NOTES ON RANDOM MATRICES

LESSON 7: RANDOM SURFACES, 2D QUANTUM GRAVITY, AND RANDOM MATRICES

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1. Euler number and Gauss-Bonnet theorem.
2. String theory as a theory of random surfaces.
3. Triangulations and the cubic matrix model.

In Lesson 2 we showed that the free energy of a cubic or quartic 1-matrix model provides the counting numbers of random closed connected graphs with coordination 3 or 4, genus h and V vertices. Such graphs are duals of polyhaedra made of V triangles or quadrangles that discretize random closed surfaces with genus h. Near $g_{\rm cr}$ the free energy has exponents independent of the potential:

$$F_n(g) = \sum_{h=0}^{\infty} \frac{1}{n^h} F_h(g) \approx a_0(g_{\rm cr}-g)^{5/2} + \frac{a_2}{n^2}\log(g_{\rm cr}-g) + \frac{a_4}{n^4}(g_{\rm cr}-g)^{-5/2} + \dots$$

The average number of vertices (or triangles/squares in the dual picture) at fixed genus h is divergent for all genera:

$$\langle N_V\rangle = -g\frac{\partial}{\partial g}\log F_h(g) \approx \frac{1}{g_{\rm cr}-g}$$

Therefore $g \to g_{\rm cr}$ is the continuum limit of the matrix model, with universal properties.

1. Euler number and Gauss-Bonnet theorem.

A surface embedded in $\mathbb{R}^n$ can be discretised by a triangulation with triangles specified by vertices belonging to the surface, and an adjacency matrix. Let N_T, N_V and N_E be the numbers of triangles, vertices and edges of the triangulation, the Euler number is

$$\chi = N_T + N_V - N_E \tag{1}$$

If in each triangle a new point of the surface is chosen and connected with the three vertices of the triangle, the new triangulation contains $3N_T$ faces, $N_V + N_T$ vertices and $N_E + 3N_T$ edges: the Euler number is unchanged. By iterating the process, the triangulation matches the surface, showing that the invariant Euler number is a property of the surface itself. Thus a sphere has the Euler number of the inscribed tetrahedron: $\chi = 2$.
Since variations of the lengths or bending of the edges do not change χ, the number is insensitive to continuous deformations of the surface.

Date: 23 mar 2018.

Removal of a triangle (but not of the boundary edges) reduces N_T and thus the Euler number by 1. If also an adjacent triangle and the common edge are removed, χ is unchanged. The result is that a hole, no matter how big, reduces χ by one. Thus a cylinder with caps removed has $\chi = 0$.
Consider two triangulated surfaces, both with two holes formed by n_i edges and n_i vertices. The two triangulations can be adjoined by matching both holes. In this surgery, $n_1 + n_2$ overlapping edges and $n_1 + n_2$ overlapping vertices are removed. The Euler number of the new surface is $\chi = \chi_1 + \chi_2 + n_1 + n_2 - (n_1 + n_2) = \chi_1 + \chi_2$. As a result, the addition of a handle to a surface of number χ requires the formation of two holes and their adjoining with a cylinder, ending with $\chi - 2$. The torus is a sphere with one handle, and has Euler number $\chi = 0$.

The Euler number of a surface is:

$$\boxed{\chi = 2 - 2\,\#\text{handles} - \#\text{holes}} \tag{2}$$

For each triangle T of the triangulation the sum of the three inner angles $\Theta(T)$ is π, then $\frac{1}{\pi}\sum_T \Theta(T) = N_T$. The sum of all angles can be also done with respect to vertices: $\frac{1}{\pi}\sum_V \Theta_V = N_T$, where Θ_V is the sum of the angles with vertex V. For triangles it is $3N_T = 2N_E$, then: $\chi = N_V - \frac{1}{2}N_T$ i.e.

$$\boxed{\chi = \frac{1}{2\pi}\sum_V (2\pi - \Theta_V)} \tag{3}$$

Thus the Euler number is expressed by the sum of the deficit angles at the vertices. The formula has a counterpart in the *Theorema Egregium* (Gauss 1827): If T is a triangle bounded by geodesics, the *spherical excess* $\Theta(T) = \theta_1 + \theta_2 + \theta_3 - \pi$ is

$$\int_T d^2\xi \sqrt{g}\, K = \Theta(T) \tag{4}$$

where $g = \det g_{ij}$ (metric tensor) and K is the Gaussian curvature. It is $K = 1/(rR)$ where r and R are the minimal and maximal radius of curvature at a point, with opposite signs if the centers are in opposite half-spaces with respect to the tangent plane. $K = R/2$, where R is the scalar curvature.
For a closed surface, summation on a covering of geodesic triangles gives, in the right hand side: $2\pi N_V - \pi N_T = 2\pi\chi$. Then, the Euler number is given by the formula by Gauss and Bonnet (1848):

