LESSON 6: TRIDIAGONAL MATRICES

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1. General properties (*transfer matrix for eigenvectors and spectral duality, characteristic polynomials, Christoffel-Darboux sums, eigenvectors*)
2. Real symmetric tridiagonal matrices (*simplicity of eigenvalues, eigenvectors, Vandermonde determinant of eigenvalues, map to eigenvalues and a unit vector, periodic matrices, bands, lemniscates*)
3.Tridiagonal models (*free particle, particle in electric field, Harper model, 1D Anderson model, Thouless formula, Hatano-Nelson model, Lloyd model*)
4. Measures of localization of eigenvectors (*probability width, participation ratio, entropy*).
5. Methods of tridiagonalization (*Lanczos and Householder*), Beta ensembles
6. Dynamical localization in quantum chaos (*the kicked rotator, the Maryland construction*)

1. General properties

We first discuss general tridiagonal matrices $n \times n$, with real or complex matrix elements. Off diagonal elements are always taken to be non-zero. It is very convenient to consider also the matrix with corner terms, with $\tau = \exp(\xi + i\varphi)$. The matrices describe some open or closed chain model, with different boundary conditions.

$$M = \begin{bmatrix} a_1 & b_1 & & 0 \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ 0 & & c_{n-1} & a_n \end{bmatrix} \quad M(\tau^n) = \begin{bmatrix} a_1 & b_1 & & \tau^{-n} \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ \tau^n & & c_{n-1} & a_n \end{bmatrix}$$

The eigenvalues of $M(\tau^n)$ are continuous functions of the parameter. An interesting question is how are they affected by it (see figg. 1 and 2). The following similarity relation holds. It balances the matrix and makes the numerical evaluation of eigenvalues feasible when n is large:

$$\begin{bmatrix} \frac{1}{\tau} & & \\ & \ddots & \\ & & \frac{1}{\tau^n} \end{bmatrix} M(\tau^n) \begin{bmatrix} \tau & & \\ & \ddots & \\ & & \tau^n \end{bmatrix} = \begin{bmatrix} a_1 & b_1\tau & & 1/\tau \\ \frac{1}{\tau}c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1}\tau \\ \tau & & \frac{1}{\tau}c_{n-1} & a_n \end{bmatrix} = M_B(\tau)$$

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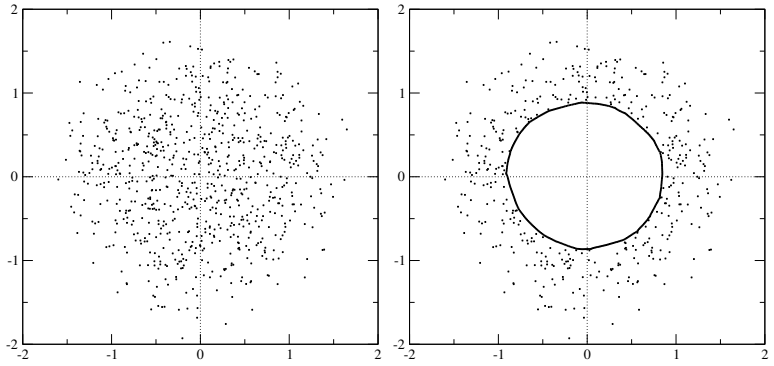

FIGURE 1. The eigenvalues of a tridiagonal matrix with independent matrix elements uniformly distributed in the unit disk with $\xi = 0$ (left) and $\xi = .5$ (right). The eigenvalues inside the 'circle' are swept to the boundary, while eigenvalues outside the circle appear to be unaffected (from [16]).

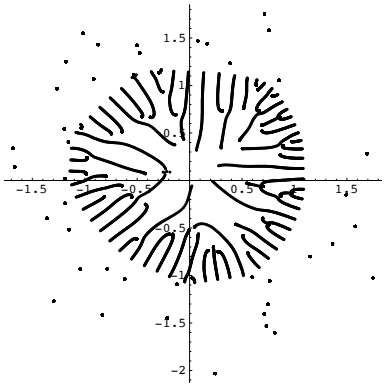

FIGURE 2. The motion of $n = 100$ eigenvalues of a tridiagonal matrix with independent matrix elements uniformly distributed in the unit disk as ξ increases from 0.3 to 0.6 Outermost ones appear to be unaffected (from [16]).

1.1. **Transfer matrix for eigenvectors and spectral duality.** The equations for the right eigenvectors of the two matrices differ in the boundary terms:

$$\begin{aligned} & u_0 + a_1 u_1 + b_1 u_2 = z u_1 \\ & c_k u_{k-1} + a_k u_k + b_k u_k = z u_k, \quad k = 2, ..., n-1 \\ & b_{n-1} u_{n-1} + a_n u_n + u_{n+1} = z u_n \end{aligned} \tag{1}$$

It is $u_0 = 0$ and $u_{n+1} = 0$ for the first matrix, and $u_0 = \tau^{-n} u_n$, $u_{n+1} = \tau^n u_1$ for the second one. The 2-term recursion can be recast as a 1-term recursion,

$$\begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} b_k^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x - a_k & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c_{k-1} \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}$$

Iteration gives the transfer matrix for vectors:

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = \mathsf{T}_n(z) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} \tag{2}$$

$$\mathsf{T}_n(z) = \begin{bmatrix} z-a_n & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b_{n-1}^{-1} & 0 \\ 0 & c_{n-1} \end{bmatrix} \dots \begin{bmatrix} b_1^{-1} & 0 \\ 0 & c_1 \end{bmatrix} \begin{bmatrix} z-a_1 & -1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

The determinant is a constant: $\det \mathsf{T}_n(z) = (b_1^{-1}c_1)...(b_{n-1}^{-1}c_{n-1})$.

If the length n can be large, the eigenvalues of the transfer matrix determine the large n behaviour of the vector components u_n, given u_0 and u_1. Typically, the eigenvalues of $\mathsf{T}_n(z)$ grow or decrease in modulus as $\exp n\xi(z)$.
As the transfer matrix is derived from M, the following spectral duality is useful.

Remark 1.1 (Spectral Duality)**.**
If u is an eigenvector of $M(\tau^n)$ with eigenvalue z, then $u_{n+1} = \tau^n u_1$ and $u_n = \tau^n u_0$, i.e. $(u_1, u_0)^T$ is eigenvector of $\mathsf{T}_n(z)$ with eigenvalue τ^n.
If $(u_1, u_0)^T$ is eigenvector of $\mathsf{T}_n(z)$ with eigenvalue τ^n, then the pair uniquely reconstructs an eigenvector of $M(\tau^n)$ via the recursion.
Therefore, z is eigenvalue of $M(\tau^n)$ iff τ^n is eigenvalue of $\mathsf{T}(z)$: the rôle of eigenvalues and parameters exchange.

Proposition 1.2 (Duality identity)**.**

$$\boxed{\det[z - M(\tau^n)] = (b_1...b_{n-1}) \operatorname{tr} \mathsf{T}_n(z) - \tau^n (b_1...b_{n-1}) - \frac{1}{\tau^n}(c_1...c_{n-1})} \tag{4}$$

Proof. According to spectral duality, $\det[z - M(\tau^n)] = 0$ iff $\det[\mathsf{T}_n(z) - \tau^n] = 0$. The second expression coincides with $\det \mathsf{T}_n(z) - \tau^n \operatorname{tr} \mathsf{T}_n(z) + \tau^{2n}$. The determinant is a number, and the trace is a polynomial with leading term $(b_1...b_{n-1})^{-1} z^n$. Then:

$$\boxed{\det[z - M(\tau^n)] = -\frac{1}{\tau^n}(b_1...b_{n-1}) \det[\mathsf{T}_n(z) - \tau^n]} \tag{5}$$

which is another form of duality, equivalent to (4). □

The exponents $\xi(z)$ of the transfer matrix can be obtained by Jensen's formula of complex analysis: *if $f(\tau)$ is holomorphic with $f(0) \neq 0$, and $\tau_1, ..., \tau_n$ are its zeros in the disk $|\tau| < r$, then:*

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \log |f(re^{i\varphi})| = \log |f(0)| - \sum\nolimits_k \log(\frac{|\tau_k|}{r})$$

The identity is applied to the log of the duality identity (5) with $r = 1$, i.e. $\xi = 0$:

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} \log |\det[z - H(e^{i\varphi})]| = \sum\nolimits_k \log |b_k| + \log |\det \mathsf{T}_n(z)| + n\xi_-(z)$$

This exact result is an expression for the positive exponent of the transfer matrix:

$$\xi_+(z) = \frac{1}{n} \int_0^{2\pi} \frac{d\varphi}{2\pi} \log |\det[z - H(e^{i\varphi})]| - \frac{1}{n} \sum\nolimits_k \log |b_k| \tag{6}$$

1.2. **Characteristic polynomials.** We adopt the notation $M_{j:k}$ to denote the principal sub-matrix of M whose diagonal elements are $a_j ... a_k$; thus $M_{1:n} = M$, $M_{2:n-1}$ is the matrix of size $n-2$ obtained from M by deleting rows and columns 1 and n, and $M_{1:k}$ is the $k \times k$ sub-matrix obtained by deleting rows and colums $k+1, ..., n$ in M.

