The QCD partition function and Chiral Random Matrices

Mauro Pastore

May 18, 2020

Contents

1 Introduction

The literature about the application of Random Matrix Theory to the study of *exact* properties of the low energy spectrum of QCD is huge and mainly due to Jacobus Verbaarschot, starting from the late '80ies. Some of the early articles are [1–8]. At the time, he gave also a lot of talks about the subject, as [9–13]. Some reviews I found useful: [14–18]. Shortly after, the interest moved to the generalization of these chiral random matrix models in order to study QCD at finite density, starting from [19]. This research line is still open, given the persisting need to understand the QCD phase diagram, but I will not review it here.

2 Symmetries of QCD

2.1 Preliminaries

The QCD partition function in Euclidean space is

$$\begin{aligned}\mathcal{Z}_{\mathrm{QCD}} &= \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\,\mathcal{D}\psi\, e^{-S_{\mathrm{D}}[\psi,\bar{\psi},A_\mu]-S_{\mathrm{YM}}[A_\mu]} \\ &= \int \mathcal{D}A_\mu \prod_{f=1}^{N_f} \det\left(\not{D}[A_\mu]+m_f\right) e^{-S_{\mathrm{YM}}[A_\mu]} \\ &= \mathcal{Z}_{\mathrm{YM}} \left\langle \prod_{f=1}^{N_f} \det\left(\not{D}[A_\mu]+m_f\right)\right\rangle_{\mathrm{YM}} ,\end{aligned} \tag{1}$$

with Dirac and Yang-Mills actions[1]

$$S_{\rm D}[\psi,\bar\psi,A_\mu] = \int \mathrm{d}^4x \sum_{f=1}^{N_f} \bar\psi(x)\left(\not{D}[A_\mu]+m_f\right)\psi(x)\,, \tag{2}$$

$$S_{\rm YM}[A_\mu] = \frac{1}{2g^2}\int \mathrm{d}^4x\ \mathrm{Tr}\left[F_{\mu\nu}(x)F_{\mu\nu}(x)\right] = \frac{1}{4g^2}\int \mathrm{d}^4x\, F^a_{\mu\nu}(x)F^a_{\mu\nu}(x)\,. \tag{3}$$

The Dirac operator is:

$$\not{D} = \gamma_\mu D_\mu = \gamma_\mu\left(\partial_\mu + iA_\mu\right)\,, \tag{4}$$

such as $\not{D}^\dagger = -\not{D}$. The covariant derivative is expressed in terms of the non-abelian, algebra-valued gauge fields:

$$A_\mu = \sum_{a=1}^{N_c^2-1} A^a_\mu \Theta^a\,, \tag{5}$$

with Θ^a (Hermitian) generators of the gauge group $\mathrm{SU}(N_c)$, chosen to comply

$$\mathrm{Tr}(\Theta^a\Theta^b) = \frac{1}{2}\delta_{ab}\,. \tag{6}$$

The corresponding, non-abelian field-strength is

$$\begin{aligned} F_{\mu\nu}(x) &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i\left[A_\mu(x),A_\nu(x)\right] = -i[D_\mu,D_\nu](x) = \sum_{a=1}^{N_c^2-1} F^a_{\mu\nu}\Theta^a\,,\\ F^a_{\mu\nu}(x) &= \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + \sum_{b,c} f_{abc}A^b_\mu(x)A^c_\mu(x)\,, \end{aligned} \tag{7}$$

with the $\mathrm{SU}(N_c)$ structure constants f_{abc} defined by

$$i[\Theta^a,\Theta^b] = \sum_c f^{ab}{}_c\Theta^c\,. \tag{8}$$

The Euclidean γ matrices are such that

$$\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}\,,\qquad \{\gamma_5,\gamma_\mu\}=0\,,\qquad \gamma_\mu^\dagger=\gamma_\mu \qquad \gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4\,. \tag{9}$$

In chiral basis,

$$\gamma_4 = \begin{pmatrix} 0 & \mathbb{I}\\ \mathbb{I} & 0\end{pmatrix},\qquad \gamma_j = i\begin{pmatrix} 0 & -\sigma^j\\ \sigma^j & 0\end{pmatrix},\qquad \gamma_5 = \begin{pmatrix} \mathbb{I} & 0\\ 0 & -\mathbb{I}\end{pmatrix}. \tag{10}$$

The right (left) Dirac spinor is the eigenvector of γ_5 with eignenvalue $+1$ (-1):

$$\gamma_5\psi^R(x) = +\psi^R(x),\qquad \gamma_5\psi^L(x) = -\psi^L(x) \tag{11}$$

and can be obtained projecting a generic spinor via

$$\psi^R(x) = \frac{1+\gamma_5}{2}\psi(x)\,,\qquad \psi^L(x) = \frac{1-\gamma_5}{2}\psi(x)\,. \tag{12}$$

Of course,

$$\bar\psi = \psi^\dagger\gamma_4 \tag{13}$$

and so

$$\bar\psi\,\frac{1+\gamma_5}{2} = \left(\psi^R\right)^\dagger\gamma_4 \equiv \bar\psi^R\,,\qquad \bar\psi\,\frac{1-\gamma_5}{2} = \left(\psi^L\right)^\dagger\gamma_4 \equiv \bar\psi^L\,. \tag{14}$$

The fermion action can be written as

$$S_{\rm D} = -\int \mathrm{d}^4x \sum_{f=1}^{N_f}\left[\bar\psi^L_f\not{D}\psi^L_f + \bar\psi^R_f\not{D}\psi^R_f + m_f\left(\bar\psi^L_f\psi^R_f + \bar\psi^R_f\psi^L_f\right)\right]\,. \tag{15}$$

The compact representation of the QCD partition function in equation (1) hides a lot of subtleties needed to define it properly:

[1]"Tr" indicates the trace over colour. Other conventions are fine-tuned to give the ones in [14].

a) If the theory is set directly in the continuum space-time, the measure over the bosonic, algebra-valued, fields A_μ have to be defined via a *gauge fixing* (Faddeev-Popov) procedure, in order to avoid infinities due to the overcounting of the gauge degrees of freedom (configurations of the fields that differ only for a gauge transformation give rise to the same physical description, so that only a representative for each gauge orbit should be kept in count). The straightforward procedure, which produces a gauge fixing term in the YM action and auxiliary fields (the Faddeev-Popov ghosts) in the description of the theory, can be not enough to avoid completely the overcounting due to gauge freedom: a gauge orbit can intersect the surface defined via gauge fixing more than one time, leaving in the measure configurations that differ for finite gauge transformations (the so-called Gribov copies).

b) The functional integral over the fields can be defined via a limit procedure over a discrete number of integral: the theory is defined on a lattice, whose spacings a_μ, which have the role of UV regulators, have to be sent to 0 at the end of the day, to obtain physical results (*continuum limit*). Problems arise when the fermion fields are set on the lattice, because of the fermion doubling problem. The action have to be modified accordingly, adding lattice artifacts that ensure the spurious degrees of freedom decouple in the continuum limit. In doing so, the axial $\mathrm{U}_A(1)$ symmetry of the classical massless theory is inevitably lost (*axial anomaly*).

c) Whenever the lattice spacing a is kept finite, the gauge integrals are not performed on algebra-valued fields A_μ, but on group-valued variables $U_\mu(x) = e^{aA_\mu(x)}$ living on the links of the lattice, because the derivatives of the fields are replaced by finite differences and the gauge connection have to transport fields that are distant for a finite amount. As the gauge group is compact, the invariant measure of integration is well defined unambiguously (it is the Haar measure). The gauge fixing procedure is needed only in perturbation theory, which can be obtained as a weak-coupling limit of the lattice theory. Perturbation theory, on the other hand, is more difficult on the lattice, because of the appearance of new vertices and divergences in the diagrammatic calculations, due to lattice artifacts both in the action and in the measure of integration.

d) The relativistic theory with infinite degrees of freedom is obtained via the *thermodynamic limit* of a system confined in a box of finite volume V, which works as an IR regulator.

e) The mass of the quarks works as an explicit breaking term for the axial chiral-flavour symmetry (loosely speaking, "$\mathrm{SU}_A(N_f)$", see below). As this symmetry is expected to be spontaneously broken even in the massless theory, the massless limit (the so-called *chiral limit*) have to be taken after the thermodynamic limit, to find a nontrivial value of the chiral condensate, in the same way as finite magnetization is obtained for a ferromagnetic system taking the thermodynamic limit before, and then sending to zero the external magnetic field.

