Generalized Robertson-Walker space-times and perfect fluids in f(R) gravity

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SIGRAV 2018, 12 sept
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Abstract

In f(R) gravity and in quadratic gravity, a perfect-fluid Ricci tensor

$$R_{ij} = \frac{R - n\xi}{n-1} u_i u_j + \frac{R - \xi}{n-1} g_{ij}$$

($u^j u_j = -1$, $R = R^k{}_k$ $R_{ij} u^j = \xi u_i$), does not in general correspond to a perfect-fluid stress-energy tensor $T_{ij} = (p + \mu) u_i u_j + p g_{ij}$.

We show that:

- If the space-time is Generalised Robertson-Walker (GRW) with $\nabla^m C_{jklm} = 0$, $\dot{R} \neq 0$, (Robertson-Walker (RW) in $n = 4$), then the two properties coexist for any $f(R)$.
- On an *n*-dimensional RW sp-time, if the Ricci tensor is perfect-fluid then the stress energy tensor is perfect-fluid in any quadratic theory of gravity.

$f[R]$ gravity, $\quad S = \frac{1}{2\kappa}\int d^n x \sqrt{g}\, f[R(x)] + S_{\rm matt}$

$$R_{kl} - \tfrac{1}{2} R g_{kl} = \kappa T_{kl}$$

$$R_{kl} \text{ perfect fluid} \iff T_{kl} \text{ perfect fluid}$$

$$f'(R)R_{kl} - [f'''(R)(\nabla_k R)(\nabla_l R) + f''(R)\nabla_k\nabla_l R]$$
$$+ g_{kl}[f'''(R)(\nabla_k R)^2 + f''(R)\nabla^2 R - \tfrac{1}{2} f(R)] = \kappa T_{kl}$$

lemma

$$\begin{cases} \nabla_k R \nabla_l R = \alpha u_i u_j + \beta g_{ij} \\ \nabla_k \nabla_l R = \alpha' u_i u_j + \beta' g_{ij} \end{cases} \iff \begin{cases} \nabla_i R = -u_i \dot R \quad (\dot R \neq 0) \\ \boxed{\nabla_i u_j = \varphi(u_i u_j + g_{ij})} \end{cases}$$

Then: $\nabla_k \nabla_l R = -\varphi \dot R g_{kl} - (\varphi \dot R - \ddot R) u_k u_l \qquad (\dot R = u^p \nabla_p R).$

A hierarchy of space-times

$$ds^2 = -dt^2 + a^2 g^*_{\mu\nu}(x) dx^\mu dx^\nu$$

1) RW: $a(t)$, (M^*, g^*) maximally symmetric.
2) GRW: $a(t)$, (M^*, g^*) a Riemannian manifold.
- studied by geometers since late '90
- on a time-slice the average of principal curvatures is const. $H(t) = \dot{a}/a$.

3) TWISTED: $a(\mathbf{x}, t)$, (M^*, g^*) a Riemannian manifold (Chen, 1979).

Theorem (Bang-Yen Chen, covariant characterizations)

A Lorentzian manifold is:
TWISTED *iff there is a time-like vector field s.t.* $\nabla_i \tau_j = \rho g_{ij} + \alpha_i \tau_j$ *with* $\alpha^k \tau_k = 0$ *(Kragujev.J.M. 2017);*
GRW *iff there is a time-like vector field s.t.* $\nabla_i X_j = \rho g_{ij}$ *(GERG, 2014)*

An unifying scheme:

Theorem (Mantica & Molinari, GERG 49, 2017)

A Lorentzian manifold is **Twisted** *iff there is a vector field (*)*

$$\boxed{u^j u_j = -1, \qquad \nabla_i u_j = \varphi(g_{ij} + u_i u_j)}$$

- $\varphi = \dot{a}/a$
- *the manifold is* **GRW** *if* $R_j{}^k u_k = \xi u_j$
- *the manifold is* **RW** *if also* $C_{ijkl} = 0$

(* acceleration, vorticity and shear -free velocity field
$\nabla_i u_j = \frac{\theta}{n-1}(g_{ij} + u_i u_j) - u_i \dot{u}_j + \omega_{ij} + \sigma_{ij})$

Twisted space-times I $\nabla_i u_j = \varphi(u_i u_j + g_{ij}),\; u^j u_j = -1$

i) u_k is "Weyl / Riemann compatible"

$$(u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm}) u^m = 0$$

$$(\text{then } C_{jklm} u^m = 0 \iff E_{kl} = u^j u^m C_{jklm} = 0)$$

$$(u_i R_{jklm} + u_j R_{kilm} + u_k R_{ijlm}) u^m = 0 \quad \textbf{(GRW)}$$

ii) Weyl tensor

$$u^m C_{jklm} = 0 \implies \nabla^m C_{jklm} = 0$$

$$\nabla^m C_{jklm} \implies u^p \nabla_p (u^m C_{jklm}) = -\varphi(n-1) u^m C_{jklm}$$

$$u^m C_{jklm} = 0 \iff \nabla^m C_{jklm} = 0 \quad \textbf{(GRW)}$$

iii) Ricci tensor

$$R_{ij} = \frac{R-\xi}{n-1} g_{ij} + \frac{R-n\xi}{n-1} u_i u_j + (n-2)(u_i v_j + v_i u_j - C_{kijm} u^k u^m)$$
$$\xi = (n-1)(\dot{\varphi} + \varphi^2),$$
$$v_k = \nabla_k \varphi + u_k \dot{\varphi}, \quad v_k u^k = 0$$