The polynomials $p_k(z) = \det[z - M_{1:k}]$ are monic of degree k, and satisfy the recurrence

$$p_k(z) = (z - a_k)p_{k-1}(z) - b_{k-1}c_{k-1}p_{k-2}(z) \tag{7}$$

with initial conditions $p_1(z) = z - a_1$ and $p_0(z) = 1$. It shows that the eigenvalues of a tridiagonal matrix depend on b_k and c_k only through their product (thus, real eigenvalues are ensured by a_k real and $(b_k c_k) > 0$ for all k).
The two-term recursion is translated to one-term

$$\begin{bmatrix} p_k(z) \\ p_{k-1}(z) \end{bmatrix} = \begin{bmatrix} z - a_k & -b_{k-1}c_{k-1} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{k-1}(z) \\ p_{k-2}(z) \end{bmatrix} = ... = \mathcal{T}_k(z) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathcal{T}_k(z) = \begin{bmatrix} z - a_k & -b_{k-1}c_{k-1} \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} z - a_2 & -b_1 c_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} z - a_1 & -1 \\ 1 & 0 \end{bmatrix}$$

Next, consider the characteristic polynomials of the sub-matrices $M_{2:k}$ where the first row and column are deleted: $q_{k-1}(z) = \det[z - T_{2:k}]$. They are monic polynomials of degree $k-1$, with $q_1(z) = z - a_2$, and obey the recursion (7) with different initial conditions:

$$\begin{bmatrix} q_{k-1}(z) \\ q_{k-2}(z) \end{bmatrix} = \mathcal{T}_k(z) \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

We obtain the result of the product that defines the transfer matrix:

$$\boxed{\mathcal{T}_k(z) = \begin{bmatrix} p_k(z) & -q_{k-1}(z) \\ p_{k-1}(z) & -q_{k-2}(z) \end{bmatrix}} \tag{8}$$

$$\det \mathcal{T}_k(z) = -p_k(z)q_{k-2}(z) + p_{k-2}(z)q_{k-1}(z) = (b_1 ... b_{k-1})(c_1 ... c_{k-1}) \tag{9}$$

Now let's turn to the characteristic polynomial of the matrix with corners. It can be evaluated with the rules for determinants[1], and expressed with the transfer matrix:

$$\begin{aligned} \det[z - M(\tau^n)] &= \det[z - M] - \det[z - M_{2:n-1}] - (b_1 ... b_{n-1})\tau^n - \frac{1}{\tau^n}(c_1 ... c_{n-1}) \\ &= \operatorname{Tr} \mathcal{T}_n(z) - (b_1 ... b_{n-1})\tau^n - \frac{1}{\tau^n}(c_1 ... c_{n-1}) \end{aligned} \tag{10}$$

1.3. **Christoffel-Darboux sums.** Multiply (7) by $p_{k-1}(z')$ and subtract the result with z and z' exchanged:

$$\begin{aligned} [p_k(z)p_{k-1}(z') - p_{k-1}(z)p_k(z')] - b_{k-1}c_{k-1}[p_{k-1}(z)p_{k-2}(z') - p_{k-2}(z)p_{k-1}(z')] \\ = (z - z')p_{k-1}(z)p_{k-1}(z') \end{aligned}$$

[1]The formula is true also with $\tau = b_n$ and $1/\tau = c_0$: $\det[z - M(c_0, b_n)] = \det[z - M_{1:n}] - c_0 b_n \det[z - M_{2:n-1}] - (b_1 ... b_n) - (c_0 ... c_{n-1}) = p_n(z) - c_0 b_n q_{n-2}(z) - (b_1 ... b_n) - (c_0 ... c_{n-1})$.

Divide by $b_{k-1}c_{k-1}...b_1c_1$ and sum for $k = 2...n$. Most terms cancel. The result is the Christoffel-Darboux summation:

$$\frac{p_n(z)p_{n-1}(z') - p_{n-1}(z)p_n(z')}{b_{n-1}c_{n-1}...b_1c_1(z-z')} = 1 + \sum_{k=2}^{n} \frac{p_{k-1}(z)p_{k-1}(z')}{b_{k-1}c_{k-1}...b_1c_1} \tag{11}$$

The limit $z = z'$ gives a second formula:

$$\frac{p'_n(z)p_{n-1}(z) - p'_{n-1}(z)p_n(z)}{b_{n-1}c_{n-1}...b_1c_1} = 1 + \sum_{k=2}^{n} \frac{p^2_{k-1}(z)}{b_{k-1}c_{k-1}...b_1c_1} \tag{12}$$

1.4. **Eigenvectors.** The eigenvalue equation $Mu = z\,u$, has solution for $z = z_j$, a zero of the characteristic polynomial $p_n(z_j) = 0$. If the recursion for the polynomials (7) is divided by $b_{k-1}b_{k-2}...b_1$, one obtains the recursion of eigenvectors (1). Therefore:

$$\boxed{u_k(z_j) = q_j \frac{p_{k-1}(z_j)}{b_{k-1}\dots b_1}, \qquad k = 2, \dots, n.} \tag{13}$$

where $u_1(z_j) \equiv q_j$ is determined by the normalization condition, and can always be chosen real and positive.
Left eigenvectors solve $v^T M = z\,v^T$ i.e. $M^T v = z\,v$. The difference is the exchange of c_k with b_k. The formulae (11) and (12) imply $v(z_j)^T u(z_k) = 0$ if $z_j \neq z_k$ and

$$v(z_j)^T u(z_j) = q_j q'_j \frac{p'_n(z_j)p_{n-1}(z_j)}{b_{n-1}c_{n-1}...b_1c_1}$$

2. Real symmetric tridiagonal matrices

$$H = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix}$$

a_k and b_k are real; the zeros of $p_k(z)$ are real.

2.1. **Simplicity of eigenvalues.** By Cauchy's theorem, the zeros of p_{k-1} intertwine those of p_k. The recursion (7) shows that intertwining is strict, otherwise all p_k would share the same zero. As a consequence, the zeros are simple. We give another proof.

Theorem 2.1. *The eigenvalues of a real symmetric tridiagonal matrix are simple.*

Proof. The second Christoffel-Darboux formula (12) in the real symmetric case is:

$$\frac{p'_m(z)p_{m-1}(z) - p_m(z)p'_{m-1}(z)}{b^2_{m-1}...b^2_1} = 1 + \sum_{k=2...m} \frac{p_{k-1}(z)^2}{b^2_{k-1}...b^2_1} > 1 \tag{14}$$

It follows that p_m and p'_m are never simultaneously zero, for all m. □

Since all roots of $p_n(x) = x^n + a_1 x^{n-1} + \cdots + a_n$ are real and simple, according to *Descartes' rule* the number of positive roots (i.e. eigenvalues of T) is equal to the number of sign differences between consecutive non-zero coefficients.
For example, the characteristic polynomial $p_3(x) = x^3 - 3x^2 - 5x + 12$ has 2 positive (and 1 negative) roots.