2.2 Dirac spectrum

The Dirac operator satisfies

$$\{\gamma_5, \not{D}\} = 0 \tag{16}$$

so, given an eigenfunction $\psi_k(x)$ such that, for a fixed gauge configuration A_μ,

$$\not{D}\psi_k(x) = i\lambda_k\psi_k(x)\,, \qquad \lambda_k = \lambda_k[A_\mu] \neq 0\,, \tag{17}$$

also $\gamma_5\psi_k(x)$ is an eigenfunction with eigenvalue $-i\lambda_k$: all the non-zero eigenvalues come in opposite pairs. Given n the number of eigenmodes with $\lambda_k > 0$, the total number for $\lambda_k \neq 0$ is thus $2n$. These eigenfunctions cannot be arranged to have definite chirality:

$$\begin{cases} \not{D}\psi_k^R(x) = i\lambda_k\psi_k^L(x) \\ \not{D}\psi_k^L(x) = i\lambda_k\psi_k^R(x) \end{cases} \qquad \lambda_k \neq 0\,. \tag{18}$$

Of course, the exception is when $\lambda_k = 0$: then the corresponding eigenfunctions $\phi_k^\pm$ can be chosen to be simultaneously eigenfunctions of γ_5 with eigenvalues ± 1 and they do not necessarily come in pairs (indeed, in this case $\phi_k^\pm$ and $\gamma_5\phi_k^\pm$ are trivially linearly dependent). Denoting the number of zero eigenmodes with positive and negative chirality N_+ and N_- respectively, because of Atiyah-Singer index theorem we know that the *winding number*

$$\nu \equiv N_+ - N_- \tag{19}$$

is a topological invariant (it does not change under continuous deformations of the gauge fields), while N_+ and N_-, separately, depends on gauge configuration. As a consequence, a small continuous deformation of the gauge field configuration *lifts the accidental zero eigenvalues*, that is it changes N_+ and N_- keeping fixed ν. For this reason, it is always possible to reduce the discussion to a case where or $N_+ = 0$ or $N_- = 0$, with exactly $|\nu|$ zero eigenvalues, including the others in n (in other words, zero eigenvalues are unpaired except for a set of zero measure, in the gauge functional integral). Moreover, because of (16),

$$\begin{aligned}
\langle \psi_j^R | \not{D} | \psi_k^R \rangle &= \langle \psi_j | \frac{1+\gamma_5}{2} \not{D} \frac{1+\gamma_5}{2} | \psi_k \rangle \\
&= \frac{1}{4} \left[\langle \psi_j | \not{D} | \psi_k \rangle + \langle \psi_j | \left\{ \gamma_5, \not{D} \right\} | \psi_k \rangle + \langle \psi_j | \gamma_5 \not{D} \gamma_5 | \psi_k \rangle \right] \\
&= \frac{1}{4} \left[\langle \psi_j | \not{D} | \psi_k \rangle - \langle \psi_j | \not{D} | \psi_k \rangle \right] = 0 \qquad\qquad j,k = 1, \cdots, n+N_+
\end{aligned} \tag{20}$$

and the same for $\langle \psi_j^L | \not{D} | \psi_k^L \rangle$ (with $j,k = 1, \cdots, n+N_-$), so, in the basis of all the eigenfunctions,

$$\not{D}_{jk} = \langle \psi_j | \not{D} | \psi_k \rangle = \begin{pmatrix} 0 & iT \\ iT^\dagger & 0 \end{pmatrix}_{jk}, \qquad (\gamma_5)_{jk} = \langle \psi_j | \gamma_5 | \psi_k \rangle = \begin{pmatrix} \mathbb{I}_{n+N_+} & 0 \\ 0 & -\mathbb{I}_{n+N_-} \end{pmatrix}_{jk}, \tag{21}$$

with T a rectangular matrix $(n+N_+) \times (n+N_-)$. In this way $\not{D}$ is a square $(2n+N_++N_-) \times (2n+N_++N_-)$ matrix with $|\nu|$ eigenvalues equal to 0 and the others $(2n+N_++N_--|\nu|)$ paired. Indeed, suppose $\lambda \neq 0$, then

$$\begin{aligned}
\lambda^{-|\nu|} \begin{vmatrix} \lambda\mathbb{I}_{n+N_+} & -T \\ -T^\dagger & \lambda\mathbb{I}_{n+N_-} \end{vmatrix} &= \lambda^{-|\nu|} \left| \begin{pmatrix} \mathbb{I}_{n+N_+} & \lambda^{-1}T \\ 0 & \mathbb{I}_{n+N_-} \end{pmatrix} \begin{pmatrix} \lambda\mathbb{I}_{n+N_+} & -T \\ -T^\dagger & \lambda\mathbb{I}_{n+N_-} \end{pmatrix} \right| \\
&= \lambda^{-|\nu|} \begin{vmatrix} \lambda\mathbb{I}_{n+N_+} - \lambda^{-1}TT^\dagger & 0 \\ -T^\dagger & \lambda\mathbb{I}_{n+N_-} \end{vmatrix} = \lambda^{n+N_- -|\nu|} \left| \lambda\mathbb{I}_{n+N_+} - \lambda^{-1}TT^\dagger \right| \\
&= \lambda^{n+N_- -|\nu|} \left| \lambda^{-1} \left(\lambda^2 \mathbb{I}_{n+N_+} - TT^\dagger \right) \right| = \lambda^{-\nu-|\nu|} \left| \lambda^2 \mathbb{I}_{n+N_+} - TT^\dagger \right| .
\end{aligned} \tag{22}$$

With the same passages,

$$\begin{aligned}
\lambda^{-|\nu|} \begin{vmatrix} \lambda\mathbb{I}_{n+N_+} & -T \\ -T^\dagger & \lambda\mathbb{I}_{n+N_-} \end{vmatrix} &= \lambda^{-|\nu|} \left| \begin{pmatrix} \mathbb{I}_{n+N_+} & 0 \\ \lambda^{-1}T^\dagger & \mathbb{I}_{n+N_-} \end{pmatrix} \begin{pmatrix} \lambda\mathbb{I}_{n+N_+} & -T \\ -T^\dagger & \lambda\mathbb{I}_{n+N_-} \end{pmatrix} \right| \\
&= \lambda^{\nu-|\nu|} \left| \lambda^2 \mathbb{I}_{n+N_-} - T^\dagger T \right|
\end{aligned} \tag{23}$$

So, according to the sign of ν, I can choose one of this relation to prove that

$$\lambda^{-|\nu|} P_{-i\not{D}}(\lambda) = \begin{cases} \left| \lambda^2 \mathbb{I}_{n+N_+} - TT^\dagger \right| & \text{if } \nu < 0 \text{ (case 1)} \\ \left| \lambda^2 \mathbb{I}_{n+N_-} - T^\dagger T \right| & \text{if } \nu > 0 \text{ (case 2)} \end{cases} \tag{24}$$

with P_A characteristic polynomial of A. As $TT^\dagger$ and $T^\dagger T$ are positive definite Hermitian matrices:[2]

$$\sum_{i,j=1}^{n+N_+} x_i^* (TT^\dagger)_{ij} x_j = \|T^\dagger x\|^2 > 0 \qquad (\text{if } \nexists\, x \text{ such that } T^\dagger x = 0), \tag{25}$$

they have a real positive spectrum, say $\{\kappa_j^1\}_{j=1}^{n+N_+}$ and $\{\kappa_j^2\}_{j=1}^{n+N_-}$, and the above equation says that the spectrum of $-i\not{D}$ is the set $\left\{ \pm\sqrt{\kappa_j^{1 \text{ or } 2}} \right\}$, with cardinality $(2n+N_++N_--|\nu|)$, plus $|\nu|$ zero eigenvalues.

