HOW TO USE RANDOM MATRIX THEORY IN THE DETECTION OF GRAVITATIONAL WAVES

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WHAT ARE RANDOM MATRICES

Random Matrix: matrix whose entries are random variables

They have many applications → nuclear physics

→ econophysics

→ biology

→ gravitational wave physics

SOME HISTORY

ON THE STATISTICAL DISTRIBUTION OF THE WIDTHS AND SPACINGS OF NUCLEAR RESONANCE LEVELS

By EUGENE P. WIGNER

Communicated by P. A. M. Dirac

Received 18 September 1950

One of the first applications of random matrices to physics is due to Jenö Pál Wigner

Problem: how to determine the energy levels of heavy nuclei (such as Uranium)

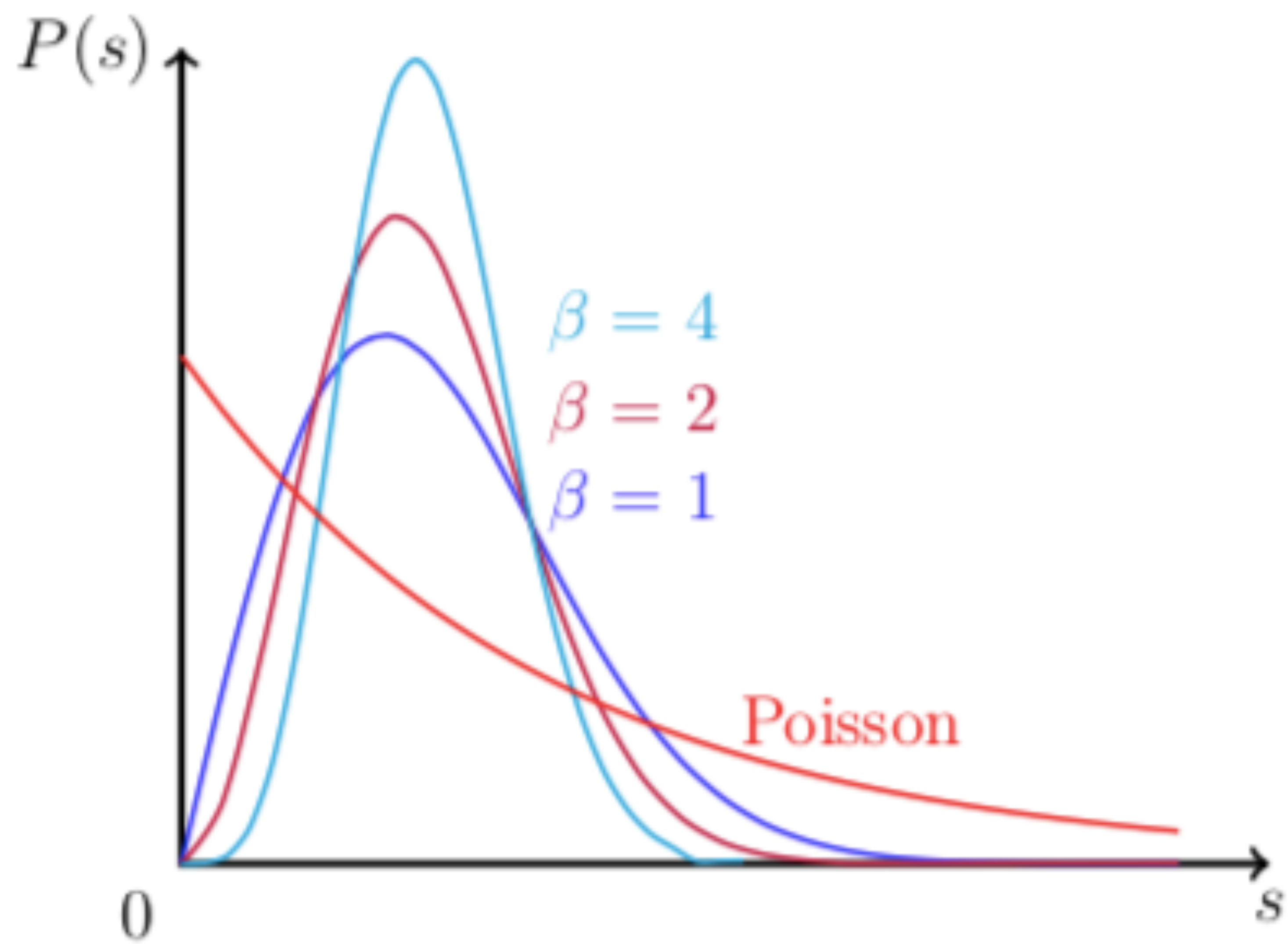
Difficulties: it is a very complicated system, we do not know how to write the Hamiltonian

$$H\Psi_n = E_n\Psi_n$$

Wigner observed that the space, s, between neighbouring energy levels were governed by the following probability density