2.2. **Eigenvectors.** The eigenvalue equation $Hu = z\,u$, is solved for real $z = x_j$, a zero of the characteristic polynomial $p_n(x_j) = 0$, and

$$u_k(x_j) = q_j \frac{p_{k-1}(x_j)}{b_1 \dots b_{k-1}}, \qquad k = 2, \dots, n. \tag{15}$$

The first component $u_1(x_j) \equiv q_j$ is determined by the normalization condition $\|u(x_j)\|^2 = 1$, which is evaluated with the formula (14):

$$1 = q_j^2 + q_j^2 \sum_{k=2\dots n} \frac{p_{k-1}^2(x_j)}{b_1^2 \dots b_{k-1}^2} = q_j^2 \, \frac{p_n'(x_j) p_{n-1}(x_j)}{b_1^2 \dots b_{n-1}^2} \quad \text{i.e.}$$

$$(b_1 \dots b_{n-1})^2 = q_j^2 \, p_{n-1}(x_j) \prod_{k, k \neq j} (x_j - x_k) \tag{16}$$

The normalised eigenvectors $u(x_j)$ are the columns of an orthogonal matrix that diagonalizes H:

$$H = QXQ^T, \quad X = \mathrm{Diag}(x_1 > x_2 > .... > x_n), \quad Q_{ij} = u_i(x_j)$$

The relations $QQ^T = 1$ and $Q^T Q = 1$ correspond to completeness and orthonormality of eigenvectors in $\mathbb{R}^n$:

$$\sum_{j=1\dots n} u_k(x_j) u_m(x_j) = \delta_{km}, \qquad \sum_{k=1\dots n} u_k(x_j) u_k(x_m) = \delta_{jm} \tag{17}$$

In particular $\sum_{j=1}^n q_j^2 = 1$.

2.3. **Vandermonde determinant of eigenvalues.**

$$\prod_{j=1}^{n-1} b_j^{n-j} = \prod_{k=1}^{n} q_k \prod_{j<k} (x_j - x_k) \tag{18}$$

Proof. The determinant $\det[u_k(x_j)] = \pm 1$ can be evaluated:

$$\det \begin{bmatrix} u_1(x_1) & \dots & u_1(x_n) \\ u_2(x_1) & \dots & u_2(x_n) \\ \vdots & & \vdots \\ u_n(x_1) & \dots & u_n(x_n) \end{bmatrix} = \frac{\prod_j q_j}{b_1^{n-1} b_2^{n-2} \dots b_{n-1}} \det \begin{bmatrix} 1 & \dots & 1 \\ p_1(x_1) & \dots & p_1(x_n) \\ \vdots & & \vdots \\ p_{n-1}(x_1) & \dots & p_{n-1}(x_n) \end{bmatrix}$$

$$= \frac{\prod_j q_j}{b_1^{n-1} b_2^{n-2} \dots b_{n-1}} \prod_{i>k} (x_i - x_k)$$

If eigenvalues are ordered $x_1 > ... > x_n$ the determinant is $+1$ □

2.4. **A map to eigenvalues and a unit vector.** We now show that the unit vector $\mathbf{q}$ with $q_j = u_1(x_j) > 0$, and the ordered vector of eigenvalues, uniquely determine the matrix H with positive b_j and the eigenvectors.

Theorem 2.2 (Parlett, [17])**.** *For a real symmetric tridiagonal matrix H, there is a one-to-one correspondence of $\{\mathbf{a}, \mathbf{b}\}$ with $\{\mathbf{q}, \boldsymbol{x}\}$ where $\mathbf{q}$ is a unit norm $n-$dimensional vector of positive real entries, and $\boldsymbol{x}$ a strictly increasingly ordered sequence of n real numbers.*

Proof. Write $QH = XQ$. Now, the eigenvectors of H are the rows of Q, and $\mathbf{q}_k$ are the columns of Q. The j-th column of the relation is the recurrence relation:

$$b_j \mathbf{q}_{j+1} = X\mathbf{q}_j - a_j \mathbf{q}_j - b_{j-1}\mathbf{q}_{j-1} \equiv \mathbf{r}_j \tag{19}$$

with $b_0 = b_n = 0$. The scalar product with $\mathbf{q}_j$ gives $a_j = \mathbf{q}_j \cdot X\mathbf{q}_j$. It is also $b_j = \|\mathbf{r}_j\|$ and $\mathbf{q}_{j+1} = \mathbf{r}_j / b_j$. Then, b_{j-1}, $\mathbf{q}_{j-1}$ determine in turn a_j, $\mathbf{r}_j$, b_j and $\mathbf{q}_{j+1}$ for $j = 1, ..., n$. Since $b_0 = 0$, the vector $\mathbf{q}_1$ determines alone $a_1 = \mathbf{q}_1 \cdot X\mathbf{q}_1$, $\mathbf{r}_1 = (X - a_1)\mathbf{q}_1$, $b_1 = \|\mathbf{r}_1\|$ and $\mathbf{q}_2 = \mathbf{r}_1 / b_1$ and so, by finite induction, it determines uniquely all the elements of H and Q.
Similarly, one may show that $\mathbf{q}_n$ and X determine T and Q uniquely. □

As a consequence, we prove the following formula for the infinitesimal volumes:

Theorem 2.3 (see Forrester, [6] p.46)**.**

$$\boxed{\prod_{k=1}^{n} da_k \prod_{j=1}^{n-1} db_j = \frac{1}{\sqrt{1 - \sum_{j=1}^{n} q_j^2}} \frac{\prod_{r=1}^{n-1} b_r}{\prod_{r=1}^{n-1} q_r} \prod_{k=1}^{n} dx_k \prod_{j=1}^{n-1} dq_j} \tag{20}$$

Proof. Let $T = Q^T X Q$, where X is the diagonal matrix of eigenvalues and Q is orthogonal with first column being the unit vector $\mathbf{q}$. Then: $(T^k)_{11} = (Q^T X^k Q)_{11} = \sum_{j=1}^{n} q_j^2 x_j^k$ and $d(T^k)_{11} = \sum_j k q_j^2 x_k^{k-1} dx_j + 2 q_j x_j^k dq_j$. The condition $\|\mathbf{q}\|^2 = 1$ gives $q_n dq_n = \sum_{j<n} q_j dq_j$, then:

$$d(T^k)_{11} = \sum_{j=1}^{n} k q_j^2 x_k^{k-1} dx_j + \sum_{j=1}^{n-1} 2 q_j (x_j^k - x_n^k) dq_j, \quad k = 1, 2, ..., 2n-1$$

$$L(d\mathbf{a}, d\mathbf{b}) = M(d\mathbf{x}, d\mathbf{q})$$

The matrix L has a triangular structure: $dT_{11} = da_1$, $d(T^2)_{11} = 2a_1 da_1 + 2b_1 db_1$, $d(T^3)_{11} = ... + b_1^2 da_2$, ... , $d(T^{2n-1})_{11} = ... + b_1^2 b_2^2 \cdots b_{n-1}^2 da_n$. Dots mean differential terms with da_k and db_k that are accounted by lower powers of T:

$$\begin{bmatrix} 1 & & & & & & & \\ ... & 2b_1 & & & & & & \\ & ... & b_1^2 & & & & & \\ & & ... & 2b_1^2 b_2 & & & & \\ & & & & \ddots & & & \\ & & & & & 2(b_1 \cdots b_{n-2})^2 b_{n-1} & & \\ ... & ... & & & ... & ... & (b_1 ... b_{n-1})^2 & \end{bmatrix} \begin{bmatrix} da_1 \\ db_1 \\ da_2 \\ db_2 \\ \vdots \\ db_{n-1} \\ da_n \end{bmatrix}$$