[2] This is not strictly true, as, by definition, $TT^\dagger$ and $T^\dagger T$ have, respectively, N_+ and N_- zero modes. However, as explained above, in the functional integral over the gauge fields, the sets of configurations where $N_+ \neq 0$ (if $\nu < 0$) or $N_- \neq 0$ (if $\nu > 0$) have zero measure.

The spectral density of the Dirac operator is defined by

$$\rho(\lambda) = \left\langle \frac{1}{V} \sum_k \delta(\lambda - \lambda_k) \right\rangle \underset{V\to\infty}{\longrightarrow} \rho_c(\lambda)\,. \tag{26}$$

For free fermions, using the fact that $\gamma_\mu \partial_\mu \gamma_\nu \partial_\nu = \Box$, the spectrum is obtained simply as the square root of the one for the potential well in 4 dimensions. The positive eigenvalues are

$$\lambda_n = \pi \sqrt{\sum_\mu \left(\frac{n_\mu}{L_\mu}\right)^2}\,, \qquad n_\mu = 0, \cdots, L_\mu - 1\,. \tag{27}$$

The total number of eigenvalues less than a certain λ goes as (simply take the "volume" of the 3-sphere with radius $\sqrt{\sum_\mu n_\mu^2}$ from the previous equation)

$$N_{\text{free}}(\lambda) \sim \frac{\lambda^4 V}{\pi^4} \tag{28}$$

so

$$\rho_{\text{free}}(\lambda) \sim \lambda^3 \tag{29}$$

and the spacing between eigenvalue goes as

$$(\Delta\lambda)_{\text{free}} \sim \frac{1}{V^{1/4}}\,. \tag{30}$$

2.3 Topology

In terms of the gauge fields, the index (19) can be evaluated as

$$\nu = \frac{1}{32\pi^2} \int \mathrm{d}^4x\, \mathrm{Tr}\left[\epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(x) F^{\rho\sigma}(x)\right] = \frac{1}{16\pi^2} \int \mathrm{d}^4x\, \mathrm{Tr}\left[{}^*F_{\rho\sigma}(x) F^{\rho\sigma}(x)\right]\,, \tag{31}$$

where

$$^*F_{\rho\sigma}(x) = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(x)\,. \tag{32}$$

This term has the same symmetries of the ones in QCD Lagrangian, *except* P and T (parity and time reversal: it is an $\mathbf{E}\cdot\mathbf{B}$ term). Note that, because of CPT symmetry, a violation of symmetry under time reversal can be seen as well as a violation under CP. Because CP is not believed to be a fundamental symmetry (it's not a symmetry of the Standard Model, because of the weak sector), one cannot in principle exclude from the action a term like

$$S_\theta = -\frac{i\theta}{16\pi^2} \int \mathrm{d}^4x\, \mathrm{Tr}\left[{}^*F_{\rho\sigma}(x) F^{\rho\sigma}(x)\right] = -i\nu\theta\,. \tag{33}$$

The integrand is proportional to a total derivative, because

$$\mathrm{Tr}\, F_{\rho\sigma}(x)\, {}^*F^{\rho\sigma}(x) = \partial_\mu \left[\epsilon^{\mu\nu\rho\sigma} A_\nu^a \left(F_{\rho\sigma}^a - \frac{g}{3} f^{abc} A_\rho^b A_\sigma^c\right)\right]\,. \tag{34}$$

This means that this term is irrelevant in a perturbative approach, but cannot be neglected in a non-perturbative analysis, because of instantons and so on. However, experimentally, θ is compatible with 0. This is the famous strong-CP problem: why QCD should have CP symmetry? The fact that this term is proportional to a topological invariant means that it is the same for gauge fields that differ only by continuous transformations, so the path-integral measure $\mathcal{D}A_\mu$, which is over all the gauge configurations, in practice factorizes in sectors with a definite value of the index ν:

$$\mathcal{Z}_{\text{QCD}}^{(\theta)} = \sum_{\nu=-\infty}^{+\infty} e^{i\nu\theta} \mathcal{Z}_{\text{QCD}}^{(\nu)}\,, \tag{35}$$

where the integral in $\mathcal{Z}^{(\nu)}_{\mathrm{QCD}}$ is now restricted to the sector with topological charge ν. Using (1), in terms of the spectrum of $\not{D}$, I have

$$\begin{aligned}\mathcal{Z}^{(\nu)}_{\mathrm{QCD}} &= \mathcal{Z}^{(\nu)}_{\mathrm{YM}} \left\langle \prod_{f=1}^{N_f} \prod_{k=1}^{n+\frac{\nu-|\nu|}{2}} (i\lambda_k + m_f)\,(-i\lambda_k + m_f) \prod_{j=1}^{|\nu|} m_f \right\rangle_{\mathrm{YM},\nu} \\ &= \mathcal{Z}^{(\nu)}_{\mathrm{YM}} \left\langle \prod_{f=1}^{N_f} m_f^{|\nu|} \prod_{k=1}^{n+\frac{\nu-|\nu|}{2}} (\lambda_k^2 + m_f^2) \right\rangle_{\mathrm{YM},\nu} .\end{aligned} \tag{36}$$

2.4 Chiral-flavour global symmetry

In the so-called chiral limit ($m_f \to 0$ for all flavours), the action (15) becomes

$$S_{\mathrm{D}} = \int \mathrm{d}^4x \sum_{f=1}^{N_f} \left[\bar{\psi}^L_f \not{D} \psi^L_f + \bar{\psi}^R_f \not{D} \psi^R_f\right] . \tag{37}$$

In this form, it is clear that a transformation of the type

$$\psi^L \longrightarrow U_L \psi^L, \qquad \psi^R \longrightarrow U_R \psi^R \tag{38}$$

with U_L, U_R independent matrices in $\mathrm{U}(N_f)$, is a symmetry of the theory. Thus, the chiral-flavour symmetry group is

$$\mathrm{U}_L(N_f) \times \mathrm{U}_R(N_f) = \mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f) \times \mathrm{U}_L(1) \times \mathrm{U}_R(1) . \tag{39}$$