$$P(s) = C_\beta s^\beta \exp(-\alpha_\beta s^2)$$

Wigner surmise



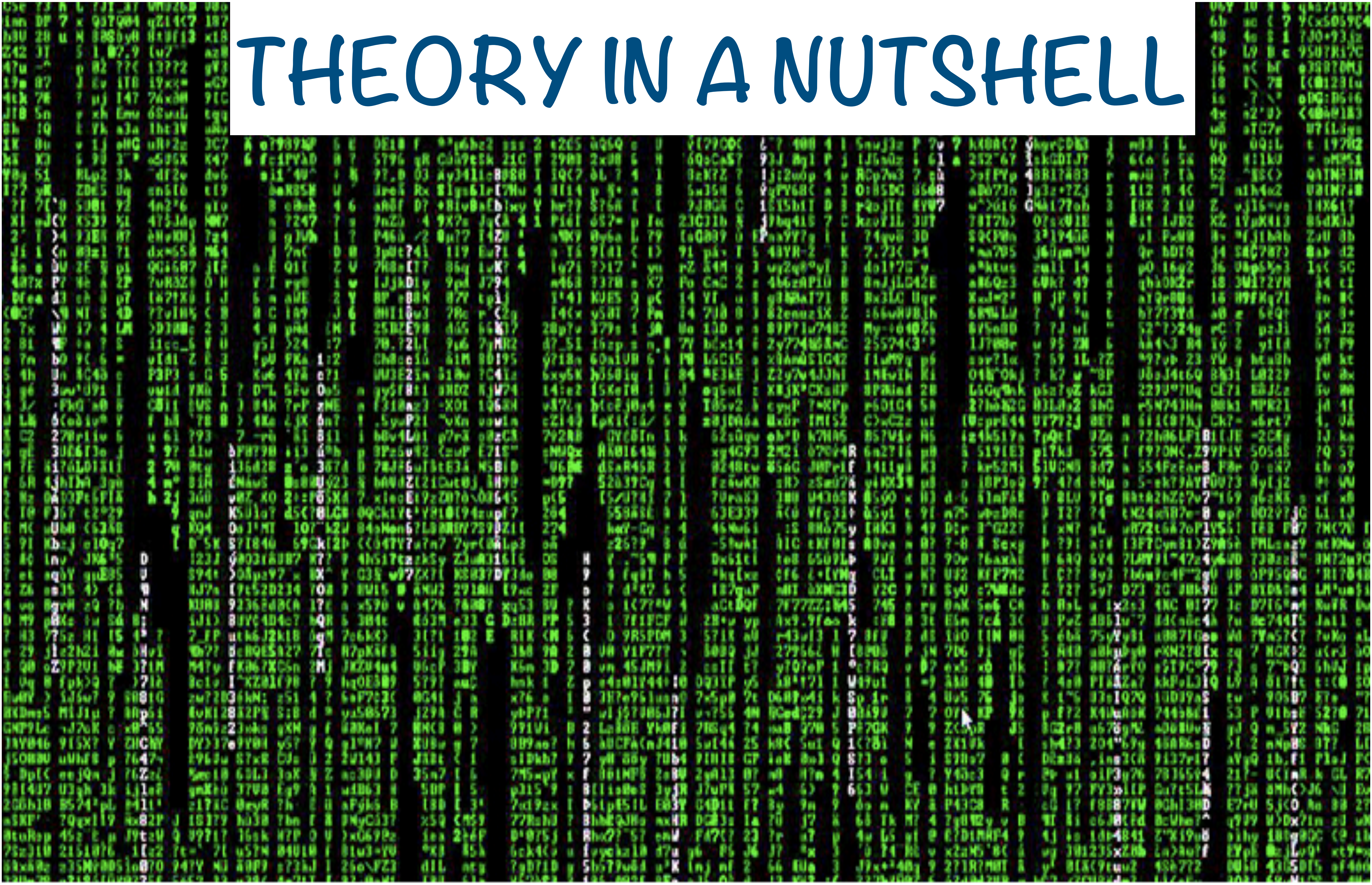
I Idea: interpret the distance between energy levels as the distance between eigenvalues of the Hamiltonian of the system

II Idea: define an ensemble of Hamiltonians with statistical properties as those that the real Hamiltonian might have if it could be written explicitly

RANDOM MATRIX THEORY

give up intrinsic details of the system in order to collect information on average properties

THEORY IN A NUTSHELL



THEORY IN A NUTSHELL

Gaussian Ensembles: independent, identically distributed Gaussian entries

Gaussian **O**rthogonal **E**nsemble: real symmetric matrices

Gaussian **U**nitary **E**nsemble: complex Hermitian matrices

Gaussian **S**ymplectic **E**nsemble: quaternionic Hermitian matrices

$$E_N^\beta$$

Dyson index β : counts the number of real components per matrix element

The eigenvalue spacing distribution of these ensembles is approximated by the Wigner surmise

These matrices can be diagonalized as

$$M = U\Lambda U^{-1}$$

angular radial decomposition

$$\Lambda = (\lambda_1, \ldots, \lambda_n) \in R^n$$

The matrix U is

- a real orthogonal matrix if $M \in$ GOE
- a complex unitary matrix if $M \in$ GUE
- a complex symplectic matrix if $M \in$ GSE

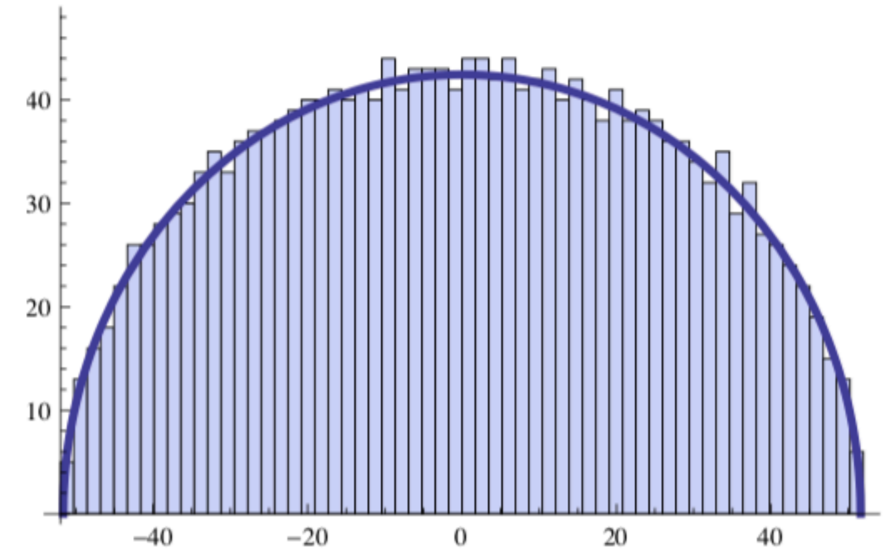
Wigner semicircle law

Let X be a n-dimensional complex Hermitian matrix with

- $(X_{ij})_{i<j}$ i.i.d. with zero mean and finite variance σ,
- (X_{kk}) i.i.d. with bounded mean and variance.