The right-side term is:

$$\left[\begin{array}{ccc|ccc} ... & q_k^2 & ... & ... & 2q_j(x_j-x_n) & ... \\ ... & 2q_kx_k & ... & ... & 2q_j(x_j^2-x_n^2) & ... \\ & \vdots & & & \vdots & \\ & \vdots & & & \vdots & \\ & \vdots & & & \vdots & \\ ... & (2n-1)q_k^2x_k^{2n-2} & ... & ... & 2q_j(x_k^{2n-1}-x_n^{2n-1}) & ... \end{array}\right]\left[\begin{array}{c} dx_1 \\ \vdots \\ dx_n \\ dq_1 \\ \vdots \\ dq_{n-1} \end{array}\right]$$

where the colums are $k=1,..,n$, $j=1,..,n-1$.
Thus we have $(d\mathbf{a},d\mathbf{b})=L^{-1}M(d\mathbf{x},d\mathbf{q})$. The ratio of volume elements is

$$\prod_{j=1...n}da_j\prod_{k=1..n-1}db_k=\frac{\det M}{\det L}\prod_{j=1...n}dx_j\prod_{k=1..n-1}dq_k$$

where $\det L=2^{n-1}\prod_{k=1}^{n-1}b_k^{4(n-k)-1}$ and $\det M=2^{n-1}(q_1...q_{n-1})^3q_n^2\det M'$,

$$M'=\left[\begin{array}{cccccc} 1 & ... & 1 & x_1-x_n & ... & x_{n-1}-x_n \\ 2x_1 & ... & 2x_n & x_1^2-x_n^2 & ... & x_{n-1}^2-x_n^2 \\ \vdots & & \vdots & \vdots & & \vdots \\ (2n-1)x_1^{2n-2} & ... & (2n-1)x_n^{2n-2} & x_1^{2n-1}-x_n^{2n-1} & ... & x_{n-1}^{2n-1}-x_n^{2n-1} \end{array}\right]$$

the result $\det M'=\prod_{i>j}(x_i-x_j)^4$ is proven in the appendix (see ref.[14]). Therefore:

$$\frac{\det M}{\det L}=q_n^2\prod_{k=1}^{n-1}\frac{q_j^3}{b_j^{4(n-j)-1}}\prod_{i>k}(x_i-x_k)^4$$

The Lemma says that: $\prod_{j=1}^{n-1}b_j^{4(n-j)}=\prod_{k=1}^{n}q_k^4\prod_{j>k}(x_j-x_k)^4$ and the Vandermonde simplifies. □

The resolvent. The resolvent of the tridiagonal matrix $G(z)=\det(z-H)^{-1}$ can be evaluated both by the formula for inversion and by the spectral representation.

$$G(z)_{ij}=\sum_a\frac{\langle i|x_a\rangle\langle x_a|j\rangle}{z-x_a}=\frac{\det(z-H_{ji})}{\det(z-H)} \tag{21}$$

where H_{ji} is the matrix H with row j and column i removed. In the limit $z=x_a$, the residues are obtained:

$$\langle i|x_a\rangle\langle x_a|j\rangle=\frac{\det(x_a-H_{ji})}{\prod_b'(x_a-x_b)} \tag{22}$$

In particular, from $G(z)_{1n}$ we reobtain (16).
The equation for the resolvent $(z-M)G(z)=1$, i.e.

$$-b_jG(z)_{j+1,k}+(z-a_j)G(z)_{j,k}-b_{j-1}G(z)_{j-1,k}=\delta_{j,k}$$

is liable of a transfer matrix solution. In particular, the components of the transfer matrix can be expressed as follows:

$$\mathsf{T}_n(z)=\frac{1}{G_{1,n}}\left[\begin{array}{cc} 1 & G_{1,1} \\ -G_{n,n} & G_{n,n}G_{1,1}-G_{n,1}G_{1,n} \end{array}\right] \tag{23}$$

The large n behaviour of the transfer matrix can be related to the decay properties of the inverse of tridiagonal matrices.

2.5. **Periodic tridiagonal matrices.** Now consider the real tridiagonal matrices with the addition of corner terms τ^n and $1/\tau^n$ that enter as b.c. in the eigenvalue equation:

$$H(\tau^n) = \begin{bmatrix} a_1 & b_1 & & 1/\tau^n \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ \tau^n & & b_{n-1} & a_n \end{bmatrix} \tag{24}$$

When $|\tau| = 1$ the matrices are Hermitian and correspond to the Bloch decomposition of infinite periodic matrices, $H_{i+n,j+n} = H_{i,j}$, with Bloch phase τ. If also $\xi \neq 0$ the matrices are no longer Hermitian, but this is advantageous for the study of localization of eigenvectors.
The Duality identities (4) and (5) become:

$$\boxed{\frac{\det[z - H(\tau^n)]}{b_1...b_{n-1}} = \operatorname{tr} \mathsf{T}_n(z) - (\tau^n + \frac{1}{\tau^n}) = -\frac{1}{\tau^n} \det[\mathsf{T_n(z)} - \tau^{\mathsf{n}}]} \tag{25}$$

Note the useful identification:

$$\boxed{\det[z - H(\pm i)] = (b_1...b_{n-1}) \operatorname{tr} \mathsf{T}_n(z)} \tag{26}$$

Proposition 2.4. *The eigenvalues of $H(\tau^n)$ solve the equation*

$$\det[z - H(i)] = (\tau^n + \frac{1}{\tau^n})(b_1...b_{n-1}) \tag{27}$$

They are determined by the intersections of the fixed line $y = \det[z - H(i)]$, with the line $y = (\tau^n + \frac{1}{\tau^n})(b_1...b_{n-1})$. This can be visualized for $z = x$ real, in the plane (x, y), and $\tau^n = \exp(i\varphi)$ or $\tau = \exp \xi$.

2.6. **Bands.** For $\tau^n = \exp(i\varphi)$ the matrix $H(e^{i\varphi})$ is Hermitian and the eigenvalues are real, periodic functions $x_j(\varphi)$ with period 2π, and solve:

$$\det[x - H(e^{i\pi/2})] = 2(b_1...b_{n-1}) \cos \varphi \tag{28}$$

They are the abscissae of the n intersections of the fixed line $y = \det[x - H(e^{i\pi/2})]$ with the line $y = 2(b_1...b_{n-1}) \cos \varphi$, and such n intersections exist and are distinct. As the angle changes, the eigenvalues sweep bands in the real axis. The extrema (inversion points) of each interval are at $\varphi = 0, \pi$ (periodic or antiperiodic b.c.). Since for any φ there are n distinct real eigenvalues, the bands do not overlap (at most may touch at extrema).

The *level velocity* $\dot{x}_j(\varphi)$ is related to the slope of the polynomial at $x = x_j(\varphi)$:

$$\dot{x}_j(\varphi) = -2 \sin \varphi \frac{b_1...b_{n-1}}{p'_n(x_j(\varphi); i)} \tag{29}$$

where $p_n(x; i) = \det[x - H(i)]$. It vanishes at the band's ends.

2.7. **Lemniscates.** For $\tau = \pm e^{\xi}$ and increasing ξ, the intersection of the polynomial line with the line $y = \pm 2(b_1...b_{n-1}) \cosh(n\xi)$ continues to have n intersections, that determine eigenvalues $x_j(n\xi)$ that now enter the gaps between bands, and eventually collide. A collision occurs at a local maximum or minimum of $p(x; i)$. After the collision they become complex.

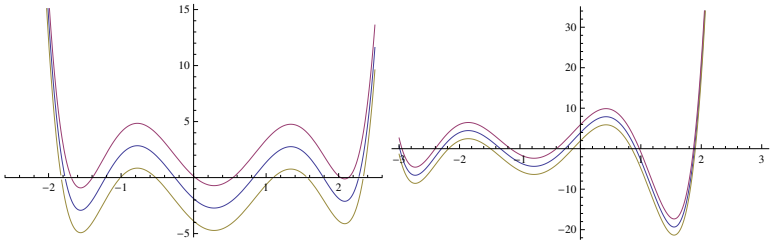

FIGURE 3. $y = \det[x - T(e^{i\varphi})]$ for 3 Anderson matrices 6×6, with $W = 1$ and $W = 2$. From above: $\varphi = 0$ (periodic b.c.), $\varphi = \frac{\pi}{2}$ and $\varphi = \pi$ (antiperiodic b.c.). The eigenvalues for $\varphi = 0$ and $\varphi = \pi$ bound the spectral bands. The lateral bands are narrower: the eigenvalues are less sensitive to variations of b.c., and eigenstates are more localized. Larger disorder (right) gives narrower bands (stronger localization).

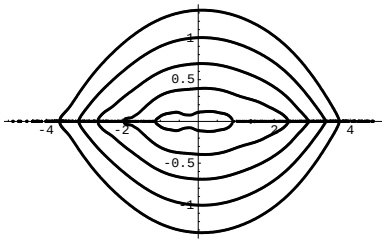

FIGURE 4. The superposition of five spectra of $n = 600$ eigenvalues of a random matrix with diagonal elements uniformly chosen in $[-3.5, 3.5]$, $b_k = 1$, for values of ξ increasing from 0 to 1. They lie on spectral curves $\xi = \frac{1}{n} \log \tau(z)$, where $\tau(z)$ is eigenvalue of the transfer matrix $\mathsf{T}_n(z)$ (from [16]).