In terms of the generators in the algebra, these transformations can be represented as

$$\psi^L \longrightarrow e^{-i\alpha_L} \exp\left(-i\frac{\tau^j}{2}\theta^j_L\right)\psi^L, \qquad \psi^R \longrightarrow e^{-i\alpha_R} \exp\left(-i\frac{\tau^j}{2}\theta^j_R\right)\psi^R \tag{40}$$

with $\tau^j/2$, $j = 1, \cdots, N_f^2 - 1$ Hermitian generators of $\mathrm{SU}(N_f)$ and α_L, α_R, θ^j_L, θ^j_R independent parameters. The $\mathrm{U}_L(1) \times \mathrm{U}_R(1)$ part can be realized as a transformation in $\mathrm{U}_V(1) \times \mathrm{U}_A(1)$, where $\mathrm{U}_V(1)$ ($\mathrm{U}_A(1)$) denotes the group of phase rotations such that the left and right components are transformed with the same (respectively, opposite) angles. On the full Dirac field, these transformations are realized as

$$\psi \longrightarrow e^{-i\alpha_V} e^{-i\alpha_A\gamma_5}\psi, \qquad e^{-i\alpha_V} \in \mathrm{U}_V(1), \quad e^{-i\alpha_A\gamma_5} \in \mathrm{U}_A(1) \tag{41}$$

with

$$\alpha_V = \frac{\alpha_R + \alpha_L}{2}, \qquad \alpha_A = \frac{\alpha_R - \alpha_L}{2} . \tag{42}$$

The vector (or diagonal) part $\mathrm{U}_V(1) \equiv \mathrm{U}_B(1)$ corresponds to the baryon number conservation. The axial part $\mathrm{U}_A(1)$, despite being a symmetry of the classical theory, cannot be maintained in the proper quantum one, because the fermionic path integral measure (or the UV regularization procedure) is not invariant under this transformation. The corresponding broken conservation law is said to be *anomalous*. Similarly, one can separate the diagonal subgroup $\mathrm{SU}_V(N_f)$ of $\mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f)$, considering transformations with $U_L = U_R$, from the axial coset $[\mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f)]/\mathrm{SU}_V(N_f)$. Note that, in the non-Abelian case, the axial transformations do not form a subgroup: indeed, the corresponding, would-be, generators can be written as $\gamma_5\tau^j/2$, but the algebra does not close:

$$i\left[\gamma_5\frac{\tau^j}{2}, \gamma_5\frac{\tau^k}{2}\right] = f^{jkl}\,\mathbb{I}\,\frac{\tau^l}{2} . \tag{43}$$

This is the reason why referring to the axial coset as "$\mathrm{SU}_A(N_f)$" is misleading: it would be a group only if $\mathrm{SU}_V(N_f)$ were a central subgroup of $\mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f)$, which is not.

It is widely believed that, due to spontaneous symmetry breaking, the axial transformations are not symmetries of the low energy spectrum of the theory. When $N_f = 2$ is considered as the number of flavours of the approximate chiral symmetry, because ($m_u \approx 2.2\,\mathrm{MeV}$) $\approx$ ($m_d \approx 4.7\,\mathrm{MeV}$) ≈ 0, the corresponding Goldstone modes due to the spontaneous breakdown are the three pions, while in the case $N_f = 3$ ($m_s \approx 96\,\mathrm{MeV}$) the Gell-Mann's Eightfold Way of the light mesons is obtained.

For all these reasons, the chiral-flavour symmetry group of zero-mass QCD is broken to

$$\mathrm{SU}_L(N_f) \times \mathrm{SU}_R(N_f) \times \mathrm{U}_L(1) \times \mathrm{U}_R(1) \quad \longrightarrow \quad \mathrm{SU}_V(N_f) \times \mathrm{U}_B(1)\,. \tag{44}$$

2.5 Banks-Casher relation

The order parameter of the spontaneous breakdown of chiral symmetry is the vacuum expectation value of the operator proportional to the mass term in the action, which plays the same role of a small external magnetic field breaking explicitly the rotational invariance of a ferromagnetic Hamiltonian (say, for example, the Heisenberg model). There, the spontaneous magnetization can be evaluated as the limit of zero external fields of the expectation value of the mean of the spin variable, evaluated *after* having performed the thermodynamic limit. Let $m_f = m$ for all f, for brevity.

$$\begin{aligned}
\langle\bar{\psi}\psi\rangle &= \lim_{m\to 0}\lim_{V\to\infty}\left[\frac{1}{\mathcal{Z}_{\mathrm{QCD}}}\int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \left(\frac{\int \mathrm{d}^4y \sum_f \bar{\psi}_f(y)\psi_f(y)}{VN_f}\right) e^{-S_{\mathrm{D}}[\psi,\bar{\psi},A_\mu]-S_{\mathrm{YM}}[A_\mu]}\right]_{\text{finite } V} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{VN_f}\frac{\partial}{\partial m}\left[\log \mathcal{Z}_{\mathrm{QCD}}\right]_{\text{finite V}} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\frac{1}{\mathcal{Z}_{\mathrm{QCD}}}\frac{\partial}{\partial m}\left[\mathcal{Z}_{\mathrm{YM}}\left\langle \det\left(\not{D}[A_\mu]+m\right)\right\rangle_{\mathrm{YM},\,\text{finite } V}\right] \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\frac{1}{\mathcal{Z}_{\mathrm{QCD}}}\frac{\partial}{\partial m}\left[\mathcal{Z}_{\mathrm{YM}}\left\langle \prod_k \left(i\lambda_k[A_\mu]+m\right)\right\rangle_{\mathrm{YM},\,\text{finite } V}\right] \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\frac{\mathcal{Z}_{\mathrm{YM}}}{\mathcal{Z}_{\mathrm{QCD}}}\left\langle \sum_j \prod_{k\neq j}\left(i\lambda_k[A_\mu]+m\right)\right\rangle_{\mathrm{YM},\,\text{finite } V} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\frac{\mathcal{Z}_{\mathrm{YM}}}{\mathcal{Z}_{\mathrm{QCD}}}\left\langle \prod_k\left(i\lambda_k[A_\mu]+m\right)\sum_j\frac{1}{i\lambda_j[A_\mu]+m}\right\rangle_{\mathrm{YM},\,\text{finite } V} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\left\langle\sum_k\frac{1}{i\lambda_k+m}\right\rangle_{\text{finite } V} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\frac{1}{V}\left\langle\sum_{k:\lambda_k>0}\left(\frac{1}{i\lambda_k+m}+\frac{1}{-i\lambda_k+m}\right)+\sum_{i=1}^{|\nu|}\frac{1}{m}\right\rangle_{\text{finite } V} \\
&= -\lim_{m\to 0}\lim_{V\to\infty}\left\langle\frac{1}{V}\sum_{k:\lambda_k>0}\frac{2m}{\lambda_k^2+m^2}+\frac{|\nu|}{mV}\right\rangle_{\text{finite } V}.
\end{aligned} \tag{45}$$

Note that, even if not explicitly stated, $\lambda_k = \lambda_k[A]$, $\nu = \nu[A]$ depend on the configuration of the gauge field, which have to be integrated with the Yang-Mills weight. Notice also that the expectation value at the end is taken with respect to the full QCD action, fermionic determinant included. It can be proven (see [20]) that the winding number distribution goes as

$$\langle\nu^2\rangle \sim V \tag{46}$$

so that the last term can be neglected. Moreover, inserting the definition of the spectral density (26),

$$\begin{aligned}
\langle\bar{\psi}\psi\rangle &= -\lim_{m\to 0}\lim_{V\to\infty}\int_0^\infty \mathrm{d}\lambda\,\frac{1}{V}\sum_{k:\lambda_k>0}\frac{2m\,\langle\delta(\lambda-\lambda_k)\rangle}{\lambda^2+m^2} \\
&= -\lim_{m\to 0}\int_0^\infty \mathrm{d}\lambda\,\frac{2m\rho_c(\lambda)}{\lambda^2+m^2} = -\pi\rho_c(0)\,.
\end{aligned} \tag{47}$$

Given that

$$|\langle\bar{\psi}\psi\rangle| = \Sigma \tag{48}$$

is a constant, known for example via lattice simulations, the spacing between eigenvalues near $\lambda = 0$ goes as

$$\Delta\lambda = \frac{1}{V\Sigma} \tag{49}$$

to be confronted with the free prediction (30).