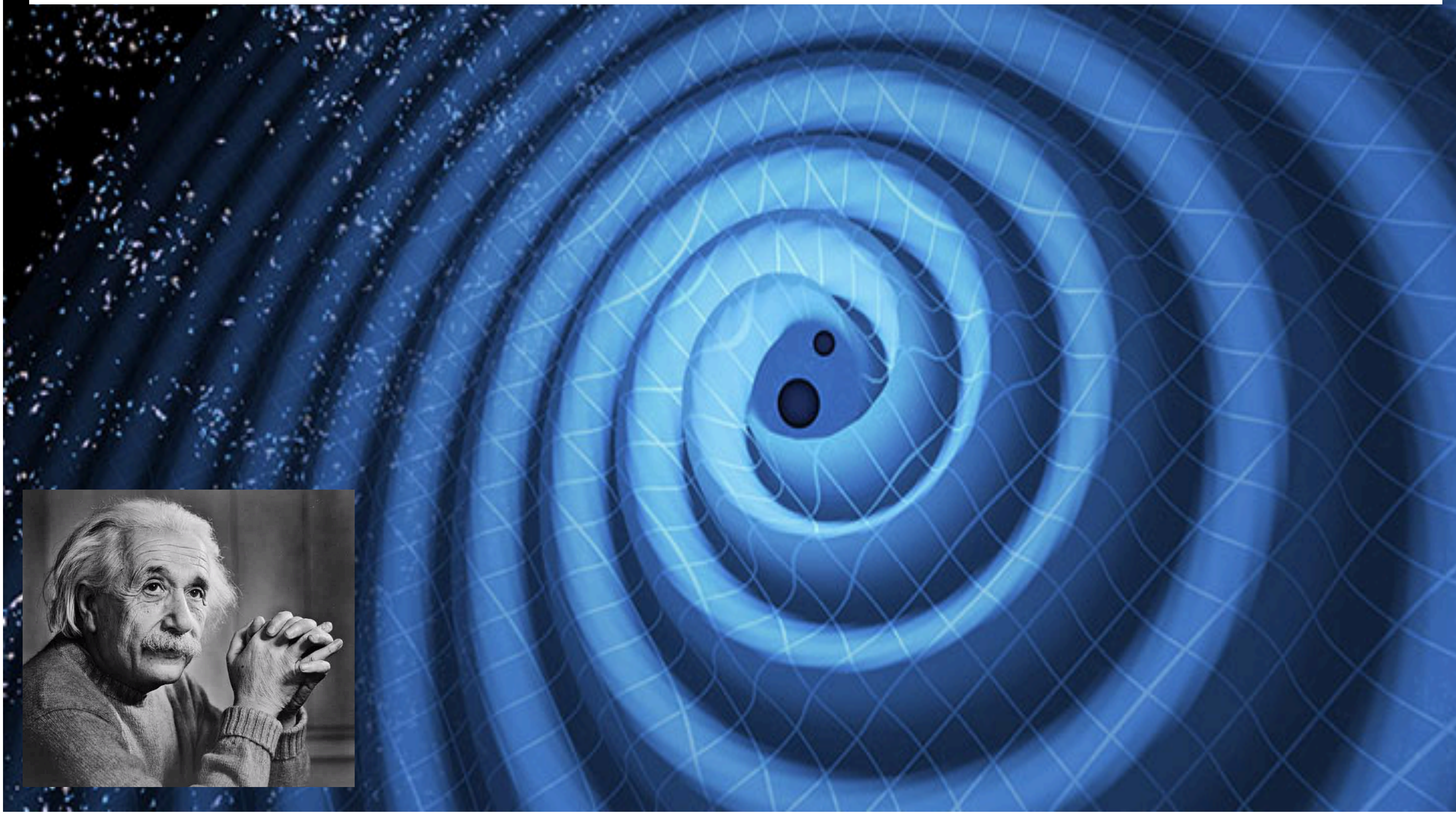
Then the eigenvalues of $n^{-1/2}X$ tend in distribution to the semicircle law

$$\rho = \frac{1}{2\pi\sigma^2}\sqrt{4\sigma^2 - x^2}$$



Gaussian Ensembles follow this eigenvalue distribution

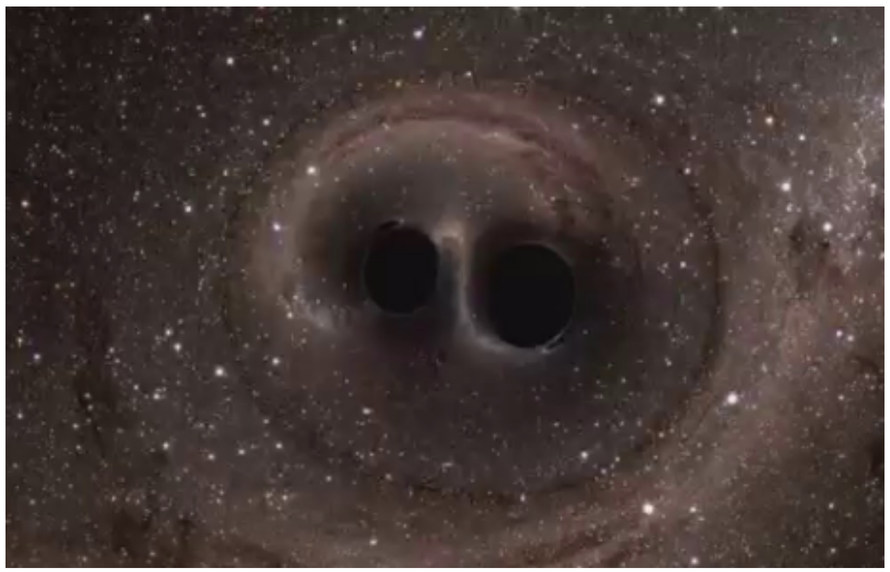
APPLICATION TO GW DETECTION



APPLICATION TO GW DETECTION

Gravitational waves: perturbations of the gravitational field that travel at the speed of light

14 September 2015: **FIRST** direct detection



LSC LIGO Scientific Collaboration

VIRGO

- two low mass black hole merger

first detection of this type of event

effective proof that these phenomena may take place within the current age of the universe

GRAVITATIONAL WAVE ASTRONOMY ERA

APPLICATION TO GW DETECTION

Gravitational wave **stochastic** background: superposition of signals from unresolved sources

Gaussian

Problem: the stochastic background data may hide a gravitational wave signal

We need a method to look for long-memory effect in the background

Idea: **RANDOM MATRICES**

Random matrix approach in search of weak signals immersed in background noise
D. Grech & J. Miśkiewicz, Europhysics Letters, Volume 97, Issue 3, February 2012

I STEP: BUILD THE MATRICES AND THE ENSEMBLE

$\{x_i\}$

- time series
- i=1,...,N+1
- N>>1

$\{\Delta x_i\}$

- increment time series
- we have $\Delta x_i = x_{i+1} - x_i$

$s_k = \left\{ \Delta x_{(k-1)L+1}, .. \Delta x_{kL} \right\}$

- k=1,...,N/L
- lenght L

Renormalize according to

$$s_k \to \hat{s}_k = \frac{s_k}{\sqrt{L}\sigma_k}$$

I STEP: BUILD THE MATRICES AND THE ENSEMBLE

- the first L subseries build the first LxL matrix
- the second L subseries build the second LxL matrix
- ...
- thus, we get N/L^2 matrices $M^{(n)}$ (n=1,...,N/L^2)