3. TRIDIAGONAL MODELS

Several models on the 1D lattice are described by Hamiltonian matrices $H = T + D$, where $(Tu)_k = u_{k+1} + u_{k-1}$ is the kinetic term and $(Du)_k = a_k u_k$ is a potential specified by numbers a_k. The eigenvalue equation is

$$u_{k+1} + u_{k-1} + a_k u_k = x u_k$$

3.1. **Free particle.** $u_{k+1} + u_{k-1} = x u_k$. General solution: $u_k(p) = Ae^{ipk} + Be^{-ipk}$, $x(p) = 2\cos p$. The parameter p is fixed by b.c. Open chain ($u_0 = u_{n+1} = 0$): $u_k(x_j) = \sqrt{\frac{2}{n}} \sin(\frac{2\pi}{n+1} jk)$, $x_j = 2\cos(\frac{2\pi}{n+1} j)$, $j = 1...n$. Periodic chain ($u_k = u_{k+n}$): $u_k(x_j) = \frac{1}{\sqrt{n}} \exp(\pm i \frac{2\pi}{n} jk)$, $x_j = 2\cos(\frac{2\pi}{n} j)$, $j = 1...n$.

3.2. **Particle in electric field.** $u_{k+1} + u_{k-1} + Ek\, u_k = x u_k$.
General solution with Bessel functions.

3.3. **Harper (or almost-Mathieu) equation.** $u_{k+1} + u_{k-1} + 2g\cos(2\pi k\alpha + \beta) u_k = x u_k$, $g \neq 0$. The case $g = 1$ arose from the study of en electron in a

square lattice and magnetic field (Hofstadter), and attracted a lot of interest. The Fourier transform

$$u_k = e^{i\beta k} \sum_{m\in\mathbb{Z}} e^{im(2\pi\alpha k+\beta)} v_m$$

reproduces the equation, $v_{k+1} + v_{k-1} + (2/g)\cos(2\pi\alpha k + \beta)v_k = xv_k$, with coupling $g \to 1/g$. The model is self-dual for $g = 1$. The spectrum of the Hamiltonian is a Cantor set for all irrational α [1].

3.4. **1D Anderson model.** Consider the ensemble of random matrices $H = T + D$ where the potential D is specified by i.i.d. random numbers a_k. In the Anderson model a_k have uniform distribution $[-W, W]$ (W is the parameter of disorder). The spectrum is contained in the interval $|x| \leq 2 + W$ and for the infinite system it remains pure point for all values $W > 0$. The eigenvectors are exponentially localized (see fig.5).

The transfer matrix $\mathsf{T}_n(x)$ is the product of n independent random matrices, and has unit determinant. For large n, with probability 1, its eigenvalues are $\exp[\pm n\gamma(x)]$, where $\gamma(x)$ is independent of n and of the realization of disorder (Furstenberg's theorem, (1963), see [13]). It is the inverse of the localization length. The exponential localization of eigenvectors for $W > 0$, implies that the matrix H has pure point spectrum in the $n \to \infty$ limit. The rate of decay is evaluated by the Herbert-Jones [12] and Thouless formula.

Proposition 3.1 (Thouless formula, [19], 1972)**.**

$$\boxed{\gamma(x) = \int dx' \log|x - x'|\rho(x') - \langle \log b \rangle} \tag{30}$$

where $\rho(x)$ is the density of eigenvalues of the tridiagonal matrix $T(i)$.

Proof. The log of the modulus of eq.(26) is

$$\tfrac{1}{n}\sum_{j=1}^{n} \log|x - x_j(i)| = \tfrac{1}{n}\sum_{k=1}^{n-1} \log|b_k| + \tfrac{1}{n}\log \mathrm{tr}\mathsf{T}_n(z)$$

The large n limit and Furstenberg's theorem, $\mathrm{tr}\mathsf{T}_n(x) \approx e^{n\gamma(x)}$, give the result. □

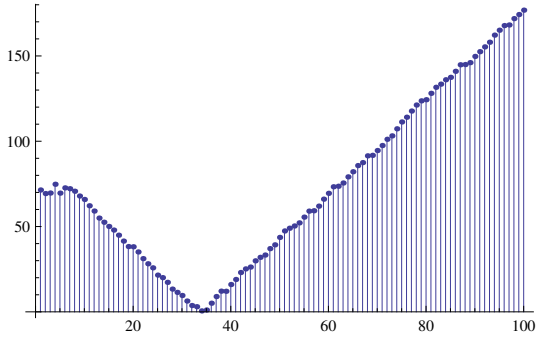

FIGURE 5. An eigenvector of the Anderson model, N=100, W=4. It plots $-\log|u_k|^2$, where $|u_k|^2$ is the probability of occupation of site k, versus $k = 1...100$. The eigenstate is exponentially localized around site $k = 35$.

3.5. **Hatano-Nelson model.** Hatano and Nelson [11] inaugurated a study of localization of eigenvectors by modifying the Anderson Hamiltonian as follows:

$$e^{\xi}u_{k+1} + e^{-\xi}u_{k-1} + a_k u_k = E(\xi)u_k \tag{31}$$

For $\xi = \xi_{cr}$, the first pair of real eigenvalues collide at value $E \approx 0$ becoming complex. This critical value is $\xi_{cr} = \gamma(0)$. As ξ grows, more eigenvalues in the center of the band turn to complex, and distribute on a single a line

$$\gamma(E_x + iE_y) = \xi$$

with tails of real eigenvalues at the edge of the spectrum, where eigenstates are more localized. The study in depth of the single line of eigenvectors was done by Goldsheid and Khoruzhenko [7], [8],[9] and several others.
The explanation is simple. Eq.(31) results after a similarity transformation of a matrix $H(e^{n\xi})$ with unit hopping, and terms $\exp(\pm n\xi)$ in the corners. For $\xi = 0$ all eigenvalues are real and eigenvectors are exponentially localized with rate $\gamma(E)$. The similarity multiplies u_k by $e^{k\xi}$. If ξ is big enough, it overcomes the exponential localization $e^{-\gamma|k-k_0|}$ and two eigenvalues jumps to complex. This means that the eigenvectors feel the non-Hermitian b.c., i.e. the exponential localization $e^{-k\gamma(E)}$ is destroyed by the factor $e^{k\xi}$.
The analogous behaviour in tridiagonal matrices with complex spectrum, was investigated by Molinari and Lacagnina [16].

3.6. **The Lloyd model.** In 1969 P. Lloyd introduced a tight-binding model with Cauchy disorder for the study of the Anderson localisation in 3D. Its main feature is that the average spectral density may be analytically evaluated [15].
The evaluation can be extended to $H = H_0 + D$, where H_0 is any Hermitian matrix of size n and D is diagonal with i.i.d. elements a_j distributed according to the Cauchy distribution

$$p(a) = \frac{\delta}{\pi}\frac{1}{a^2 + \delta^2} \tag{32}$$

Spectral density. Let H be a $n \times n$ Hermitian random matrix, with resolvent $G(z) = (z - H)^{-1}$ and eigenvalues x_k. The averaged density of the eigenvalues is

$$\rho(x) = \Big\langle \frac{1}{n}\sum_k \delta(x - x_k) \Big\rangle = \frac{1}{\pi}\lim_{\epsilon\to 0} \mathrm{Im}\,\langle \frac{1}{n}\mathrm{tr}\, G(x - i\epsilon)\rangle \tag{33}$$

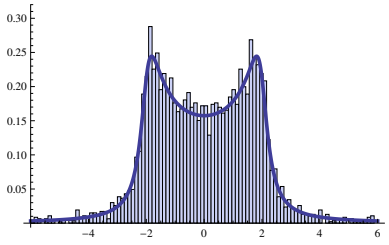

FIGURE 6. The eigenvalues of a random matrix $(Hu)_k = u_{k+1} + u_{k-1} + \epsilon_k u_k$ of size $n = 4000$, with Cauchy disorder $\delta = 0.3$. The line is the theoretical distribution eq.(38).