3 Chiral Random Matrix Theory

Being not pedagogical, Verbaarschot's works are not so immediate as a reference on the Random Matrix side of the correspondence (with maybe the exception of [16], which is still far from complete). I found good references in [21] (chapter 3) and [22]. In the mathematical literature the chRMT ensembles are usually called Wishart-Laguerre ensembles (LOE, LUE, LSE).

3.1 Singular value decomposition

The model is defined by the partition function[3]

$$\mathcal{Z}_{\text{chRMT}}^{(\nu,N_f)} = \int \mathrm{D}W \prod_{f=1}^{N_f} \begin{vmatrix} m_f & iW \\ iW^\dagger & m_f \end{vmatrix} e^{-\operatorname{Tr} v(W^\dagger W)} \qquad (m_f \to 0)\,, \tag{50}$$

where W is a $n \times m$ complex matrix, $N = n + m$, $|\nu| = |n - m|$, with rank

$$r = \min(n, m)\,. \tag{51}$$

The integration measure is a product of ordinal Lebesgue integrals in the $2nm$ real degrees of freedom of W:

$$\mathrm{D}W = \prod_{i=1}^{n}\prod_{j=1}^{m} \mathrm{d}(\operatorname{Re} W_{ij})\, \mathrm{d}(\operatorname{Im} W_{ij})\,. \tag{52}$$

This measure is invariant under the transformation

$$W \longrightarrow UWV^\dagger\,, \tag{53}$$

with $U \in \mathrm{U}(n)$, $V \in \mathrm{U}(m)$. Confronting with with (1), the identifications are

$$\not{D} \longrightarrow \begin{pmatrix} 0 & iW \\ iW^\dagger & 0 \end{pmatrix}, \qquad S_{\text{YM}} \longrightarrow \frac{N}{2} \operatorname{Tr} v(W^\dagger W), \qquad \mathcal{D}A_\mu \longrightarrow \mathrm{D}W\,. \tag{54}$$

Given a generic complex matrix W $n \times m$, it is always possible the *singular value decomposition*

$$W = UXV^{-1} \qquad W^\dagger = VX^TU^{-1} \tag{55}$$

with U, V as before and X, X^T rectangular matrices respectively $n \times m$ and $m \times n$ with entries $x_i > 0$ $(i = 1, \cdots, r)$ on the principal diagonal (and 0 otherwise), called singular values. In order to count the independent degrees of freedom of the resulting matrices, note that, as the rank of W is only r, it has r singular values, so the sums can be truncated to r:

$$\sum_{j=1}^{n}\sum_{k=1}^{m} U_{ij}X_{jk}V_{kl}^{-1} = \sum_{j,k=1}^{r} U_{ij}X_{jk}V_{kl}^{-1} \tag{56}$$

[3] I report here results suitable for the study of QCD with 3 or more colours and quarks in the fundamental representation of the gauge group. The corresponding matrix model has Dyson index $\beta = 2$ and is called (when $v(M) = M$) chGUE (chiral Gaussian Unitary Ensemble). Other models can be studied with $\beta = 1$ (chGOE, $N_c = 2$ and quark in the fundamental representation) and $\beta = 4$ (chGSE, any N_c and quarks in the adjoint representation).

so that the matrices U, V are effectively $n \times r$ and $m \times r$. We are interested in rewriting the partition function (50) in terms of the singular values x_i. As usual, we note that the infinitesimal length element invariant under the transformation (53) is

$$\mathrm{d}s^2 = \mathrm{tr}\left(\mathrm{d}W\,\mathrm{d}W^\dagger\right) \equiv \sum_{i=1}^{n}\sum_{j=1}^{m}\mathrm{d}W_{ij}\,\mathrm{d}W^\dagger_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\mathrm{d}(\mathrm{Re}\,W_{ij})^2 + \mathrm{d}(\mathrm{Im}\,W_{ij})^2\right]. \tag{57}$$

From (55),

$$\begin{aligned}\mathrm{d}W &= \mathrm{d}UXV^{-1} + U\,\mathrm{d}XV^{-1} - UXV^{-1}\mathrm{d}VV^{-1}\,,\\ \mathrm{d}W^\dagger &= \mathrm{d}VX^TU^{-1} + V\,\mathrm{d}X^TU^{-1} - VX^TU^{-1}\mathrm{d}UU^{-1}\end{aligned} \tag{58}$$

so that, with the anti-Hermitian matrices $\delta U = U^{-1}\,\mathrm{d}U$ and $\delta V = V^{-1}\,\mathrm{d}V$,

$$\begin{aligned}\mathrm{tr}\left(\mathrm{d}W\,\mathrm{d}W^\dagger\right) &= \mathrm{tr}\left(U^{-1}\,\mathrm{d}WVV^{-1}\,\mathrm{d}W^\dagger U\right)\\ &= \mathrm{tr}\left\{\left(\delta UX - X\delta V + \mathrm{d}X\right)\left(\delta VX^T - X^T\delta U + \mathrm{d}X^T\right)\right\}\\ &= \mathrm{tr}\left[2\delta UX\delta VX^T - \delta VX^TX\delta V - \delta UXX^T\delta U + \mathrm{d}X\,\mathrm{d}X^T\right.\\ &\qquad \left. + \delta V\left(X^T\,\mathrm{d}X - \mathrm{d}X^TX\right) + \delta U\left(X\,\mathrm{d}X^T - \mathrm{d}XX^T\right)\right].\end{aligned} \tag{59}$$

The last line is 0, because

$$\begin{aligned}\mathrm{tr}\left[\delta V\left(X^T\,\mathrm{d}X - \mathrm{d}X^TX\right)\right] &= \sum_{i,j=1}^{m}\sum_{k=1}^{n}\left[\delta V_{ij}\left(X^T_{jk}\,\mathrm{d}X_{ki} - \mathrm{d}X^T_{jk}X_{ki}\right)\right]\\ &= \sum_{i,j,k=1}^{r}\left[\delta V_{ij}\left(x_j\delta_{jk}\,\mathrm{d}x_k\delta_{ki} - \mathrm{d}x_j\delta_{jk}x_k\delta_{ki}\right)\right]\\ &= \sum_{i,j=1}^{r}\left[\delta V_{ij}\left(x_j\,\mathrm{d}x_j - \mathrm{d}x_jx_j\right)\delta_{ji}\right] = 0\,.\end{aligned} \tag{60}$$

Moreover,

$$\begin{aligned}\mathrm{tr}\left(\delta UX\delta VX^T\right) &= \sum_{i,j=1}^{n}\sum_{k,l=1}^{m}\delta U_{ij}X_{jk}\delta V_{kl}X^T_{li} = \sum_{i,j,k,l=1}^{r}x_ix_j\delta U_{ij}\delta_{jk}\delta V_{kl}\delta_{li} = \sum_{i,j=1}^{r}x_ix_j\delta U_{ij}\delta V_{ji}\,,\\ \mathrm{tr}\left(\delta VX^TX\delta V\right) &= \sum_{i,j,l=1}^{m}\sum_{k=1}^{n}\delta V_{ij}X^T_{jk}X_{kl}\delta V_{li} = \sum_{i=1}^{m}\sum_{j,k,l=1}^{r}x_jx_k\delta V_{ij}\delta_{jk}\delta_{kl}\delta V_{li} = \sum_{i=1}^{m}\sum_{j=1}^{r}x_j^2\delta V_{ij}\delta V_{ji}\,,\\ \mathrm{tr}\left(\delta UXX^T\delta U\right) &= \sum_{i,j,l=1}^{n}\sum_{k=1}^{m}\delta U_{ij}X_{jk}X^T_{kl}\delta U_{li} = \sum_{i=1}^{n}\sum_{j,k,l=1}^{r}x_jx_k\delta U_{ij}\delta_{jk}\delta_{kl}\delta U_{li} = \sum_{i=1}^{n}\sum_{j=1}^{r}x_j^2\delta U_{ij}\delta U_{ji}\,.\end{aligned} \tag{61}$$