Strategy: Further steps rely on examination of eigenvalue spectra properties for the ensamble of symmetrized matrices with (i,j) entries $(M_{ij}^{(n)} + M_{ji}^{(n)})/2$ and on a comparison with a spectrum known ***a priori***. Any distortion from this ***a priori*** spectrum is interpreted as a weak signal hidden in the background

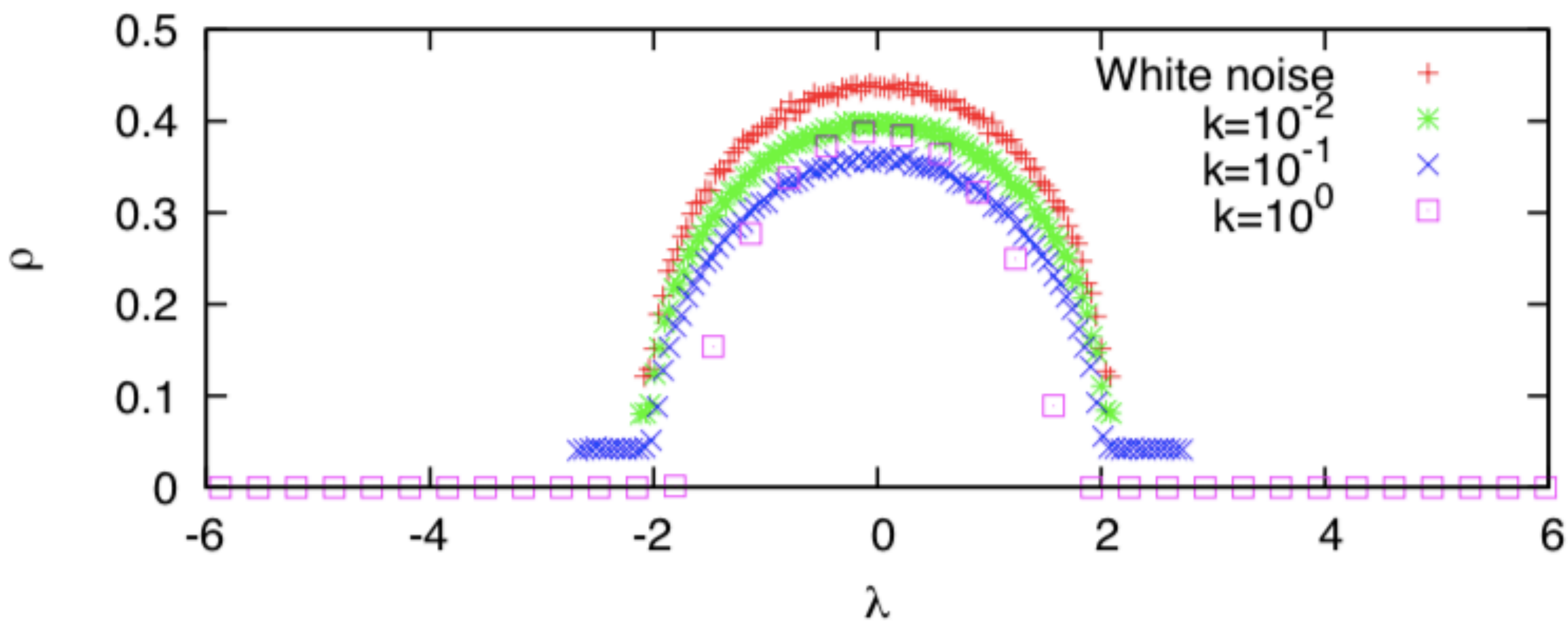
II STEP:SIMPLE EXAMPLE

- assume that the noise is pure white noise
- add a sinusoidal signal for different s/n
- if only white noise the eigenvalue spectrum should disappear for $|\lambda| > 2$

$$\rho(\lambda) = \frac{1}{2\pi}\sqrt{-\lambda^2 + 4}$$

II STEP:SIMPLE EXAMPLE

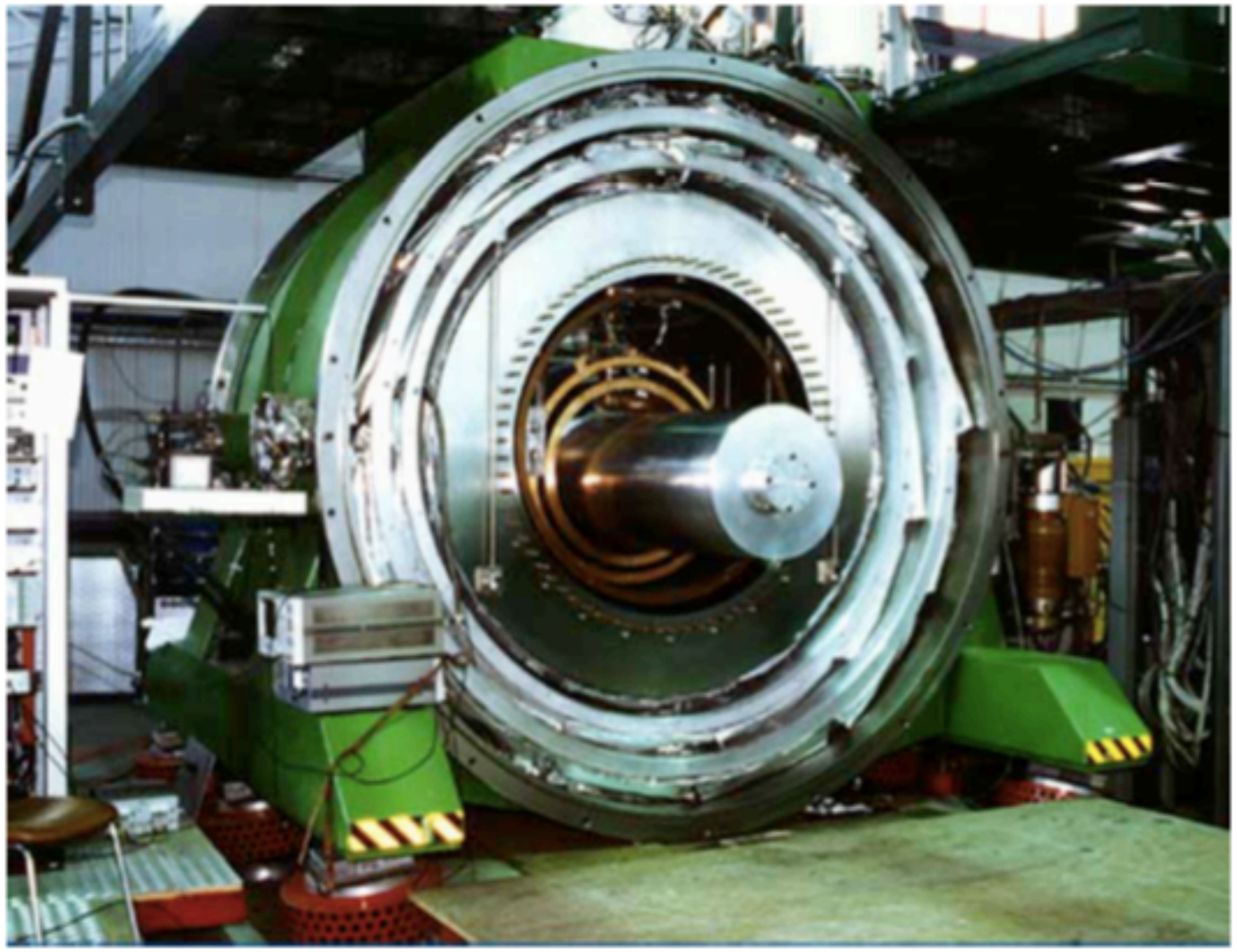
ensemble of 1000 matrices of size 200x200