The following formula holds:

$$\operatorname{tr} G(z) = -\frac{d}{dz}\frac{\det(z'-H)}{\det(z-H)}\Big|_{z'=z} \tag{34}$$

The ratio of determinants can be expressed as a supersymmetric integral, which makes the averaging possible:

$$\frac{\det(x'-H)}{\det(x-i\epsilon-H)} = \int \prod_{k=1}^{n} d\bar{\psi}_k d\psi_k \frac{d^2\varphi_k}{\pi}\; e^{-i\bar{\psi}(x'-H)\psi - i\varphi^\dagger(x-i\epsilon-H)\varphi} \tag{35}$$

The Lloyd model is simplest example of evaluation of an average on disorder.

The average of (35) factors because the a_j are independent. A factor is

$$\langle e^{ia\,(\bar{\psi}_j\psi_j + |\varphi_j|^2)}\rangle = \int da\, p(a)\, e^{ia\,|\varphi_j|^2}(1 + ia\bar{\psi}_j\psi_j) = e^{-\delta(\bar{\psi}_j\psi_j + |\varphi_j|^2)}$$

$$\begin{aligned}\Big\langle \frac{\det(x'-H)}{\det(x+i\epsilon-H)}\Big\rangle &= \int \prod_{k=1}^{n} d\bar{\psi}_k d\psi_k \frac{d^2\varphi_k}{\pi}\; e^{-i\bar{\psi}(x'-H_0-i\delta)\psi - i\varphi^\dagger(x-H_0-i\delta)\varphi} \\ &= \frac{\det(x'-H_0-i\delta)}{\det(x-H_0-i\delta)}\end{aligned}$$

Then $\langle\rho(x)\rangle = \frac{1}{\pi}\mathrm{Im}\,\frac{1}{n}\operatorname{tr} G_0(x-i\delta)$ where $G_0(z)$ is the resolvent of H_0. If x_k are the eigenvalues of H_0, the spectrum of H averaged on Cauchy disorder is just a superposition of Cauchy distributions:

$$\rho(x) = \frac{1}{n}\sum_{k=1\ldots n}\frac{1}{\pi}\frac{\delta}{(x-x_k)^2+\delta^2} \tag{36}$$

If H_0 is the adjacency matrix for the cubic lattice $\mathbb{Z}^d$ with spacing L, the spectral density per unit volume is

$$\begin{aligned}\rho_0(x) =& \frac{1}{L^d}\sum_{1\le k_i\le L}\delta(x - 2\cos(\tfrac{2\pi k_1}{L}) - \ldots - 2\cos(\tfrac{2\pi k_d}{L})) \\ =& \int_{\mathbb{R}^d}\frac{d\mathbf{k}}{(2\pi)^d}\delta(E - 2\sum_j \cos k_j) = \int_0^\infty \frac{ds}{\pi} J_0(2s)^d\cos(xs)\end{aligned}$$

The spectral density of $H_0 + D$ evaluated as a convolution integral,

$$\rho(x) = \int_{-\infty}^{+\infty} dy \frac{\delta}{\pi}\frac{\rho_0(y)}{(x-y)^2+\delta^2} = \int_0^\infty \frac{ds}{\pi} J_0(2s)^d e^{-\delta s}\cos(xs) \tag{37}$$

The integral for $d = 1$ is known (GR 6.751.3). See fig.6 for a plot:

$$\rho(x) = \frac{1}{\pi\sqrt{2}}\frac{\sqrt{4+\delta^2-x^2+\sqrt{(4+\delta^2-x^2)^2+4x^2\delta^2}}}{\sqrt{(4+\delta^2-x^2)^2+4x^2\delta^2}} \tag{38}$$

The Lyapunov exponent of the 1D Lloyd model can be analytically evaluated:

$$\cosh\gamma(x) = \tfrac{1}{4}\sqrt{(x-4)^2+\delta^2} + \tfrac{1}{4}\sqrt{(x+4)^2+\delta^2} \tag{39}$$

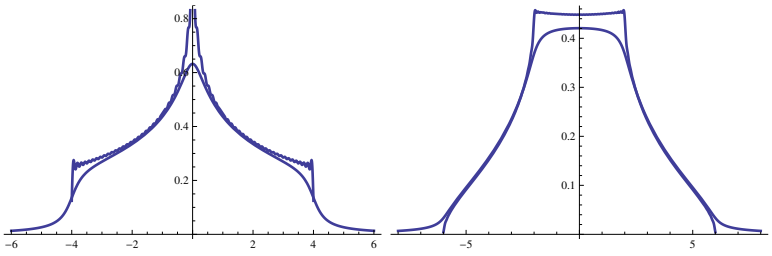

FIGURE 7. The distributions of eigenvalues of the Laplacian and of the Laplacian with Cauchy diagonal (Lloyd model, $\delta = 0.3$) on the square and cubic lattices. The van Hove singularities are smoothed and the disorder adds tails to the sharp band $[-2d, 2d]$.

4. Measures of localization of eigenvectors

Consider a Hermitian $n \times n$ matrix $H_{ij} = \sum_a x_a \langle i|x_a\rangle\langle x_a|j\rangle$, with propagator $U(t) = \exp(-itH)$, resolvent $G(z) = (z - H)^{-1}$.
The spectrum of the matrix says nothing about the eigenvectors, unless one perturbs the matrix to detect the shift of the eigenvalues: linear response relates the shift to an expectation on the unperturbed state. Other measures of localization require the eigenvector components. Usually, the measures have to be normalized with respect to some reference system. Below, we list some.

Width of probability. The numbers $|\langle j|E_a\rangle|^2$, $j = 1...n$ are probabilities for the occupation of "site" j. The perturbation of H with $\delta H_{ij} = \epsilon\delta_{ij}j$ gives, by linear response, the shifts $\delta E_a = \epsilon \sum_{j=1}^n j|\langle j|E_a\rangle|^2$ i.e. the baricenters j_a of the probability distribution. The perturbation $\delta H_{ij} = \epsilon\delta_{ij}j^2$ provides the values $(j^2)_a$ and, finally $\Delta_a = \sqrt{(j^2)_a - (j_a)^2}$ measures the width of the probability distribution: i.e. where the particle is most likely to be.

Inverse participation ratio. The inverse participation ratio (IPR) of a normalized vector u is the sum $\sum_{j=1}^n |u_j|^4$. Its inverse (the participation ratio) is a measure of

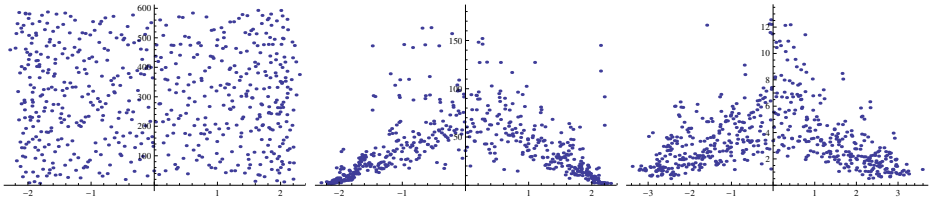

FIGURE 8. Anderson model, $n = 600$, $W = 0.5$. Left: the baricenters of the eigenstates are almost uniformly scattered in (0,600) for all values of the energy in $[-2.5, 2.5]$. Middle: the eigenstates near the edges of the spectrum are localized, while at the center of spectrum they are still delocalized. Right: at $W = 2$ all states are localized (variance localization Δ).

localization of the vector:

$$\xi_2 = (\sum_{j=1}^{n} |u_j|^4)^{-1} \tag{40}$$

The inverse participation ratio measures the time-average of the probability of return:

$$\begin{aligned}\pi_{jj} &= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt |\langle j|U(t)|j\rangle|^2 \\ &= \sum_{rs} |\langle j|x_a\rangle|^2 |\langle j|x_b\rangle|^2 \lim_{T\to\infty}\frac{1}{T}\int_0^T dt\, e^{-it(x_a-x_b)} = \sum_a |\langle j|x_a\rangle|^4\end{aligned}$$

Since $U(t)\theta(t) = -\int_{-\infty}^{+\infty} \frac{dx}{2\pi i} G(x+i\epsilon)e^{-ixt}$, it is $\pi_{mn} = \int_{-\infty}^{+\infty}\frac{dx}{2\pi}|G(x+i\epsilon)_{n,m}|^2$.

