LESSON 4: HAAR MEASURES

LUCA GUIDO MOLINARI

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1. Euler angles for $\mathsf{SO}(n)$, Example $\mathsf{SO}(3)$.
2. Invariant measures on the linear groups of invertible real and complex matrices.
3. Haar measures on classical groups (*Euler angles, Weyl parametrization, Cayley transform*).

Haar measures on classical groups

The first appearance of Random Matrices in mathematics occurred in relation to the Haar measure on classical groups. A nice account is given in [2].
The story begins in 1897, when Adolf Hurwitz parameterised orthogonal and unitary matrices in terms of Euler angles, and computed the volumes of the groups. In 1933 Alfred Haar (a doctorate student of David Hilbert) proved the existence of an invariant measure on any separable compact topological group. Soon after, von Neumann proved uniqueness[1].

There are three and only three associative division algebras[2] over the field of the real numbers: $\mathbb{R}$, $\mathbb{C}$ and quaternions $\mathbb{H}$ (Frobenius, 1877). The corresponding compact continuous groups are the *classical groups*: the orthogonal $\mathsf{O}(n)$, the unitary $\mathsf{U}(n)$, and the symplectic $\mathsf{Sp}(2n)$ groups. The latter was introduced by Hermann Weyl [10].

1. Euler angles for $\mathsf{SO}(n)$

Hurwitz's construction is simple enough to be presented (taken from the book by Girko [5]). Given a $n\times n$ rotation matrix Q, consider the product of Q with a simple (inverse) rotation:

$$Q' = QR_{12}^T(\theta_{12}) = \begin{bmatrix} q_{11} & q_{12} & \dots \\ q_{21} & q_{22} & \dots \\ \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} \cos\theta_{12} & \sin\theta_{12} & \\ -\sin\theta_{12} & \cos\theta_{12} & \\ & & I_{n-2} \end{bmatrix}$$

The product Q' is again a rotation. Choose the angle θ_{12} in $[0,\pi]$ such that $q'_{12} = q_{11}\sin\theta_{12} + q_{12}\cos\theta_{12} = 0$. Next multiply Q' by $R_{13}^T(\theta_{13})$:

$$Q'' = Q'R_{13}^T(\theta_{13}) = \begin{bmatrix} q'_{11} & 0 & q'_{13} & \dots \\ q'_{21} & q'_{22} & \dots & \\ \vdots & \vdots & & \\ & & & \end{bmatrix} \begin{bmatrix} \cos\theta_{13} & 0 & \sin\theta_{13} & \\ 0 & 1 & 0 & \\ -\sin\theta_{13} & 0 & \cos\theta_{13} & \\ & & & I_{n-3} \end{bmatrix}$$

Date: 18 jan 2018.

[1] J. Diestel and A. Spalsbury, The joys of Haar measure (AMS, 2014)

[2] In a division algebra, for every elements a and $b\neq 0$ there exist precisely one element c such that $a = bc$ and one element c' such that $a = c'b$.


Figure 1. Adolf Hurwitz (Hildesheim 1859, Zurich 1919) studied in Munich and was doctoral student of Felix Klein in Leipzig. As professor in Königsberg he influenced the career of the young Hilbert and Minkowski. After the departure of Frobenius he moved to the University of Zurich (now ETH).

Figure 2. Hermann Weyl (Germany 1885, Zurich 1955) had a great impact on the development of theoretical physics and mathematics. He took his doctorate in Gottingen with David Hilbert. While professor of mathematics in Zurich, he was colleague of A. Einstein, who was working on General Relativity, and E. Schrödinger. In 1930 he left to become successor of Hilbert in Gottingen, but three years later the racial laws forced him to flee to U.S. where he permanently held a position in the new Institute for Advanced Studies in Princeton.

This leaves the element $q''_{12} = 0$. Choose the angle θ_{13} in $[0, \pi]$ such that $q''_{13} = 0$. Proceeding in this way a matrix $Q(1) = QR^T_{12}(\theta_{12}) \dots R^T_{1n}(\theta_{1n})$ is obtained with row elements $q(1)_{12} = \cdots = q(1)_{1n} = 0$. Since $Q(1)$ is a rotation, rows and columns have norm 1. Then $q(1)_{11} = \pm 1$ and, necessarily, $q(1)_{21} = \cdots = q(1)_{n1} = 0$:

$$Q(1) = \begin{bmatrix} \pm 1 & 0 \\ 0 & \tilde{Q} \end{bmatrix}$$

where $\tilde{Q}$ is orthogonal and $(\pm 1) \det \tilde{Q} = 1$. If the matrix element -1 occurs, the change $\theta_{1n} \to \theta_{1n} + \pi$ makes it equal to 1. Therefore:

$$\mathsf{SO}(n) = \begin{bmatrix} 1 & 0 \\ 0 & \mathsf{SO}(n-1) \end{bmatrix} R_{1n} \dots R_{12}$$

where $\theta_{1n} \in [0, 2\pi]$ and the other angles are in $[0, \pi]$.