Summing all these terms and isolating the sums up to r from the rest, the result is

$$\sum_{i,j=1}^{r}\left(2x_ix_j\delta U_{ij}\delta V_{ji} - x_j^2\delta U_{ij}\delta U_{ji} - x_j^2\delta V_{ij}\delta V_{ji}\right) - \sum_{i=r+1}^{n}\sum_{j=1}^{r}x_j^2\delta U_{ij}\delta U_{ji} - \sum_{i=r+1}^{m}\sum_{j=1}^{r}x_j^2\delta V_{ij}\delta V_{ji}\,. \tag{62}$$

Using anti-Hermiticity $\delta U_{ji} = -\delta U_{ij}^*$,

$$\begin{aligned}
\sum_{i,j=1}^{r} x_j^2 \delta U_{ij} \delta U_{ji} &= \sum_{i<j=1}^{r} x_j^2 \delta U_{ij} \delta U_{ji} + \sum_{i>j=1}^{r} x_j^2 \delta U_{ij} \delta U_{ji} + \sum_{i=1}^{r} x_i^2 \delta U_{ii} \delta U_{ii} \\
&= \sum_{i<j=1}^{r} \left(x_i^2 + x_j^2\right) \delta U_{ij} \delta U_{ji} + \sum_{i=1}^{r} x_i^2 \delta U_{ii} \delta U_{ii} \\
&= -\sum_{i<j=1}^{r} \left(x_i^2 + x_j^2\right) \delta U_{ij} \delta U_{ij}^* + \sum_{i=1}^{r} x_i^2 \delta U_{ii} \delta U_{ii} \,, \\
\sum_{i,j=1}^{r} x_j^2 \delta V_{ij} \delta V_{ji} &= -\sum_{i<j=1}^{r} \left(x_i^2 + x_j^2\right) \delta V_{ij} \delta V_{ij}^* + \sum_{i=1}^{r} x_i^2 \delta V_{ii} \delta V_{ii} \,, \\
\sum_{i,j=1}^{r} x_i x_j \delta U_{ij} \delta V_{ji} &= \sum_{i<j=1}^{r} x_i x_j \delta U_{ij} \delta V_{ji} + \sum_{i>j=1}^{r} x_i x_j \delta U_{ij} \delta V_{ji} + \sum_{i=1}^{r} x_i^2 \delta U_{ii} \delta V_{ii} \\
&= -\sum_{i<j=1}^{r} x_i x_j \left(\delta U_{ij} \delta V_{ij}^* + \delta U_{ij}^* \delta V_{ij}\right) + \sum_{i=1}^{r} x_i^2 \delta U_{ii} \delta V_{ii} \,,
\end{aligned} \tag{63}$$

so

$$\begin{aligned}
&\sum_{i,j=1}^{r} \left(2 x_i x_j \delta U_{ij} \delta V_{ji} - x_j^2 \delta U_{ij} \delta U_{ji} - x_j^2 \delta V_{ij} \delta V_{ji}\right) \\
&= \sum_{i<j=1}^{r} (x_i - x_j)^2 \left|\frac{\delta U_{ij} + \delta V_{ij}}{\sqrt{2}}\right|^2 + \sum_{i<j=1}^{r} (x_i + x_j)^2 \left|\frac{\delta U_{ij} - \delta V_{ij}}{\sqrt{2}}\right|^2 + \sum_{i=1}^{r} x_i^2 \left|\delta U_{ii} - \delta V_{ii}\right|^2 \,.
\end{aligned} \tag{64}$$

Definitely, calling $\mathrm{d}T^{\pm} = (\delta U_{ij} \pm \delta V_{ij})/\sqrt{2}$,

$$\begin{aligned}
\operatorname{tr}\left(\mathrm{d}W \,\mathrm{d}W^\dagger\right) &= \sum_{i=1}^{r} (\mathrm{d}x_i)^2 + \sum_{i<j=1}^{r} (x_i - x_j)^2 \left[\left(\operatorname{Re} \mathrm{d}T_{ij}^+\right)^2 + \left(\operatorname{Im} \mathrm{d}T_{ij}^+\right)^2\right] \\
&+ \sum_{i<j=1}^{r} (x_i + x_j)^2 \left[\left(\operatorname{Re} \mathrm{d}T_{ij}^-\right)^2 + \left(\operatorname{Im} \mathrm{d}T_{ij}^-\right)^2\right] + \sum_{i=1}^{r} x_i^2 \left[\operatorname{Im}\left(\delta U_{ii} - \delta V_{ii}\right)\right]^2 \\
&+ \sum_{i=r+1}^{n} \sum_{j=1}^{r} x_j^2 \left[(\operatorname{Re} \delta U_{ij})^2 + (\operatorname{Im} \delta U_{ji})^2\right] + \sum_{i=r+1}^{m} \sum_{j=1}^{r} x_j^2 \left[(\operatorname{Re} \delta V_{ij})^2 + (\operatorname{Im} \delta V_{ji})^2\right] \,.
\end{aligned} \tag{65}$$

For a similar calculation in full detail for the chGOE ensemble see [21], section 3.1.1. Thus, the square root of the determinant of the metric tensor is

$$\sqrt{g} = \prod_{i<j=1}^{r} \left(x_i^2 - x_j^2\right)^2 \prod_{k=1}^{r} x_k^{1+2(n-r)+2(m-r)} = \prod_{i<j=1}^{r} \left(x_i^2 - x_j^2\right)^2 \prod_{k=1}^{r} x_k^{1+2|\nu|} \tag{66}$$

where I used the fact that

$$n + m - 2r = n + m - 2\min(n, m) = |\nu| \,. \tag{67}$$

Factorizing the measure over the symmetry group spaces in a numerical normalization factor, the partition function becomes, in terms only of the singular values,

$$\mathcal{Z}_{\text{chRMT}}^{(\nu,N_f)} \propto \int \left[\prod_{k=1}^{r} \mathrm{d}x_k\right] \left[\prod_{i<j=1}^{r} \left(x_i^2 - x_j^2\right)\right]^2 \left[\prod_{k=1}^{r} x_k^{1+2|\nu|} \prod_{f=1}^{N_f} m_f^{|\nu|} \left(m_f^2 + x_k^2\right) e^{-x_k^2}\right] \,. \tag{68}$$

With a change of variables $\lambda_k = x_k^2$, I obtain

$$\mathcal{Z}_{\mathrm{chRMT}}^{(\nu,N_f)} \propto \int \left[\prod_{k=1}^{r} \mathrm{d}\lambda_k\right] \left[\prod_{i<j=1}^{r} (\lambda_i - \lambda_j)\right]^2 \left[\prod_{k=1}^{r} \lambda_k^{|\nu|} \prod_{f=1}^{N_f} m_f^{|\nu|} \left(m_f^2 + \lambda_k\right) e^{-\lambda_k}\right] . \tag{69}$$

Note that λ_k are the eigenvalues of the matrix $W^\dagger W$, which clearly has entries not independent from each other. This is one of the few cases where exact results about the eigenvalues distribution are known for matrices with correlated entries.