$$\boxed{\chi = \frac{1}{2\pi}\int d^2\xi \sqrt{g}\, K} \tag{5}$$

Example: for a sphere of radius r the Gaussian curvature is $K = 1/r^2$. The Gauss-Bonnet formula gives $\chi = \frac{1}{2\pi}K \cdot 4\pi r^2 = 2$. The area of a spherical triangle is $r^2\Theta_T$, where $\Theta_T = \theta_1 + \theta_2 + \theta_3 - \pi$ is its spherical excess (Albert Girard, 1629). A torus with radii $r_M > r_m$ has coordinates $(r_M\varphi, r_m\theta)$; the Gaussian curvature is $K^{-1} = \pm r_m(r_M + r_m\cos\theta)$ with positive (negative) sign if the curvature centers are in the same (opposite) sides of the tangent plane. The result is $\chi = 0$.

The number of inequivalent triangulations of a surface $M(h, b)$ (with h handles and b holes) with N_T triangles, scales for large N_T as:

$$\mathcal{N}_{h,b}(N_T) \sim N_T^{\gamma_h + b - 3} e^{\mu_0 N_T}(1 + \mathcal{O}(1/N_T)) \tag{6}$$

where $\gamma_h = \frac{1}{2}(5h-1)$ and $\mu_0 > 0$ is independent of h and b.

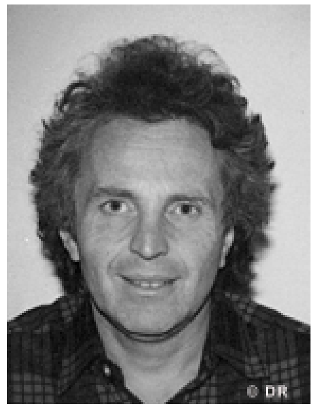

FIGURE 1. Alexander Polyakov (Russia 1945). He left the Landau Inst. in Moscow in 1990, to Princeton I.A.S. He has given basic contributions to QFT, string theory, CFT. He received important awards: Dirac medal, Heinemann prize, Lorentz medal, Klein medal, Harvey prize, Onsager prize, Fundamental Physics prize.

2. String theory as a theory of random surfaces

A string motion in space-time is a surface $X^\mu(\sigma,\tau)$, where σ is the coordinate of a point of the string and τ is its proper time. The Nambu-Goto action of the string is proportional to the area: $S_{NG}[X] = \int d\sigma d\tau \sqrt{\dot X_\mu \dot X^\mu - X'_\mu X'^\mu}$. The same dynamics results from the Polyakov action (1981) [6], with independent fields g_{ij} and X^μ:

$$S_P[g,X] = \int d^2\xi \sqrt{g}\; g^{ij}\partial_i X^\mu \partial_j X^\nu \eta_{\mu\nu} \tag{7}$$

$X^\mu(\xi^1,\xi^2)$ are coordinate fields from a $d=2$ Riemann surface to a world-sheet in D dimensional Minkowski space. The action also describes D scalar fields living on a surface. The equations of motion are:

$$T_{ij} = \partial_i X^\mu \partial_j X_\mu - \tfrac{1}{2} g_{ij} g^{kl}\partial_k X^\mu \partial_l X_\mu = 0$$
$$\nabla^2 X^\mu = \tfrac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j X^\mu) = 0.$$

Elimination of g_{ij} recovers the Nambu-Goto action.

The action is invariant under reparametrization ($\xi'^i = f^i(\xi^1,\xi^2)$), conformal transformations ($g_{ij}(\xi) \to e^{\phi(\xi)} g_{ij}(\xi)$), Poincaré group ($X'^\mu = \Lambda^\mu{}_\nu X^\nu + b^\mu$). A metric on a Riemannian surface has three independent components, a general diffeomorphism depends on two arbitrary functions, and the Weyl rescaling provides another arbitrary function. Thus any $d=2$ metric can be written $g = e^\phi f(\hat g(t))$ where $\hat g(t)$ represents some slice in the space of metrics cutting once each orbit of the group of diffeomorphisms and Weyl transformations. The parameters t are the moduli of the surface (Teichmüller). The sphere has no moduli, the torus has one complex modulus[1].