The process is repeated on $\tilde{Q}$ with rotations $R^T_{23} \dots R^T_{2n}$ that make all elements in the second row and column equal to zero, except for the element (22) that equals 1 for a suitable $\theta_{2n} \in [0, 2\pi]$. In the end, the factorisation of a rotation matrix Q

into $\frac{1}{2}n(n-1)$ simple rotations has been obtained,

$$Q = [R_{n-1,n}]\cdots[R_{2n}\cdots R_{23}][R_{1n}\cdots R_{12}] \tag{1}$$

with $n-1$ *Euler angles* $\theta_{j,n}\in[0,2\pi]$ and *Euler angles* $\theta_{j,k}\in[0,\pi]$, $j=1,\ldots,n-1$, $k=j+1,\ldots,n-1$.

Example 1.1. A matrix in $\mathsf{SO}(3)$ factors into three simple rotations:

$$R = ABC = \begin{bmatrix} 1 & 0 & 0\\ 0 & c_a & -s_a\\ 0 & s_a & c_a\end{bmatrix}\begin{bmatrix} c_b & 0 & -s_b\\ 0 & 1 & 0\\ s_b & 0 & c_b\end{bmatrix}\begin{bmatrix} c_c & -s_c & 0\\ s_c & c_c & 0\\ 0 & 0 & 1\end{bmatrix}$$

with Euler angles $\theta_a,\theta_b\in[0,2\pi]$, and $\theta_c\in[0,\pi]$ and $c_a=\cos\theta_a$, $s_a=\sin\theta_a$.
Let us evaluate the Haar measure. Since $R(dR^T)+(dR)R^T=0$ then $ds^2=\mathrm{tr}(dR)(dR^T)$, where $dR=(dA)BC+A(dB)C+AB(dC)$ and $[dA,A]=[dA,A^T]=0$ etc.

$$A^TdA=\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{bmatrix}d\theta_a,\ B^TdB=\begin{bmatrix} 0 & 0 & -1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}d\theta_b,\ C^TdC=\begin{bmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}d\theta_c$$

$$\begin{aligned} ds^2 =& \mathrm{tr}[dAdA^T+dBdB^T+dCdC^T-2(A^TdA)(B^TdB)\\ &-2(C^TdC)(B^TdB)-2(A^TdA)B(C^TdC)B^T]\\ =&2(d\theta_a^2+d\theta_b^2+d\theta_c^2)-4(\sin\theta_b)d\theta_a d\theta_c\\ =&\ 2\,[\,d\theta_a\ d\theta_b\ d\theta_c\,]\begin{bmatrix} 1 & 0 & -\sin\theta_b\\ 0 & 1 & 0\\ -\sin\theta_b & 0 & 1\end{bmatrix}\begin{bmatrix} d\theta_a\\ d\theta_b\\ d\theta_c\end{bmatrix}\end{aligned}$$

Then $\det g = 2^3\cos^2\theta_b$. The Haar measure is $d\mu = 2\sqrt{2}\,|\cos\theta_b|\,d\theta_a d\theta_b d\theta_c$. The volume of $\mathsf{SO}(3)$ is: $V=2\sqrt{2}\int_0^{2\pi}|\cos\theta_b|d\theta_b\int_0^{2\pi}d\theta_a\int_0^{\pi}d\theta_c=16\sqrt{2}\pi^2$. □

2. Invariant measures on the linear groups of invertible real and complex matrices

$\mathsf{GL}(n,\mathbb{R})$ is the linear group of real invertible $n\times n$ matrices. The column vectors of a matrix X are images of the canonical basis in $\mathbb{R}^n$, and span a volume $|\det X|$. An invertible linear transformation $X\to X'=XY$, has Jacobian matrix

$$\frac{\partial X'_{ia}}{\partial X_{jb}}=\delta_{ij}Y_{ba}\quad\Longrightarrow\quad \det\frac{\partial X'}{\partial X}=\det(I_n\otimes Y)=(\det Y)^n$$