3.2 Spectral density

The partition function can be evaluated using the orthogonal polynomial method. Factorizing the $m_f^{|\nu|}$, which trivially goes to zero in the chiral limit, in the above formulas, the (unnormalized) joint eigenvalue distribution:

$$p^{(\nu,N_f)}(\lambda_1, \cdots, \lambda_r) = [\Delta_r(\{\lambda\})]^2 \left[\prod_{k=1}^{r} \lambda_k^{|\nu|} \prod_{f=1}^{N_f} \left(m_f^2 + \lambda_k\right) e^{-\lambda_k}\right] \tag{70}$$

where $\Delta_r(\{\lambda\})$ is the Vandermonde determinant:

$$\Delta_r(\{\lambda\}) = \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{r-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_r & \lambda_r^2 & \cdots & \lambda_r^{r-1} \end{vmatrix} = \prod_{i<j=1}^{r} (\lambda_i - \lambda_j) . \tag{71}$$

Being a determinant, I can take linear combinations of the columns of the corresponding matrix without changing the result:

$$\begin{aligned} \begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{r-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{r-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_r & \lambda_r^2 & \cdots & \lambda_r^{r-1} \end{vmatrix} &= \begin{vmatrix} P_0(\lambda_1) & P_1(\lambda_1) & P_2(\lambda_1) & \cdots & P_{r-1}(\lambda_1) \\ P_0(\lambda_2) & P_1(\lambda_2) & P_2(\lambda_2) & \cdots & P_{r-1}(\lambda_2) \\ \vdots & \vdots & \vdots & & \vdots \\ P_0(\lambda_r) & P_1(\lambda_r) & P_2(\lambda_r) & \cdots & P_{r-1}(\lambda_r) \end{vmatrix} \\ &= \frac{1}{r!} \sum_{i_1,\cdots,i_r} \sum_{j_1,\cdots,j_r} \epsilon_{i_1\cdots i_r} \epsilon_{j_1\cdots j_r} P_{j_1-1}(\lambda_{i_1}) \cdots P_{j_r-1}(\lambda_{i_r}) \end{aligned} \tag{72}$$

where $P_\alpha(\lambda)$ is a monic polynomial of degree α:

$$P_\alpha(\lambda) = \lambda^\alpha + O(\lambda^{\alpha-1}) . \tag{73}$$

If these polynomial are chosen to be *orthogonal* with respect to the rest of the measure in the eigenvalues, then the problem can be solved exactly. The corresponding weight function for each eigenvalue is

$$w^{(\nu,N_f)}(\lambda_k) = \lambda_k^{|\nu|} \prod_{f=1}^{N_f} \left(m_f^2 + \lambda_k\right) e^{-\lambda_k} . \tag{74}$$

Note that the product of the N_f factors $(m_f^2 + \lambda_k)$, which comes from the determinant in (50), is *included in the weight*. In case N_f is taken equal to 0, this term drops out and the resulting theory is called *quenched approximation* (thinking to QCD, it is like the theory without dynamical quarks). When the form above above is applied to QCD, it is clear that a theory with index $|\nu|$ and N_f massless quark is equivalent to a quenched theory with index $\nu + N_f$. This property is called flavour-topology duality.

The suitable orthogonal polynomial $P_\alpha^{(\nu,N_f)}(\lambda)$ to solve the model must comply

$$\int \mathrm{d}\lambda\, w^{(\nu,N_f)}(\lambda)\, P_\alpha^{(\nu,N_f)}(\lambda) P_\beta^{(\nu,N_f)}(\lambda) = h_\alpha^{(\nu,N_f)} \delta_{\alpha\beta} . \tag{75}$$

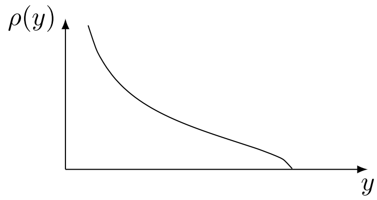

Figure 1: Marchenko-Pastur law

It can be shown that they are (rescaled) generalized Laguerre polynomials: in the simplest quenched case $N_f = 0$

$$P_\alpha^{(\nu,0)}(\lambda) = (-)^\alpha \alpha!\, L_\alpha^{(\nu)}(\lambda), \qquad h_\alpha = \alpha!\,\Gamma(\alpha+\nu+1), \qquad (\text{for } \nu > -1)\,, \tag{76}$$

where

$$\begin{aligned} L_0^{(\nu)}(\lambda) &= 1\,, \\ L_1^{(\nu)}(\lambda) &= 1+\nu-\lambda\,, \\ L_{\alpha+1}^{(\nu)}(\lambda) &= \frac{(2\alpha+1+\nu-\lambda)L_\alpha^{(\nu)}(\lambda) - (\alpha+\nu)L_{\alpha-1}^{(\nu)}(\lambda)}{\alpha+1} \end{aligned} \tag{77}$$

are the generalize Laguerre polynomials, solution of the differential equation

$$\lambda y'' + (\nu+1-\lambda)y' + \alpha y = 0\,. \tag{78}$$

The case for N_f flavours with masses is slightly more complicated (see [22]) and I will not discuss it here. Once the polynomials are known, the Vandermonde squared written as a sum over permutation of products of these polynomials can be integrated exactly with the weights w, because the integrals factorize and the orthogonality condition can be used. The partition function is simply

$$\mathcal{Z}_{\text{chRMT}}^{(\nu,N_f)} = r! \prod_{\alpha=1}^{r} h_{\alpha-1}\,. \tag{79}$$

All the k-point eigenvalue density, defined integrating out the others $r-k$ eigenvalues from (70), can be expressed as

$$\begin{aligned} R_k^{(\nu,N_f)}(\lambda_1,\cdots,\lambda_k) &\propto \int_0^{+\infty} \mathrm{d}\lambda_{k+1} \cdots \int_0^{+\infty} \mathrm{d}\lambda_r p^{(\nu,N_f)}(\lambda_1,\cdots,\lambda_r) \\ &= \prod_{j=1}^{k} w^{(\nu,N_f)}(\lambda_j) \det_{1\le i,j\le r} [K_r^{(\nu,N_f)}(\lambda_i,\lambda_j)]\,, \end{aligned} \tag{80}$$

where the kernel K_r is

$$K_r(x,y) = \sum_{\alpha=0}^{r-1} h_\alpha^{-1} P_\alpha(x) P_\alpha(y) \tag{81}$$

and, for $k=1$, we find the spectral density

$$\rho(\lambda) = w(\lambda)K_r(\lambda,\lambda) = h_r \left[P_r(\lambda)P'_{r-1}(\lambda) - P_{r-1}(\lambda)P'_r(\lambda)\right]\,. \tag{82}$$

The result is a complicated kernel with Laguerre polynomials.