Entropy length. The Shannon entropy associated to a normalized probability is $S[u] = -\sum_{j=1}^n |u_j|^2 \log|u_j|^2$. The entropy length is

$$\xi_0 = \exp S \tag{41}$$

The last two measures are special cases of $\xi_q = (\sum_{k=1}^n |u_i|^{2q})^{1/(1-q)}$.
The table summarizes some special distributions:

(42)

Vector	Δ	ξ_2	ξ_0	Notes
Box	$\frac{L}{2\sqrt{3}}$	L	L	
Exponential	$\frac{1}{\sqrt{2\mathrm{sh}\gamma}}$	$\frac{\mathrm{th}(2\gamma)}{\mathrm{th}^2\gamma}$	$\frac{\exp[2\gamma/\mathrm{sh}(2\gamma)]}{\mathrm{th}\gamma}$	
Constant	$\frac{1}{2\sqrt{3}}n$	n	n	
Open chain	$\frac{1}{2\sqrt{3}}n$	$\frac{2}{3}n$		large n
GOE(n)		$\frac{1}{3}n$	$\frac{1}{2}n$	large n

- Box state, $|u_k|^2 = 1/L$ for $k = 1...L$, and 0 otherwise.
- Exponential state, $u_k = \sqrt{\tanh\gamma}\exp(-\gamma|k|)$.
- GOE (n) eigenvectors are uniformly distributed on the sphere $\|u\| = 1$ in $\mathbb{R}^n$. The average values are: $\langle S\rangle = \psi(\frac{n}{2}) - \psi(\frac{3}{2}) \to \log n - 0.7296$, $\langle\sum_{k=1}^n |u_i|^4\rangle \to \frac{3}{n}$.
- Open chain, $u_k(j) = \sqrt{\frac{2}{n+1}}\sin(\frac{kj\pi}{n+1})$.

Note: The eigenvectors of GOE (n) matrices are uniformly distributed on the surface of the unit sphere in $\mathbb{R}^n$. With $x = |u_k|^2$, it is:

$$p(x) = \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(\frac{1}{2})} x^{-1/2}(1-x)^{\frac{n-3}{2}} \to p(y) = \frac{1}{\sqrt{2\pi y}}e^{-y/2} \tag{43}$$

The large n expression (for $y = x/n$) is the *Porter - Thomas* distribution.

5. Methods of tridiagonalization

5.1. **Lanczos.** With the Lanczos algorithm, a real symmetric matrix S is brought to tridiagonal by an iterative process that only involves matrix multiplication of vectors. The steps are:
i) Choose a normalized vector with $S\mathbf{q_1} \neq 0$ and put $a_1 = \mathbf{q_1}^T S\mathbf{q_1}$ (this is T_{11}).
ii) The vector $\mathbf{q_2'} = (S - a_1)\mathbf{q_1}$ is orthogonal to $\mathbf{q}_1$. Put $b_1 = \|\mathbf{q_2'}\|$ (i.e. T_{12}). The

vector $\mathbf{q_2} = \mathbf{q'_2}/b_1$ is normalized; put $a_2 = \mathbf{q_2}^T S \mathbf{q_2}$.
iii) continue with the steps:

$$\begin{aligned} &\mathbf{q'_{k+1}} = (S - a_k)\mathbf{q_k} - b_{k-1}\mathbf{q_{k-1}}, \\ &b_k = \|\mathbf{q'_{k+1}}\|, \\ &\mathbf{q_{k+1}} = \mathbf{q'_{k+1}}/b_k, \qquad a_{k+1} = \mathbf{q_{k+1}}^T S \mathbf{q_{k+1}} \end{aligned}$$

The vectors $\mathbf{q_k}$ are orthonormal and are, in the order, the colums of an orthogonal matrix O such that $SO = O^T T$, where T is generated step by step.

5.2. **Householder transformation.** A symmetric matrix S can be reduced to tridiagonal form by a sequence of special transformations. It is sufficient to discuss the first step: the others produce similar results of smaller and smaller blocks.
Given a normalized vector $\mathbf{v}$ define the symmetric orthogonal matrix $H(\mathbf{v})_{ij} = \delta_{ij} - 2v_i v_j$. Its action on a vector is a reflection with respect to the hyperplane orthogonal to $\mathbf{v}$. The action on S is:

$$S'_{ij} = (HSH)_{ij} = S_{ij} - 2v_i(S\mathbf{v})_j - 2(S\mathbf{v})_i v_j + 4v_i v_j(\mathbf{v}^T S \mathbf{v})$$

Choose $v_1 = 0$ and denote the first row of S as $(S_{11}, \mathbf{s})$. It is:

$$S'_{11} = S_{11}, \quad \mathbf{s}'_j = \mathbf{s}_j - 2(\mathbf{s}\cdot\mathbf{v})v_j$$

The vector $\mathbf{v}$ is chosen such that $\mathbf{s}' = (S'_{12}, \mathbf{0})$. Then: $v_3 = \lambda s_3, ..., v_n = \lambda s_n$ where $0 = 1 - 2\lambda S_{12} v_2 - 2\lambda^2(\|\mathbf{s}\|^2 - S_{12}^2)$ and the normalization condition is $1 = v_2^2 + \lambda^2(\|\mathbf{s}\|^2 - S_{12}^2)$. Now, S' has the form

$$S = \begin{bmatrix} S_{11} & \mathbf{s} \\ \mathbf{s}^T & [S]_{11} \end{bmatrix} \to S' = \begin{bmatrix} S_{11} & S'_{12} & \mathbf{0} \\ S'_{12} & ... & ... \\ \mathbf{0} & ... & ... \end{bmatrix} \qquad S'_{12} = \|\mathbf{s}\| = \sqrt{\sum_{j=2}^{n} S_{1j}^2}$$

Another transformation is made, acting on the block $[S']_{11}$, with a new vector $(0, 0, \mathbf{v}')$. In the end, a tridiagonal symmetric marix is obtained, with off diagonal elements being positive.
Each Housholder transformation is orthogonal, then the eigenvalues of the tridiagonal matrix coincide with those of S.

5.3. **Beta ensembles.** If the symmetric matrix S belongs to $\mathsf{GOE}(n)$, the matrix elements are Gaussian random variables. After one Housholder step, $S'_{11} = S_{11}$ is N[0,1] while $S'_{12} \in \chi_{n-1}[\frac{1}{\sqrt{2}}]$, where $\chi^2_{n-1}[\frac{1}{\sqrt{2}}]$ is the probability distribution for the sum of $n-1$ independent squared Gaussian variables in N$[0, \frac{1}{\sqrt{2}}]$. Being the vector $\mathbf{v}$ independent of the matrix elements in $[S]_{11}$, the submatrix $[S]_{11}$ is a member of $\mathsf{GOE}(n-1)$.
Another Housholder step gives $S''_{22} = S'_{22} \in$ N[0,1] and $S''_{23} \in \chi_{n-2}[\frac{1}{\sqrt{2}}]$.
In the end we obtain a random tridiagonal matrix with a probability densities for matrix elements

$$T = \begin{bmatrix} N[0,1] & \chi_{n-1}[\frac{1}{\sqrt{2}}] & & \\ \chi_{n-1}[\frac{1}{\sqrt{2}}] & \ddots & \ddots & \\ & \ddots & \ddots & \chi_1[\frac{1}{\sqrt{2}}] \\ & & \chi_1[\frac{1}{\sqrt{2}}] & N[0,1] \end{bmatrix} \tag{44}$$

The ensemble of matrices T has the same joint probability for eigenvalues as GOE:

$$p(x_1, ..., x_n) = \frac{1}{2^{3n/2}} \prod_{k=1...n} \frac{1}{\Gamma(1+k/2)} |\Delta(\mathbf{x})| \exp(-\tfrac{1}{2}\|\mathbf{x}\|^2)$$

It has been proven [3][4][6] that the set of random tridiagonal matrices with diagonal elements $a_k \in N[0,1]$ and subdiagonal elements $b_k \in \chi_{\beta k}$ has joint probability density for the eigenvalues proportional to

$$\exp[-\tfrac{1}{2}\sum \lambda_k^2] \prod_{j<k} |\lambda_k - \lambda_j|^\beta$$

