^2 + \frac{1}{2MG^2}l(l+1) + \frac{3}{8(MG)^2}\right] h - \gamma\frac{2}{(2MG)^2}h = \lambda h \tag{4.2.52}$$

With the zeta function renormalization, we find

$$\Lambda_{eff}^{TT}(r) = \frac{G}{16\pi}\left[\frac{3-4\gamma}{8(MG)^2}\right]^2\left[\ln\frac{\Omega^2}{\left|\frac{3-4\gamma}{8(MG)^2}\right|}+2\ln 2-\frac{1}{2}\right] \tag{4.2.53}$$

However, it isn't clear if this approach is actually meaningful, as the correction $\Delta_{TT}-\Delta_L$ should be projected on the TT subspace, that is our effective Hilbert space, and this operation could change its contribution to the eigenvalue equation. With a value of $m(r)$ near to Ω, that is the limit we are interested in studying, a small variation in the correction inserted with the parameter γ could change the sign of the logarithm and consequently of the energy contribution.
Moreover, by considering the extra Lichnerowicz term as a perturbation, we can see that the parameter γ cannot be small, therefore we conclude that an appropriate technical approach should be fully variational according with the whole approach of this thesis. This means that the analysis of the modes in this context should be non-perturbative.
On the other hand, we can immediately observe that, near the throat, all the components of the divergence of the Lichnerowicz operator are null in the odd mode, since we have

$$h_3^1 = \frac{r-2MG}{r}f_1(r)\sin\theta\partial_\theta Y_{l,0}(\theta) \tag{4.2.54}$$

that is zero when $r \to 2MG$. So the odd mode is TT in this spatial region. With these conditions we do not need any auxiliary vectors η and the eigenvalue equation we have to solve is

$$\begin{aligned}\left[-\frac{r-2MG}{r}\partial_r^2-\frac{2r-MG}{r^2}\partial_r+\frac{1}{r^2}l(l+1)+\frac{8r-20MG}{r^3}\right]\frac{1}{r^2\sin\theta}f_1(r)\partial_\theta = \\ = \lambda\frac{1}{r^2\sin\theta}f_1(r)\partial_\theta\end{aligned} \tag{4.2.55}$$

From this expression emerges a new problem: the term containing the first derivative is the leading one and can't be absorbed in the second derivative with a non-singular redefinition of the function $f_1(r)$. If one decides to study the equation for h_3^1 in spite of h_1^3, it is equivalent to accomplish a change of variable $F_1 = (r-2MG)f_1$. With this choice, the eigenvalue equation for the Lichnerowicz operator near the throat is identical to the one associated to the even mode (4.2.53). However, a not null F_1 near the throat induces a divergence in the component h_1^3 of the metric perturbation.
Anyway, we can suppose that odd and even modes have the same energy spectrum, as stated by Chandrasekhar in classical GR[40].

Chapter 5

Conclusion and perspectives

From the euclidean point of view, the Minkowski spacetime is unstable respect to thermal nucleation of black holes. The same unstable mode appears also when we introduce a temperature and we look at the thermodynamic stability of a S-AdS black hole within isothermal cavities [41][42][43]. It is interesting to note that the same pattern appears for the de Sitter (dS) space [44][45][46]. This quantum instability is related to the $S^2 \times S^2$ instantons. This instanton, termed the Nariai instanton[47], is nothing but the extreme Schwarzschild-de Sitter (SdS) solution written in another system of coordinates. This instability leads to spontaneous nucleation of black holes signaling a transition from a false vacuum to a true one[39]. This transition is possible when the energy stored in the boundaries is the same for both spaces[48]. Therefore, it seems that the presence of a black hole or a pair of black holes leads to an instability of the corresponding asymptotic space.
The case of $T = 0$ temperature examined with a Hamiltonian approach has been considered in [49] and [50], where the black hole has been substituted by a wormhole.
Discussing the Minkowski background case, we found the operator we wanted to study is not always an endomorphism of the TT Hilbert space, hence it must be projected over this subspace before some actual calculation.
In a spacetime that has not an Einstein space as spatial submanifold, this projection is highly non trivial. We tried to consider only the region near the wormhole throat, since we expect to find quantum effects mainly in this region, where gravitational interaction is stronger. In this spatial limit the equations become simpler and the Lichnerowicz operator is TT preserving, however it is still complicated to insert the difference between the Lichnerowicz and the operator Δ_{TT} as a perturbation, since this difference presents the same difficulties explained above.
If the quantum energy density would result negative after the solution of the problems described above, we would have a back ground configuration that has a pure gravity zero point energy smaller than a couple of Minkowski

spacetimes. Obviously a single wormhole wouldn't be homogeneous and isotropic as we observe to be our universe, but we could consider a spacetime with a great number of Planckian or quasi Planckian size wormhole homogeneously distributed all around the spacetime[51].

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