$s/n = 10^k$ $\quad$ $k = -2, -1, 0$

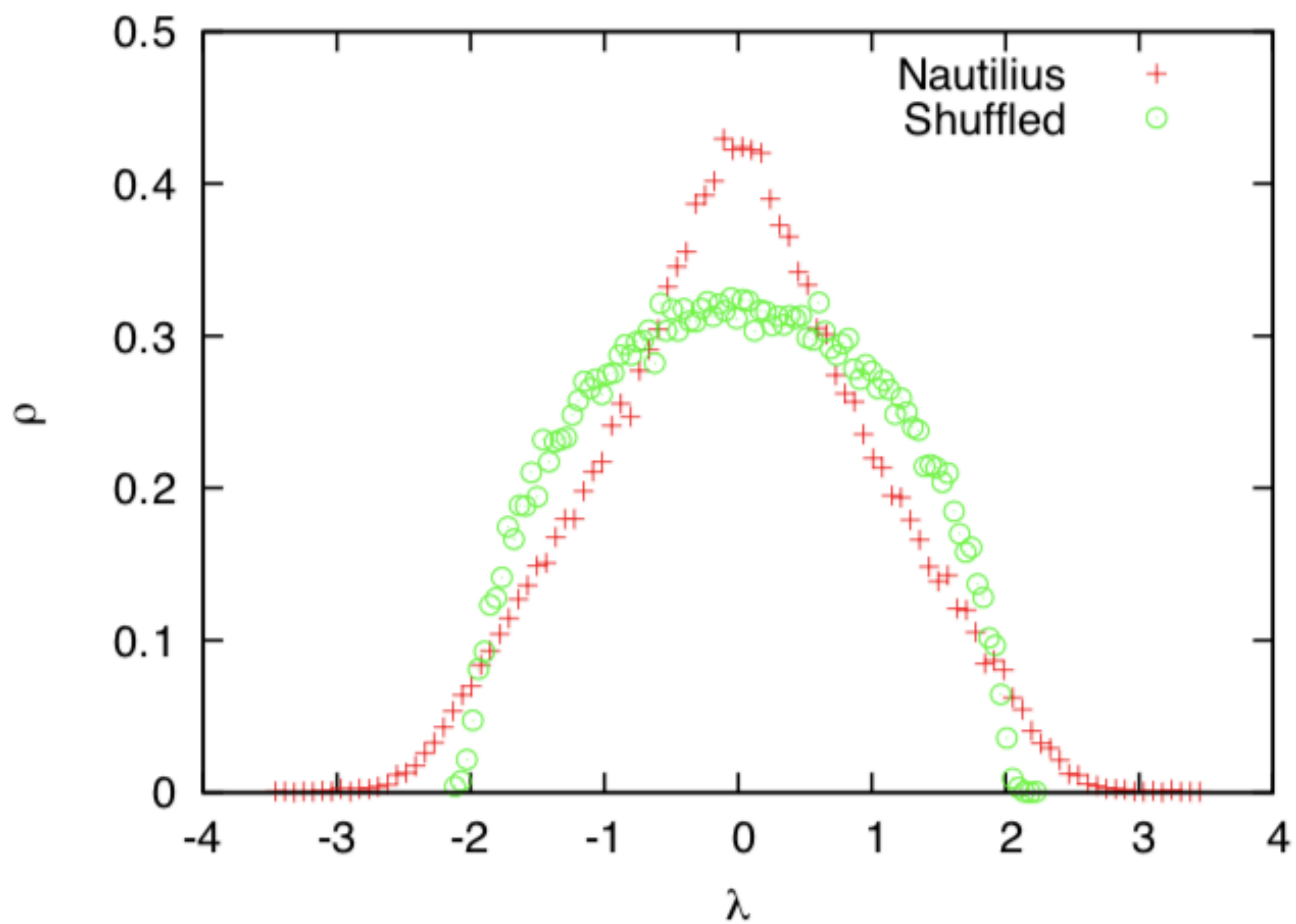
III STEP: CHARACHTERIZE THE NOISE OF A REAL EXPERIMENT

- NAUTILUS, ultra-cryogenic resonant gravitational wave detector
- built to detect gravitational bursts
- no longer running, yet it collected some data