Therefore, if $dX=\prod_{i,j=1}^n dX_{ij}$, then $d(XY)=dX\,|\det Y|^n$ and, similarly, $d(YX)=dX\,|\det Y|^n$. It follows that the invariant measure for left or right matrix multiplication of $\mathsf{GL}(n,\mathbb{R})$ is [8]

$$d\mu(X)=\frac{\prod_{ij}dX_{ij}}{|\det X|^n} \tag{2}$$

$\mathsf{GL}(n,\mathbb{C})$ is the linear group of complex invertible $n\times n$ matrices $Z=X+iY$. By representing vectors in $\mathbb{C}^n$ as vectors in $\mathbb{R}^{2n}$, the matrix Z corresponds to a block matrix RZ with real blocks

$$\begin{pmatrix} X_{ij} & -Y_{ij}\\ Y_{ij} & X_{ij}\end{pmatrix}$$

An invertible linear transformation $Z' = ZW$, $W = T + iS$, corresponds to a transformation $RZ' = (RZ)(RW)$ of real matrices, with Jacobian matrix $(RW) \otimes I_n$. By permuting colums and rows, up to unit factors:

$$\det(RW) = \det \begin{bmatrix} T & -S \\ S & T \end{bmatrix} = \det \begin{bmatrix} \bar{W} & -S \\ i\bar{W} & T \end{bmatrix} = \det \begin{bmatrix} i\bar{W} & -iS \\ 0 & W \end{bmatrix} = |\det W|^2$$

Therefore, if $d^2 Z = \prod_{i,j=1}^n dX_{ij} dY_{ij}$, then $d^2(ZW) = d^2 Z\, |\det W|^{2n}$ and, similarly, $d^2(WZ) = d^2 Z\, |\det W|^{2n}$. The invariant measure for left or right matrix multiplication of $\mathsf{GL}(n, \mathbb{C})$ is

$$d\mu(Z) = \frac{\prod_{ij} dX_{ij} dY_{ij}}{|\det Z|^{2n}} \tag{3}$$

A transformation of similarity $Z' = W^{-1} Z W$ does not change the invariant measure. Any invertible matrix is diagonalized by similarity $Z = V \Lambda V^{-1}$, where V is the matrix whose columns are the eigenvectors of Z, and Λ is the diagonal matrix of eigenvalues.

3. Haar measure of classical groups

The Haar measure of $\mathsf{SO}(n)$ is induced by the requirement that the action of $\mathsf{SO}(n)$ on a single unit vector of $\mathbb{R}^n$ uniformly covers the unit sphere. Hurwitz gave the invariant measure [2]:

$$d\mu = 2^{n(n-1)/4} \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (\sin \theta_{j,k})^{j-1} d\theta_{j,k} \tag{4}$$

The volume of the rotation group is

$$\mathrm{Vol}[\mathsf{SO}(n)] = 2^{\frac{n(n-1)}{4}} (2\pi)^{n-1} \prod_{j=2}^{n-1} \left[\int_0^\pi d\varphi (\sin \varphi)^{j-1} \right]^{n-j} = 2^{\frac{n}{2}-1} \prod_{k=1}^{n} \frac{(2\pi)^{k/2}}{\Gamma(k/2)}. \tag{5}$$

The volume of $\mathsf{O}(n)$ is twice the value. The set $\mathsf{O}(n)/\mathsf{O}(1)^n$ is realized by requiring that the first entry in each column be positive. This reduces the volume of $\mathsf{O}(n)$ by 2^n.

Euler angles can also be introduced for unitary matrices. The result is

$$\mathrm{Vol}[\mathsf{U}(n)] = 2^{n(n+1)/2} \prod_{k=1}^{n} \frac{\pi^k}{\Gamma(k)}$$

The set $\mathsf{U}(n)/\mathsf{U}(1)^n$ is realized by requiring that the first component of each column be real positive, thus reducing the volume of $\mathsf{U}(n)$ by $(2\pi)^n$.