The quantization of the bosonic string is done by integrating on inequivalent metrics:

$$Z = \sum_{h=0}^{\infty} \lambda^h \int \frac{\mathscr{D}g}{\mathrm{Vol\,Diff}} \int dX^\mu e^{iS[g,X]} \tag{8}$$

In $D = 26$ (critical string) the Weyl field decouples and the Weyl volume can be factored out, leaving the integration over the moduli space of Riemann surfaces. $D = 0$ corresponds to $2d$ gravity. Then, in $D \neq 0$, the model also describes D scalar bosonic fields (conformal matter) coupled to $2d$ quantum gravity.

For fixed but large area $A = \int d^2\xi \sqrt{g}$ and fixed topology h, it is:

$$Z_h(A) = \int \frac{\mathscr{D}g}{\mathrm{Vol\,Diff}} \int dX^\mu e^{iS[g,X]} \approx A^{\gamma_h - 3} \tag{9}$$

where $\gamma_h = 2h - \gamma_0(h-1)$ is the string susceptibility and, because of matter [7]:

$$\gamma_0 = \tfrac{1}{12}[D - 1 - \sqrt{(D-25)(D-1)}]$$

For pure gravity (D=0) $\gamma_0 = -\frac{1}{2}$ and $\gamma_h = \frac{1}{2}(5h-1)$ as in (6).

3. Triangulations and the cubic matrix model

In 1985 Francois David, Vladimir Kazakov, Ivan Kostov, David Gross and Alexander Migdal and others realised that a cubic matrix model could represent a regularisation of Polyakov's model of random surfaces (or bosonic string). The duals of cubic diagrams are triangulations of a random surface of same genus.

Kazakov proposed all triangles to be equilateral, with area a^2. The curvature of the surface is concentrated at the vertices of the triangulation, and is measured by deficit angles. If n_V is the number of triangles meeting at vertex V, the area and curvature elements are replaced by:

$$\sqrt{g}\, d^2\xi \to \tfrac{1}{3} n_V a^2, \qquad K\sqrt{g}\, d^2\xi \to \tfrac{\pi}{3}(6 - n_V)$$

Then: Area=$a^2 \frac{1}{3}\sum_V n_V = a^2 N_T$ and $2\pi\chi = \frac{\pi}{3}\sum_V(6 - n_V) = 2\pi N_V - \pi N_T$ (discrete Gauss-Bonnet).

The partition function of Polyakov's string becomes a sum on: topologies (h), number of triangles (N_T), inequivalent triangulations ($\mathcal{T}$) with N_T triangles and genus h, and matter fields (living at the vertices of the triangulation, see notes in the end):

$$Z = \sum_h \lambda^h \sum_{N_T} e^{-\Lambda(a^2 N_T)} \sum_{\mathcal{T}} \int \prod_{V\in\mathcal{T}} \frac{d^D X_V}{(2\pi)^D} e^{-\sum_{\langle i,j\rangle}(X_i - X_j)^2}$$

Pure gravity is the case $D = 0$:

$$Z = \sum_h \lambda^h \sum_{N_T} e^{-\Lambda a^2 N_T} n(h, N_T) \tag{10}$$

where $n(h, N_T)$ is the number of inequivalent triangulations of genus h and N_T triangles. The sum on triangulations with fixed number of triangles and genus is realized by a sum on trivalent graphs of the cubic model. This was done exactly with

[1] for a compact surface with h handles and b punctures the number of complex moduli is $m = 3h - 3 + b$ if $3h - 3 + b > 0$, $m = 0$ if $3h - 3 + b = 0$, $m = 1$ if $h = 0$, $b = 3$ or $h = 1$, $b = 0$.

random triangulations and in the double scaling limit, simultaneously discovered in 1990 by three different groups: E. Brézin and V. Kazakov in Paris, M. Douglas and S. Shenker at Rutgers, and D. Gross and A. A. Migdal at Princeton [5].
It was surprising that for the statistical models on random lattices one could obtain more results than the models on regular lattices. In 1988, Polyakov [1?] and Knizhnik, Polyakov and Zamolodchikov [7] were able to obtain the exact critical exponents of the minimal conformal theories coupled to two-dimensional gravity. In other words, they were able to obtain exact information about the behaviour of some conformal field theory coupled to the Liouville field.