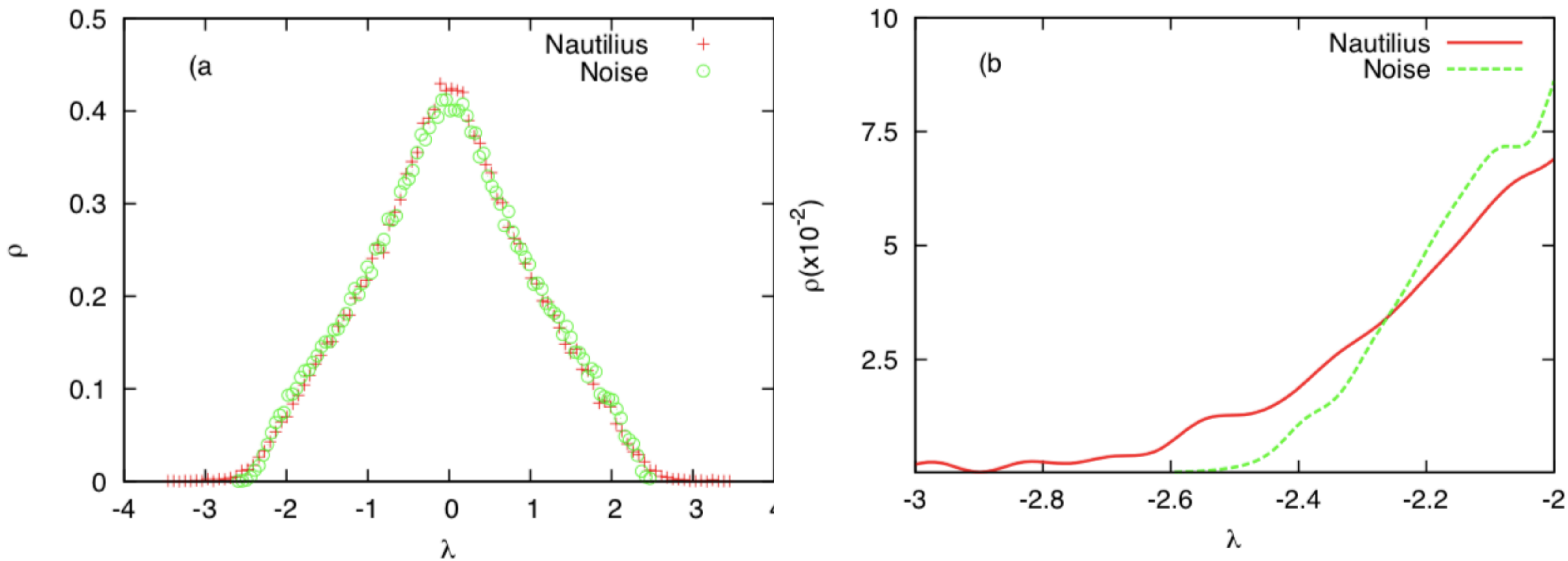
III STEP: CHARACHTERIZE THE NOISE OF A REAL EXPERIMENT

- ensemble of 1000 matrices of dimension 200x200
- triangular shape in the spectrum due to the specifics of the instrument
- if we shuffle the data we get Wigner