3.3 Microscopic limit

In the following, I will use the standard $N = r$ (r was the rank). In order to study the large N limit, we have to decide how to scale quantities to get the scaling regime we are interested in. The global spectral statistic describe the correlations between eigenvalues that have a finite fraction

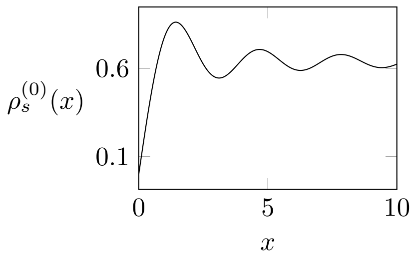

Figure 2: Microscopical spectral density.

of the other in between them. In the case of chGUE it tends to the Marchenko-Pastur law (see Fig. 1), which is the analogue of the Wigner semicircle for this ensemble. A positive eigenvalue described by the MP distribution can have three different statistical behaviour, depending where it is: it can be at the hard edge (near the origin), in the bulk or at the soft edge (far on the right). We are interested in studying the fluctuation of eigenvalues at the hard edge separated by a distance of $1/N$. Using the asymptotics

$$\lim_{N\to\infty} N^{-\nu} L_N^{(\nu)}\left(\frac{\lambda}{N}\right) = \lambda^{-\nu/2} J_\nu(2\sqrt{\lambda}) \tag{83}$$

where J_ν denotes a Bessel function, we find the *microscopic spectral density*:

$$\rho_s^{(\nu)}(x) = (x/2)\left[J_\nu^2(x) - J_{\nu+1}(x)J_{\nu-1}(x)\right] \tag{84}$$

where $x = \lim_{\substack{N\to\infty \\ \lambda\to 0}} 2\sqrt{N\lambda}$. Reintroducing in the model the parameter Σ rescaling the variance of the gaussian distribution in (50), we find

$$\rho_s^{(\nu)}(x) = (\Sigma^2 x/2)\left[J_\nu^2(\Sigma x) - J_{\nu+1}(\Sigma x)J_{\nu-1}(\Sigma x)\right] . \tag{85}$$

This is the microscopical spectral density that fits so well the unfolded density of the smallest eigenvalues of the Dirac operator, obtained with lattice simulations (see Fig. 2).

References

[1] E. V. Shuryak and J. J. M. Verbaarschot. "Random matrix theory and spectral sum rules for the Dirac operator in QCD". In: *Nucl. Phys.* A560 (1993), pp. 306–320. DOI: 10.1016/0375-9474(93)90098-I. arXiv: hep-th/9212088 [hep-th].

[2] J. J. M. Verbaarschot and I. Zahed. "Spectral density of the QCD Dirac operator near zero virtuality". In: *Phys. Rev. Lett.* 70 (1993), pp. 3852–3855. DOI: 10.1103/PhysRevLett.70.3852. arXiv: hep-th/9303012 [hep-th].

[3] J. J. M. Verbaarschot. "Chiral random matrix theory and the spectrum of the Dirac operator near zero virtuality". In: *Acta Phys. Polon.* B25 (1994), pp. 133–149. arXiv: hep-th/9310049 [hep-th].

[4] J. J. M. Verbaarschot. "The Spectrum of the Dirac operator near zero virtuality for N(c) = 2 and chiral random matrix theory". In: *Nucl. Phys.* B426 (1994), pp. 559–574. DOI: 10.1016/0550-3213(94)90021-3. arXiv: hep-th/9401092 [hep-th].

[5] J. J. M. Verbaarschot. "The Spectrum of the QCD Dirac operator and chiral random matrix theory: The Threefold way". In: *Phys. Rev. Lett.* 72 (1994), pp. 2531–2533. DOI: 10.1103/PhysRevLett.72.2531. arXiv: hep-th/9401059 [hep-th].

[6] J. J. M. Verbaarschot. "Spectrum of the Dirac operator in a QCD instanton liquid: Two versus three colors". In: *Nucl. Phys.* B427 (1994), pp. 534–544. DOI: 10.1016/0550-3213(94)90638-6. arXiv: hep-lat/9402006 [hep-lat].

[7] A. V. Smilga and J. J. M. Verbaarschot. “Spectral sum rules and finite volume partition function in gauge theories with real and pseudoreal fermions”. In: *Phys. Rev.* D51 (1995), pp. 829–837. DOI: 10.1103/PhysRevD.51.829. arXiv: hep-th/9404031 [hep-th].

[8] A. M. Halasz and J. J. M. Verbaarschot. “Universal fluctuations in spectra of the lattice Dirac operator”. In: *Phys. Rev. Lett.* 74 (1995), pp. 3920–3923. DOI: 10.1103/PhysRevLett.74.3920. arXiv: hep-lat/9501025 [hep-lat].

[9] J. J. M. Verbaarschot. “Chiral random matrix theory and QCD”. In: *Workshop on Continuous Advances in QCD Minneapolis, Minnesota, February 18-20, 1994*. 1994, pp. 195–210. arXiv: hep-th/9405006 [hep-th].

[10] J. J. M. Verbaarschot. “Random matrix model approach to chiral symmetry”. In: *Lattice’96. Proceedings, 14th International Symposium on Lattice Field Theory, St. Louis, USA, June 4-8, 1996*. Vol. 53. 1997, pp. 88–94. DOI: 10.1016/S0920-5632(96)00602-0. arXiv: hep-lat/9607086 [hep-lat].

[11] J. J. M. Verbaarschot. “On the spectrum of the QCD Dirac operator”. In: *Continuous advances in QCD 1996. Proceedings, Conference, Minneapolis, USA, March 28-31, 1996*. 1996, pp. 325–337. arXiv: hep-lat/9606009 [hep-lat].

[12] J. J. M. Verbaarschot. “Spectral fluctuations of the QCD dirac operator”. In: *New non-perturbative methods and quantization on the light cone. Proceedings, School, Les Houches, France, February 24-March 7, 1997*. 1997, pp. 97–104. arXiv: hep-ph/9705455 [hep-ph].

[13] J. J. M. Verbaarschot. “Universal behavior in Dirac spectra”. In: *Confinement, duality, and nonperturbative aspects of QCD. Proceedings, NATO Advanced Study Institute, Newton Institute Workshop, Cambridge, UK, June 23-July 4, 1997*. 1997, pp. 343–378. arXiv: hep-th/9710114 [hep-th].

[14] J. J. M. Verbaarschot and T. Wettig. “Random matrix theory and chiral symmetry in QCD”. In: *Ann. Rev. Nucl. Part. Sci.* 50 (2000), pp. 343–410. DOI: 10.1146/annurev.nucl.50.1.343. arXiv: hep-ph/0003017 [hep-ph].

[15] R. A. Janik et al. “Random matrices and chiral symmetry in QCD”. In: *Properties of hadrons in matter. Proceedings, APCTP Workshop on astro-hadron physics, Seoul, Korea, October 25-31, 1997*. 1998, pp. 21–44. arXiv: hep-ph/9806370 [hep-ph].

[16] M. A. Stephanov, J. J. M. Verbaarschot, and T. Wettig. “Random matrices”. In: (2005). arXiv: hep-ph/0509286 [hep-ph].

[17] J. J. M. Verbaarschot. “Handbook Article on Applications of Random Matrix Theory to QCD”. In: (2009). arXiv: 0910.4134 [hep-th].

[18] P. H. Damgaard. “Chiral Random Matrix Theory and Chiral Perturbation Theory”. In: *J. Phys. Conf. Ser.* 287 (2011), p. 012004. DOI: 10.1088/1742-6596/287/1/012004. arXiv: 1102.1295 [hep-ph].

[19] M. A. Stephanov. “Random matrix model of QCD at finite density and the nature of the quenched limit”. In: *Phys. Rev. Lett.* 76 (1996), pp. 4472–4475. DOI: 10.1103/PhysRevLett.76.4472. arXiv: hep-lat/9604003 [hep-lat].

[20] H. Leutwyler and A. V. Smilga. “Spectrum of Dirac operator and role of winding number in QCD”. In: *Phys. Rev.* D46 (1992), pp. 5607–5632. DOI: 10.1103/PhysRevD.46.5607.

[21] P. J. Forrester. *Log-Gases and Random Matrices*. Princeton University Press, 2010.

[22] G. Akemann. “Random Matrix Theory and Quantum Chromodynamics”. In: 2016. DOI: 10.1093/oso/9780198797319.003.0005. arXiv: 1603.06011 [math-ph].