($R_{ij} = \frac{R-\xi}{n-1} g_{ij} + \frac{R-n\xi}{n-1} u_i u_j + u_i q_j + q_i u_j + \Pi_{ij}$)

in GRW:

i) $v_k = 0$ (equivalent to $R_{ij} u^j = \xi u_i$)
ii) $-(n-2) C_{0\mu\nu 0} = R^*_{\mu\nu} - \frac{R^*}{n-1} g^*_{\mu\nu}$, (comoving frame)
iii) the Ricci tensor is perfect fluid iff $\nabla^m C_{jklm} = 0$

Twisted space-times III. Questions

1) Is the velocity field $\nabla_i u_j = \varphi(u_i u_j + g_{ij})$ unique? (Yes in GRW)
2) A recurrent generalized curvature tensor (it is zero in $n = 4$)

$$\Gamma_{jklm} =: C_{jklm} - \tfrac{n-2}{n-3}(u_j u_m E_{kl} - u_k u_m E_{jl} - u_j u_l E_{km} + u_k u_l E_{jm}) - \tfrac{1}{n-3}(g_{jm} E_{kl} - g_{km} E_{jl} - g_{jl} E_{km} + g_{kl} E_{jm})$$

$$\boxed{\Gamma_{jklm} u^m = 0, \quad u^p \nabla_p \Gamma_{jklm} = -2\varphi \Gamma_{jklm}}$$

3) What properties survive acceleration? $\nabla_i u_j = \varphi(u_i u_j + g_{ij}) - u_i \dot{u}_j$

Theorem (unpublished)

IF sp-time is GRW with $\nabla^m C_{jklm} = 0$, $\dot{R} \neq 0$ *(in* $n = 4$ *the sp-time is RW), THEN perfect fluid Ricci tensor* $\iff$ *perfect fluid matter tensor* $\forall f(R)$

Proof (always $\dot{R} \neq 0$):

$$\begin{cases} \nabla_k R \nabla_l R = \alpha u_i u_j + \beta g_{ij} \\ \nabla_k \nabla_l R = \alpha' u_i u_j + \beta' g_{ij} \end{cases} \iff \begin{cases} (1)\ \nabla_i u_j = \varphi(u_i u_j + g_{ij}) \\ (2)\ \nabla_i \dot{R} + u_i \dot{R} = 0 \quad (\dot{R} \neq 0) \end{cases}$$

- If R_{jk} perfect fluid & sp-time GRW, THEN (1) is true, $\nabla^m C_{jklm} = 0$ THEN (2) is true $\Rightarrow T_{jk}$ perfect fluid
- If sp-time GRW & $\nabla^m C_{jklm} = 0$ THEN R_{jk} is perfect fluid & (1,2) are true $\Rightarrow T_{jk}$ perfect fluid

Stress-energy tensor $T_{jk} = (p+\mu)u_j u_k + p g_{jk}$

$$\kappa p = -\tfrac{1}{2}f + \frac{R-\xi}{n-1}f' - [(n-2)\varphi\dot R + \ddot R]f'' - \dot R^2 f'''(R)$$
$$\kappa\mu = \tfrac{1}{2}f - \xi f' + (n-1)\varphi\dot R f''$$
$$\boxed{\varphi = \dot a/a = H, \qquad \xi = (n-1)\ddot a/a}$$

- Vacuum solution $p=0$, $\mu=0$ possible for $R_{ij}\neq 0$.
- **Friedmann equations:**

$$\kappa\left[p + \frac{n-3}{n-1}\mu\right] = \frac{Rf'-f}{n-1} - (n-2)(\dot H + H^2)f' - (H\dot R + \ddot R)f'' - \dot R^2 f'''$$
$$\kappa\mu = \tfrac{1}{2}(f - Rf') + \tfrac{1}{2}\left[\frac{R^*}{a^2} + (n-1)(n-2)H^2\right]f' + (n-1)H\,\dot R\,f''$$

($n=4$ and $R^*=0$, Sotiriou & Faraoni, 2010)

Quadratic gravity (Deser & Tekin, 2003)

$$S = \int d^n x \sqrt{-g}\Big[\frac{R-2\Lambda}{\kappa} + a\,R^2 + b\,R_{ij}R^{ij} + c\,(R_{jklm}R^{jklm} - 4R_{jk}R^{jk} + R^2)\Big] + S_m$$

$$\begin{aligned} T_{kl} =& \frac{1}{\kappa}(R_{kl} - \tfrac{1}{2}Rg_{kl} + \Lambda g_{kl}) + 2aR(R_{kl} - \tfrac{1}{4}Rg_{kl}) \\ &+ (2a+b)(g_{kl}\nabla^2 - \nabla_k\nabla_l)R + b\nabla^2(R_{kl} - \tfrac{1}{2}Rg_{kl}) \\ &+ 2b(R_{akbl} - \tfrac{1}{4}g_{kl}R_{ab})R^{ab} \\ &+ 2c[RR_{kl} - 2R_{akbl}R^{ab} + R_{kcde}R_l{}^{cde} \\ &- 2R_{kj}R_l^j - \tfrac{1}{4}g_{kl}(R_{jklm}R^{jklm} - 4R_{kl}R^{kl} + R^2)] \end{aligned}$$

Theorem (unpublished)

On an n-dimensional Robertson-Walker space-time:
R_{ij} perfect fluid $\Longrightarrow$ T_{ij} perfect fluid, in any quadratic theory of gravity.

References

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