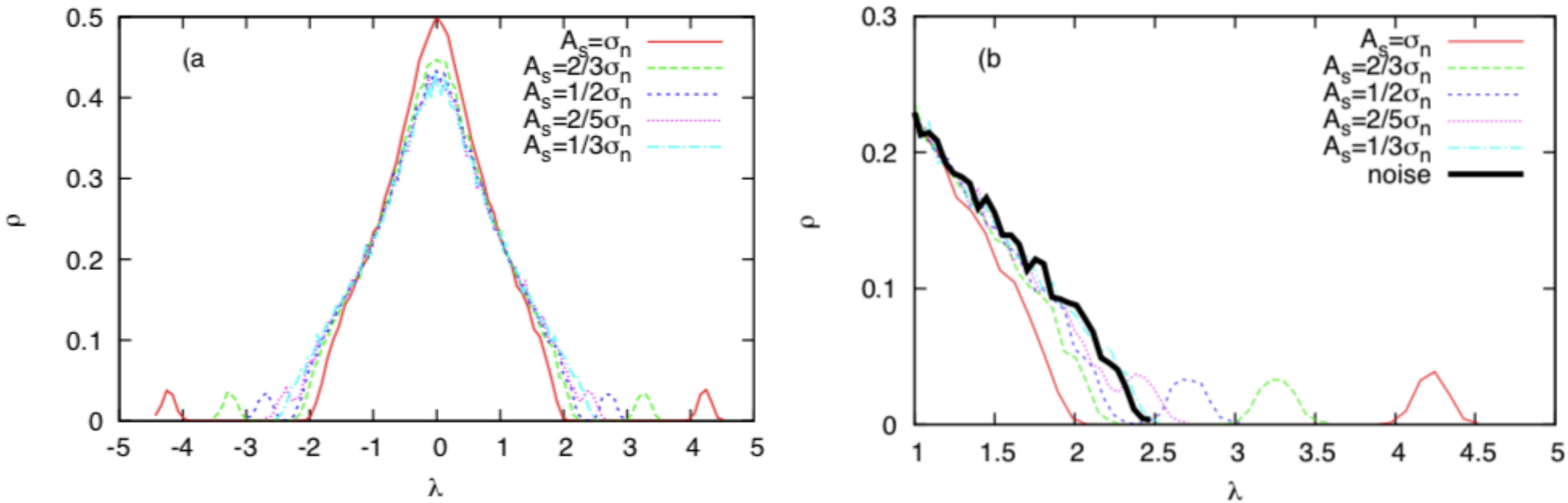
III STEP: CHARACHTERIZE THE NOISE OF A REAL EXPERIMENT

if we compare real noise and simulated noise we see a difference in the tail



this can be important for the detection of periodic weak signals

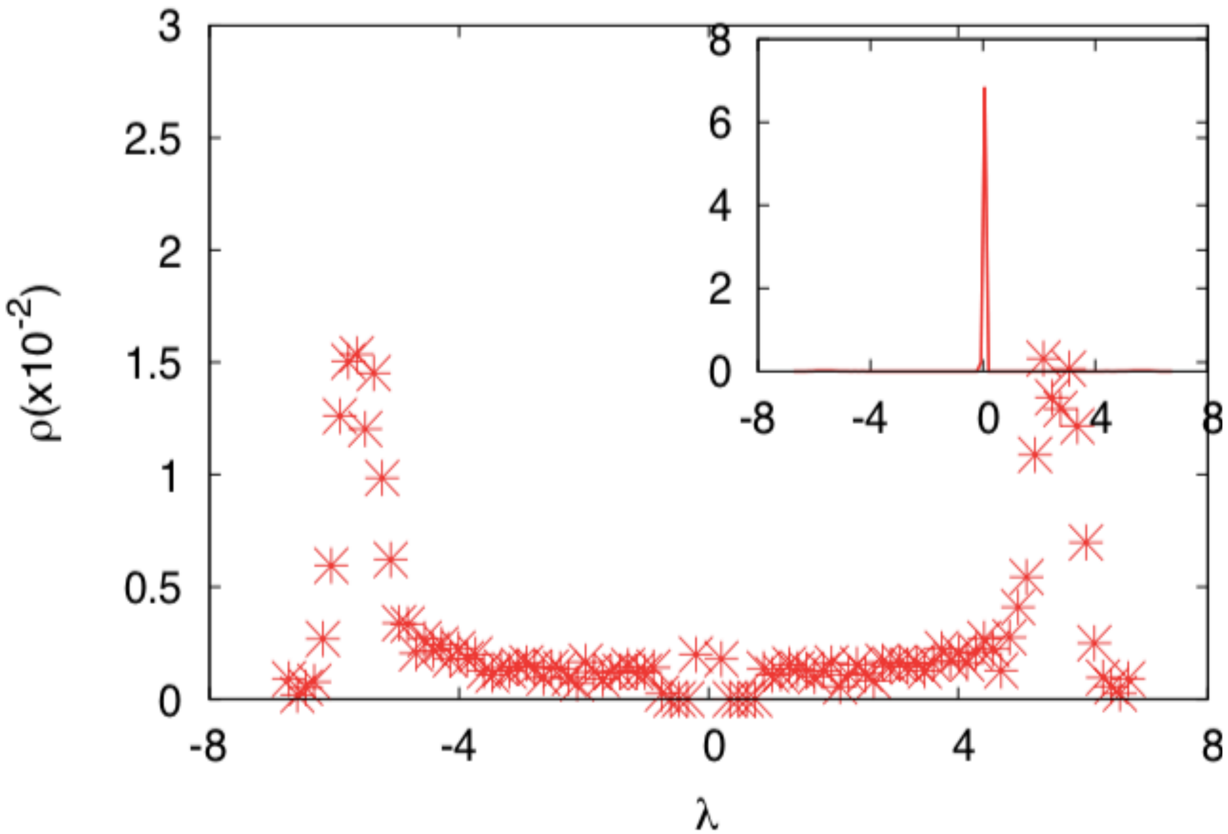
IV STEP: A SIMPLE EXAMPLE



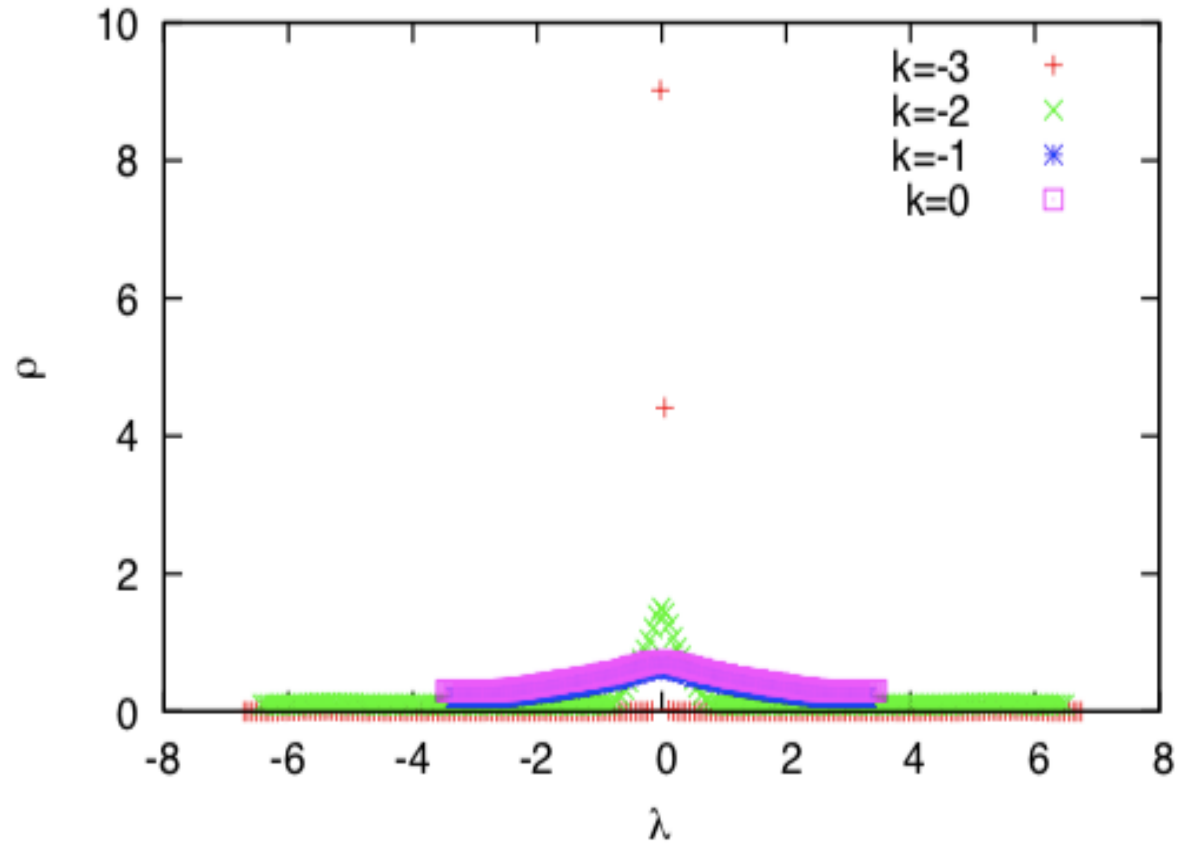
tails matter!