Given a parameterization of the group elements and the composition law $U(x)U(x')$ $= U(\varphi(x, x'))$, the Haar measure can be constructed as shown in Hamermesh or Shilov. An alternative route is the evaluation the metric tensor $g_{ab}(x)$ for the line element in parameter space, that is invariant for left and right multiplication by a matrix. Since $(UV)^{-1} d(UV) = V^{-1}(U^{-1} dU) V$, the invariant line element is

$$ds^2 = -\mathrm{tr}(U^{-1} dU)(U^{-1} dU) = g_{ab}(x) dx_a dx_b \tag{6}$$

Then, the Haar measure is

$$\int d\mu(U) f(U) = \int dx \sqrt{\det g(x)}\, f(x) \tag{7}$$

Hermann Weyl. In his book on the classical groups (1939, [10]), Weyl decomposed the invariant measures for $\mathsf{SO}(n)$ and $\mathsf{U}(n)$ in terms of eigenvalues and eigenvectors. Let $V \in \mathsf{U}(n)$, by writing $V = U^\dagger L U$ where $U \in \mathsf{U}(n)/\mathsf{U}(1)^n$ is the matrix of eigenvectors and $L = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ is the matrix of ordered eigenvalues, one has

$$d\mu(V) = \prod_{1 \le j < k \le n} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_n \, d\mu'(U). \tag{8}$$

Since the eigenvalue and eigenvector sectors factor, one reads off the *eigenvalue probability function* for matrices chosen with Haar measure from $\mathsf{U}(n)$

$$d\mu(\theta) = \frac{1}{(2\pi)^n n!} \prod_{1 \le j < k \le n} |e^{i\theta_k} - e^{i\theta_j}|^2 d\theta_1 \cdots d\theta_n \tag{9}$$

$0 \le \theta_j < 2\pi$. Here, the ordering of the eigenvalues implicit in (8) has been relaxed.

4. Cayley transform

Toyama (1948, [9]) provided the Haar measure for the classical compact groups by parameterizing the elements via the Cayley transform.
SU(n). The Cayley transform of a unitary matrix requires a Hermitian matrix:

$$U = \frac{I_n + iH}{I_n - iH} \tag{10}$$

The infinitesimal volume element of Haar measure of the unitary group $\mathsf{U}(n)$ is

$$d\mu(U) = \frac{dH}{\det(I_n + H^2)^n} \tag{11}$$

where $dH = \prod_j dH_{jj} \prod_{i<j} d\mathrm{Re}H_{ij} \, d\mathrm{Im}H_{ij}$.
SO(n). The Cayley transform of a rotation matrix requires a real anti-symmetric $n \times n$ matrix (not having eigenvalue -1, see [4]):

$$O = \frac{I_n - A}{I_n + A} \tag{12}$$

The infinitesimal volume element of the Haar measure is :

$$d\mu(O) = \frac{dA}{\det(I_n + A^2)^{\frac{1}{2}(n+1)}} \tag{13}$$

Sp(2n). A unitary symplectic matrix has Cayley's representation (10) with matrix

$$H = \begin{bmatrix} A & B \\ B^\dagger & -A \end{bmatrix} \tag{14}$$

where $A = A^\dagger$ and B is complex. The infinitesimal volume element of Haar measure of $\mathsf{Sp}(2n)$ is

$$d\mu(U) = \frac{dH}{\det(I_{2n} + H^2)^{n+\frac{1}{2}}} \tag{15}$$

Let us give a proof for the unitary group. It is $ds^2 = -\mathrm{tr}(U^{-1}dU)(U^{-1}dU) = \mathrm{tr}(dU dU^\dagger)$. If $U = (1+iH)(1-iH)^{-1}$, then it is: $U^\dagger dU = 2i(1+iH)^{-1}dH(1-iH)^{-1}$ and $ds^2 = 4\mathrm{tr}[(1+H^2)^{-1}dH]^2$. If $dT = (1+H^2)^{-1}dH$, then the flat measure for T is the Haar measure and: $d\mu(U) = \det(1+H^2)^{-n}dH$.
Note that in the three cases the power of the determinant is the dimension of the group divided by the matrix size n.

A construction of the Haar measure. [L. G. Molinari]
Consider a Lie group with elements $U = \exp(iH)$, where H belongs to the Lie algebra. If T_a are the generators, with normalisation $\mathrm{tr}(T_aT_b) = \delta_{ab}$ and structure constants $[T_a, T_b] = if_{abc}T_c$, the elements of the Lie algebra are the linear combinations $H = x_aT_a$, with $x_a = \mathrm{tr}(HT_a)$. The invariant line element is

$$ds^2 = i^2\mathrm{tr}(U^{-1}dUU^{-1}dU) = g_{ab}(x)dx_adx_b$$

By means of the Lie-Trotter formula $d(e^A) = \int_0^1 dt\, e^{(1-t)A}(dA)e^{tA}$, let's evaluate:

$$U^{-1}dU = e^{-iH}\int_0^1 dte^{i(1-t)H}(idH)e^{itH} = idx_a\int_0^1 dte^{-itH}T_ae^{itH}$$

$$g_{ab} = \iint_0^1 dt_1dt_2\ \mathrm{tr}[e^{-it_1H}T_ae^{i(t_1-t_2)H}T_be^{it_2H}] = \int_{-1}^{+1} dt(1-|t|)\mathrm{tr}[e^{iHt}T_ae^{-iHt}T_b]$$

where we made the change of variables $t = t_2 - t_1$, $2s = t_1 + t_2$ and integrated in s. The expansion $e^{iHt}T_ae^{-iHt} = c_{ab}(tx)T_b$ has coefficients $c_{ab}(tx) = \mathrm{tr}[e^{iHt}T_ae^{-iHt}T_b]$ that may be obtained by solving the equation of motion:

$$\frac{d}{dt}c_{ab}(tx) = -i\mathrm{tr}(e^{iHt}T_ae^{-iHt}[H, T_b]) = c_{ad}(tx)f_{cbd}x_c$$

with $c_{ab}(0) = \delta_{ab}$. The solution is $c_{ab}(xt) = \exp[tM(x)]_{ab}$, where $M(x) = x_cf_{cba}$ is an antisymmetric matrix whose size is the linear dimension of the algebra.

$$g_{ab} = \int_{-1}^{+1} dt(1-|t|)e^{tM(x)}$$

Being $M = -M^t$, the non-zero eigenvalues of M come in pairs $\pm i\lambda$ with real λ. The matrices g and M are diagonalized by the same rotation matrix, therefore if $i\lambda_k$ is an eigenvalue of M, the corresponding eigenvalue of g is:

$$g_k = \int_{-1}^{1} dt(1-|t|)e^{it\lambda_k} = 2\int_0^1 dt(1-t)\cos(t\lambda_k) = \frac{\sin^2(\lambda_k/2)}{(\lambda_k/2)^2}$$

The eigenvalue equation $M(x)_{ab}v_b = i\lambda v_a$ corresponds to $[V, H(x)] = i\lambda V$, where $V = v_aT_a$. The eigenvalues that contribute to the Haar measure are the differences $\{h_i - h_j\}_{i<j}$ of eigenvalues of $H(x) = x_aT_a$. Since $\sin^2(h_i/2 - h_j/2) \approx |e^{h_i} - e^{h_j}|^2$, up to a normalization the invariant measure for the group is:

$$\sqrt{g} = \prod_{\lambda_k>0}\frac{\sin^2(\lambda_k/2)}{(\lambda_k/2)^2}$$

$$d\mu(U) = \frac{\prod_{i<j}|e^{h_i} - e^{h_j}|^2}{\Delta(h_1, \ldots, h_n)^2}\prod_a dx_a$$

The Lie algebra of $\mathsf{U}(n)$ is the linear space of Hermitian matrices. In the basis of generators $T(r)_{ij} = \delta_{ir}\delta_{jr}$, $T(rs)_{ij} = \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}$ and $T'(rs)_{ij} = i\delta_{ir}\delta_{js} - i\delta_{is}\delta_{jr}$ the set $\{x_a\}$ of parameters is $\{H_{ii}, \mathrm{Re}H_{ij}, \mathrm{Im}H_{ij}, i < j\}$. The invariant measure is (up to constants)

$$d\mu(U) = \frac{\prod_{i<j}|e^{h_i} - e^{h_j}|^2}{\Delta(h_1, \ldots, h_n)^2}\, dH \approx \prod_{i<j}|e^{h_i} - e^{h_j}|^2dh_1\ldots dh_n \tag{16}$$

Proposition 4.1 (Borel's lemma). *Let* $\mathbf{u} = (u_1, \ldots, u_n)$ *be a random point of the sphere* $\|\mathbf{u}\| = 1$ *in* $\mathbb{R}^n$. *Then,*

$$P(\sqrt{n}\, u_1 \le t) \to \int_{-\infty}^{t} \frac{dx}{\sqrt{2\pi}} e^{-x^2/2} \quad (n \to \infty) \tag{17}$$

With the same hypothesis:

$$\mathbb{E}[|u_1|^{\alpha_1} \ldots |u_n|^{\alpha_n}] = \Gamma(\frac{n}{2}) n^{\frac{1}{2}\sum \alpha_k} \frac{\prod_k \Gamma(\frac{\alpha_k+1}{2})}{\Gamma(\sum_k \frac{\alpha_k+1}{2})} \tag{18}$$

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