6. Dynamical localization in quantum chaos

6.1. **The kicked rotator.** (see [2][10][18]). Quantum chaos is the study of quantum manifestations of classical chaos, i.e. a dynamics characterised by exponential instability of the time evolution as initial conditions are varied. The kicked rotator is one of the simplest models exhibiting classical chaos (1979). It is a particle on the circle, that is periodically kicked, with strength $v(\theta + 2\pi) = v(\theta)$:

$$H(\theta, p, t) = \tfrac{1}{2}p^2 + v(\theta) \sum_n \delta(t-n) \tag{45}$$

The classical dynamics is described by the *standard map* (Chirikov-Taylor) in $[-\pi, \pi] \times \mathbb{R}$, for the evolution of the state (θ, p) in one time-period:

$$\begin{cases} \theta_{n+1} = \theta_n + p_{n+1} \mod 2\pi \\ p_{n+1} = p_n + v'(\theta_n) \end{cases}$$

The map is area-preserving.
For $v(\theta) = \kappa \cos\theta$, there are non KAM curves for $\kappa > \frac{63}{64}$. The map is then chaotic, with shrinking stability islands around the fixed point $(\pi, 0)$. Iteration gives $p_n = p_0 + \kappa \sum_{k=0}^{n-1} \sin\theta_k$. Squaring: $(p_n - p_0)^2 = \kappa^2 \sum_k \sin^2\theta_k + \kappa^2 \sum_{k\neq k'} \sin\theta_k \sin\theta_{k'}$. By assuming that the angles are randomly distributed, one obtains that the energy increases linearly in time (diffusion):

$$\overline{(p_n - p_0)^2} = \tfrac{1}{2}\kappa^2 n$$

where the average is on a bunch of initial angular conditions.
The quantum dynamics is described by the unitary time-evolution on a period (Floquet operator) given by a free evolution of duration $T = 1$ with Hamiltonian $-\frac{\hbar^2}{2}\partial_\theta^2$, and a kick:

$$F = \exp(-\frac{i}{\hbar}\kappa\cos\theta)\exp(-\frac{i}{2\hbar}p_\theta^2) \tag{46}$$

The eigenvalues of F, $\exp(-\frac{i}{\hbar}T\epsilon)$, define the "quasi-energies" ϵ (here $T = 1$).
In the basis of angular momentum, F is the matrix $F_{n,m} = J_{|n-m|}(\frac{\kappa}{\hbar})e^{\frac{i}{2}\hbar m^2}$. Since for large ν, fixed x: $J_\nu(x) \approx \frac{1}{\sqrt{2\pi\nu}}(\frac{ex}{2\nu})^\nu$, then $F_{n,m}$ is exponentially small for $|n-m| > \kappa/\hbar$. $F_{n,m}$ is a "random" banded matrix with bandwidth $b \approx \kappa/\hbar$.

For $\kappa > \kappa_c$ it was discovered that the energy $\langle \frac{p_\theta^2}{2} \rangle$, increases in time n as in the classical map up to a time n_c, when it reaches a saturation value. This phenomenon is a consequence of *dynamical localization*: an initial eigenstate $|m\rangle$ of angular momentum evolves in time n to $\sum_{m'} |m'\rangle\langle m'|F^n|m\rangle$. Localization means that, for $n > n_c$ the number of states coupled by F^n does not increase with n. The number

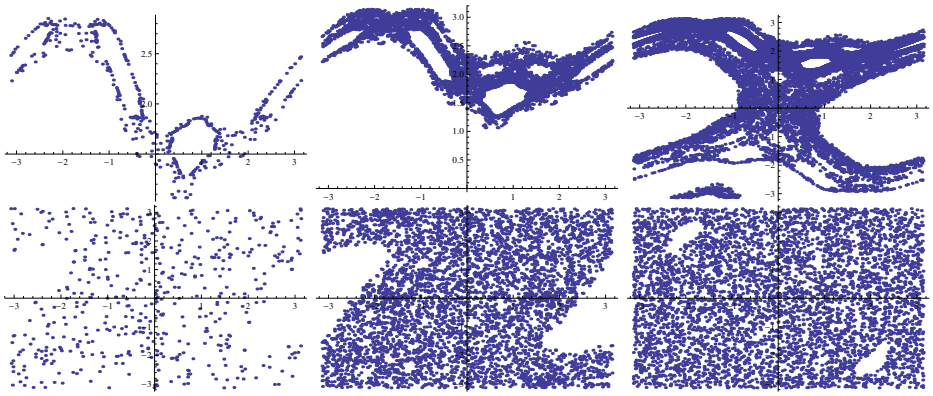

FIGURE 9. The standard map on the torus. The dynamics of a single initial point $\theta = 0.83$, $p = 1.32$. $\kappa = 1.1$ with 500, 5000 and 20.000 iterations; $\kappa = 3$, with 500, 5.000; $\kappa = 5$ with 5.000 iterations. For $\kappa = 3$ the point explores all the available surface except the island around the stable fixed point $(\pi, 0)$, which is invaded at $\kappa = 5$.

of coupled states Δn is the localization length.
Qualitatively, $\Delta n = \sqrt{Dn}$ where $D = k^2/2$. Diffusion stops at time n_c when discreteness of the quasi-energy spectrum is resolved, i.e. $\frac{2\pi}{\sqrt{Dn_c}} \cdot n_c \approx \hbar$. The relation gives $n_c \approx \hbar^2 D/4\pi^2$, and $\Delta n = \sqrt{Dn_c} \approx \hbar\kappa^2/2\pi$. This relation gave an important insight, that was later proved: *a banded random matrix with band-width b has eigenstates that are localized in a length b^2.*

6.2. **The Maryland construction.** The eigenvalue equation of the Floquet operator (46) can be mapped to a 1D tight-binding model, thus relating the dynamical localization to Anderson localization [5]. The angle-dependent unitary factor can be represented as the Cayley transform of a Hermitian (multiplication) operator:

$$e^{-\frac{i}{\hbar}v(\theta)} = \frac{1 - iW(\theta)}{1 + iW(\theta)} \Longrightarrow W(\theta) = \tan[\tfrac{1}{2\hbar}v(\theta)]$$

The eigenvalue equation $Fu = e^{-i\epsilon/\hbar}u$ becomes:

$$(e^{-\frac{i}{\hbar}(p^2/2-\epsilon)} - 1)u - iW(e^{-\frac{i}{\hbar}(p^2/2-\epsilon)} + 1)u = 0$$

By renaming $(e^{-\frac{i}{\hbar}(p^2/2-\epsilon)} + 1)u = v$ it is: $\tan[\frac{1}{2\hbar}(\frac{p^2}{2} - \epsilon)]v - Wv = 0$. In the basis of p^2 (free rotator), it reads: $W_{n,m}v_m + \tan(\frac{\hbar n^2}{4} - \frac{\epsilon}{2\hbar})v_n = 0$. With the choice $v(\theta) = 2\hbar \arctan(\cos\theta)$ it is:

$$v_{n+1} + v_{n-1} + \tan(\tfrac{\hbar n^2}{4} - \tfrac{\epsilon}{2\hbar})v_n = 0$$

APPENDIX

6.3. **Normal variable.** A random variable x belongs to $N[\mu, \sigma]$ if it has Gaussian probability density

$$p(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

The variable $x = x_1^2 + \cdots + x_n^2$, where all $x_k \in N[0, \sigma]$, belongs to $\chi_n^2[\sigma]$:

$$p(x) = \frac{1}{2\sigma^2\Gamma(n/2)} \left(\frac{x}{2\sigma^2}\right)^{\frac{n}{2}-1} \exp(-\frac{x}{2\sigma^2})$$

The variable $x = \sqrt{x_1^2 + \cdots + x_n^2}$, where all $x_k \in N[0, \sigma]$, belongs to $\chi_n[\sigma]$:

$$p(x) = \frac{2}{\sqrt{2\sigma^2}\Gamma(n/2)} \left(\frac{x}{\sqrt{2\sigma^2}}\right)^{n-1} \exp(-\frac{x^2}{2\sigma^2})$$

6.4. **Cauchy variable.** A random variable x belongs to $C[a + ib]$ $(b > 0)$ if it has Cauchy probability density

$$p(x) = \frac{b}{\pi} \frac{1}{(x-a)^2 + b^2}$$

The random numbers can be generated by $x = a + b\tan\theta$, with θ uniformly distributed in $[-\pi/2, \pi/2]$.
If x belongs to $C[z]$ then the variable $1/x$ belongs to $C[1/z^*]$.
If $x_1 \in C[z_1]$ and $x_2 \in C[z_2]$, then $x_1 + x_2 \in C[z_1 + z_2]$.

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