V STEP: A PHYSICAL EXAMPLE

- gravitational wave signal from a freely precessing anti-symmetric star
- the level of background noise was reduced by a factor 10^k (k=0,-1,-2,-3)



response of a detector with background switched off



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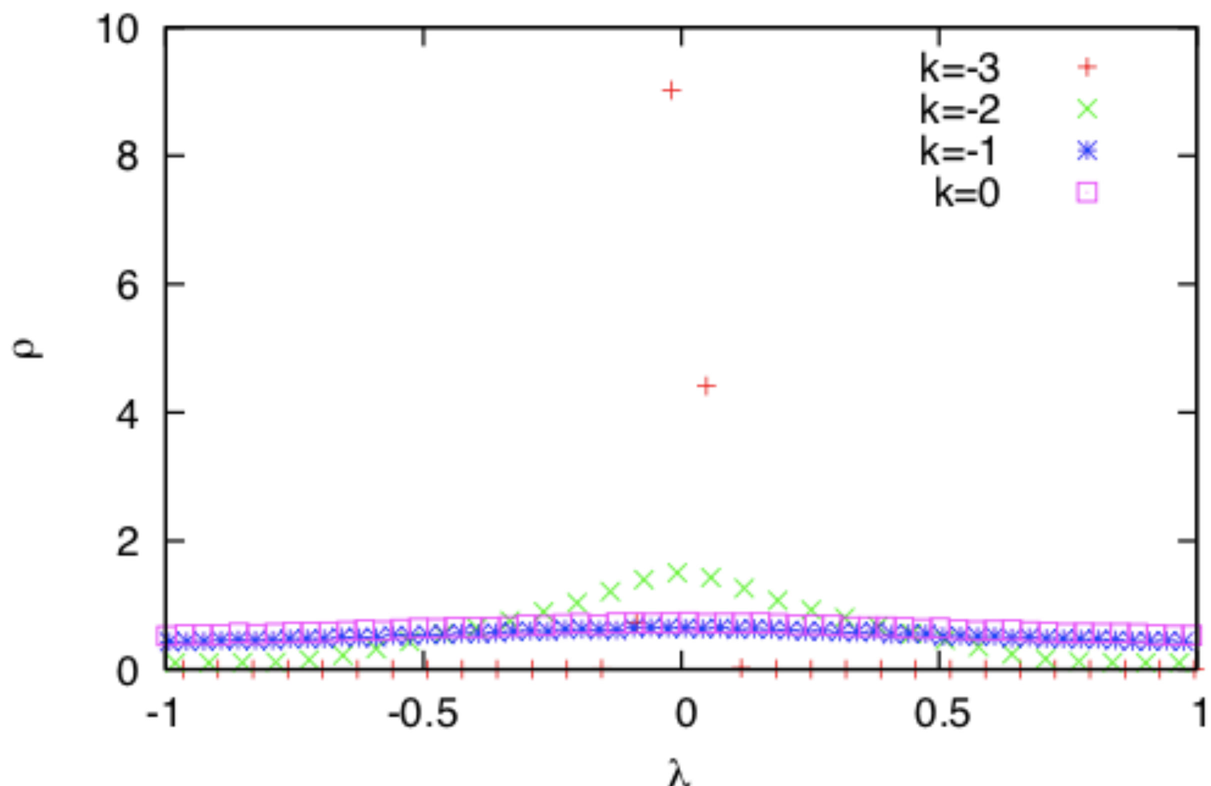
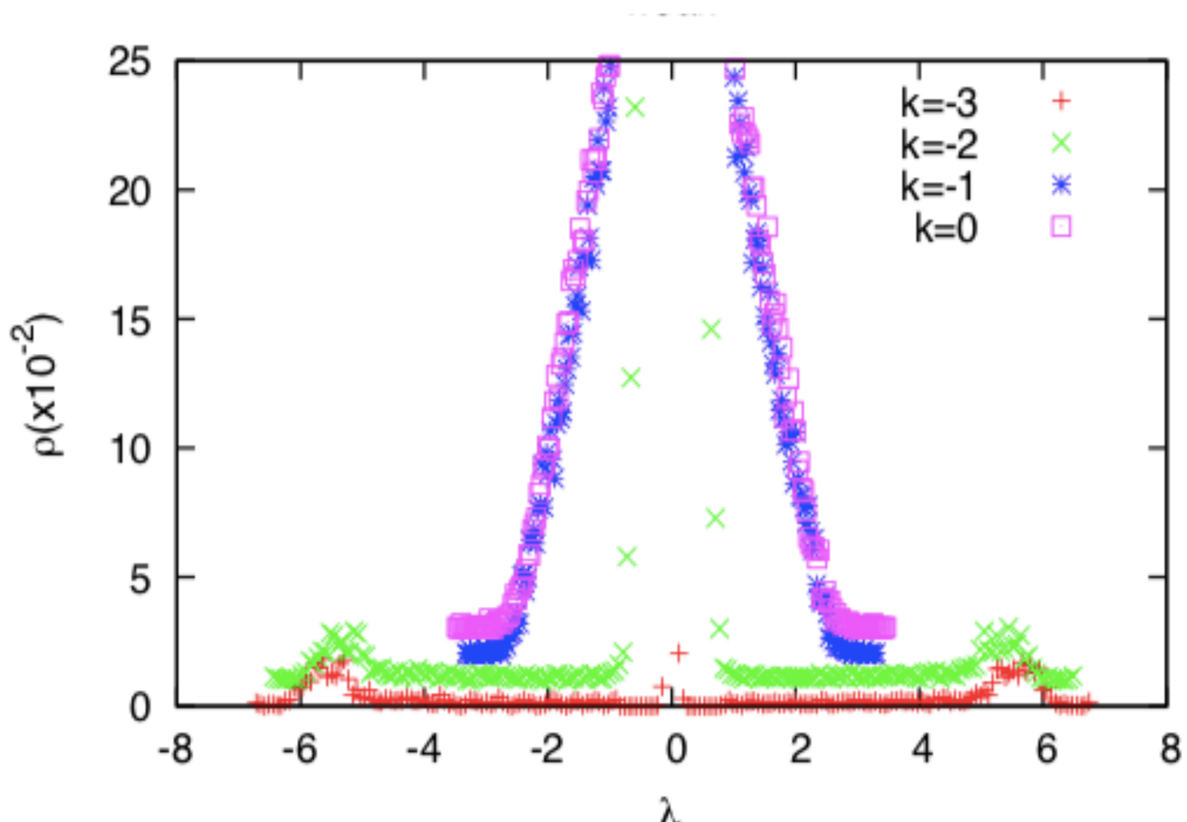
V STEP: A PHYSICAL EXAMPLE



Magnified central part of the spectra



Magnified tail part of the spectra

good performance of the method

CONCLUSIONS

- weak signal may be hidden in the background data
- random matrix approach can help in revealing this coded signal
- compare the density distribution of the eigenvalues with a known one
- pay attention to the tails

FIN