4. Notes on triangulations

(see [3])
Given a triangulation of a surface embedded in $\mathbb{R}^n$, with vertices $\mathbf{x}_k$ in the surface, each link in the triangulation has length $\ell_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$. On each triangle (ijk) one may introduce local coordinates $(\xi^1, \xi^2) \in [0,1]^2$ such that a point in the triangle has position $\mathbf{x} = \xi^1 \mathbf{x}_i + \xi^2 \mathbf{x}_j + (1 - \xi^1 - \xi^2)\mathbf{x}_k$. The Euclidean distance of two points in the same triangle is $ds^2 = g_{ab}(ijk) d\xi^a d\xi^b$ with local metric tensor

$$g(ijk) = \begin{bmatrix} \ell_{ik}^2 & \frac{1}{2}(\ell_{ik}^2 + \ell_{jk}^2 - \ell_{ij}^2) \\ \frac{1}{2}(\ell_{ik}^2 + \ell_{jk}^2 - \ell_{ij}^2) & \ell_{jk}^2 \end{bmatrix} \tag{11}$$

Heron's formula gives the area of the triangle: $\ell_{ijk} = \sqrt{p(p-\ell_{ij})(p-\ell_{jk})(p-\ell_{ik})}$ where $2p = \ell_{ij} + \ell_{jk} + \ell_{ik}$. The value coincides with: $\ell_{ijk} = \sqrt{\det g(ijk)}$.

Let us consider the dual lattice. Each triangle (ijk) has a center $O(ijk)$ equidistant from the vertices of the triangle (circumcenter). Such centers are the vertices of a trivalent dual lattice, with N_T vertices, N_V polygonal faces and N_L links. The Euler number is unchanged. The radius of the circle inscribing the triangle (ijk) is

$$\rho_{ijk} = \frac{1}{4}\frac{\ell_{ij}\ell_{ik}\ell_{jk}}{\ell_{ijk}} = \frac{1}{2}\frac{\ell_{ij}}{\sin\theta_k} = \frac{1}{2}\frac{\ell_{ik}}{\sin\theta_j} = \frac{1}{2}\frac{\ell_{jk}}{\sin\theta_i}$$

where θ_k is the angle opposite to ℓ_{ij} (sine law).

A field φ on the surface is represented by the values $\varphi_i = \varphi(\mathbf{x}_i)$ at the vertices of the triangulation. It can be extended to the interior of each triangle by the interpolation $\varphi(\mathbf{x}) = \xi^1\varphi_i + \xi^2\varphi_j + (1 - \xi^1 - \xi^2)\varphi_k$. If $g^{ab}(ijk)$ is the inverse of the metric matrix, it is:

$$\begin{aligned}
& g^{ab}\partial_a\varphi\partial_b\varphi = g^{11}(\varphi_i - \varphi_k)^2 + 2g^{12}(\varphi_i - \varphi_k)(\varphi_j - \varphi_k) + g^{22}(\varphi_j - \varphi_k)^2 \\
& = (g^{11} + g^{12})(\varphi_i - \varphi_k)^2 + (g^{22} + g^{12})(\varphi_j - \varphi_k)^2 - g^{12}(\varphi_i - \varphi_j)^2 \\
& = \frac{1}{2\ell_{ijk}^2}[(\ell_{jk}^2 + \ell_{ij}^2 - \ell_{ik}^2)(\varphi_i - \varphi_k)^2 + (\ell_{ik}^2 + \ell_{ij}^2 - \ell_{jk}^2)(\varphi_j - \varphi_k)^2 + (\ell_{ik}^2 + \ell_{jk}^2 - \ell_{ij}^2)(\varphi_i - \varphi_j)^2] \\
& = \frac{\ell_{ik}\ell_{jk}\ell_{ij}}{\ell_{ijk}^2}[\frac{1}{\ell_{ik}}\cos\theta_j(\varphi_i - \varphi_k)^2 + \frac{1}{\ell_{jk}}\cos\theta_i(\varphi_j - \varphi_k)^2 + \frac{1}{\ell_{ij}}\cos\theta_k(\varphi_i - \varphi_j)^2] \\
& = \frac{2}{\ell_{ijk}}[\cot\theta_j(\varphi_i - \varphi_k)^2 + \cot\theta_i(\varphi_j - \varphi_k)^2 + \cot\theta_k(\varphi_i - \varphi_j)^2]
\end{aligned}$$

If the triangulation is made of equilateral triangles, the Laplacian is $\sum_{\langle i,j\rangle}(\varphi_i - \varphi_j)^2$.

Two triangles (ijk) and (ijk') sharing the link (ij), have centers with distance (measured on the shortest line in the triangles)

$$\sigma_{ij} = h + h' = \frac{\ell_{ij}}{2}(\mathrm{cotg}\theta_k + \mathrm{cotg}\theta_{k'})$$

where θ_k and $\theta_{k'}$ are the angles not adjacent to ℓ_{ij}.

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