UNIVERSITÀ DEGLI STUDI DI MILANO

Random surfaces applied

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10th October 2019

The core of string theory is to replace particles by string-like 1-dimensional objects describing some two dimensional surface, the *worldsheet*, when evolving in time.

Modern quantization strongly relies on the computation of the *path integral*, the analog of the *partition function* in statistical mechanics.

Schematically, we will discretize the integral over the geometries of the worldsheet as a sum over randomly triangulated surfaces, whose partition function will be expressed as the free energy of an associated Hermitian matrix model *(Kazakov, Kostov and Migdal, '85)*.

We replace the (conformal) string theory with a statistical mechanical system at criticality and the sum over metrics with a sum over random triangulated surfaces of arbitrary topology.

From strings to matrices

The partition function for string theory is

$$Z_{string} = \sum_{\text{genera}} \int \frac{\mathcal{D}g}{\text{Vol}(Diff)} \mathcal{D}X \, e^{-S_P[g,X] - \Lambda \int d^2\xi \sqrt{-g} - \frac{\gamma}{4\pi} \int d^2\xi \sqrt{-g} R}$$

- The matter action for a bosonic string is the *Polyakov action*

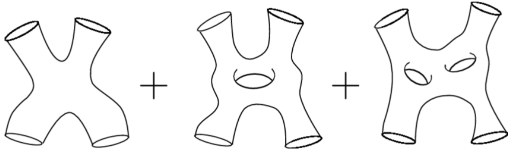
$$S_P[g, X] = \frac{1}{8\pi} \int_{\Sigma} d^2\xi \sqrt{-g} \, g^{ab} \partial_a X^{\mu} \partial_b X^{\nu} G_{\mu\nu} \, .$$

- This system can be viewed as a non-linear σ-model with matter fields $X : \Sigma \to \mathcal{M}_D$.
- $g_{ab}(\xi)$ metric on the 2-dimensional worldsheet.
- $G_{\mu\nu}(X)$ metric on the D-dimensional target space.
- $\text{Vol}(Diff)$ is needed to not overcount all the metrics g_{ab} equivalent under diffeomorphisms.
- The second term in Z_{string} is the area of the worldsheet and Λ plays the role of chemical potential.

The partition function for string theory is

$$Z_{string} = \sum_{\text{genera}} \int \frac{\mathcal{D}g}{\mathrm{Vol}(Diff)} \mathcal{D}X\, e^{-S_P[g,X] - \Lambda \int d^2\xi \sqrt{-g} - \frac{\gamma}{4\pi} \int d^2\xi \sqrt{-g} R}$$

- The last term in Z_{string} is related to the genus of the worldsheet and encodes the interactions.



It is a topological term thanks to the *Gauss-Bonnet theorem*[1]:

$$\int_\Sigma d^2\xi \sqrt{-g}\, R = 4\pi\chi\,, \qquad \chi := V - L + F = 2 - 2h\,,$$

where χ is the *Euler characteristic*.

[1] We consider only compact surfaces without boundary.

So far, string theory has been (sometimes even not completely) solved only in $D = 1, 2$ embedding dimensions, in which the methods we will employ can be applied.

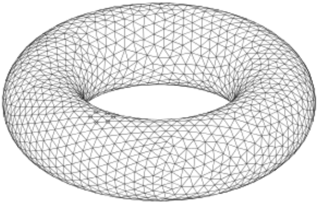
In order to make the dissertation more transparent we will focus on $D = 0$, the *pure gravity*, in which the string partion function becomes

$$Z_{string} = \sum_{\text{genera}} \int \frac{\mathcal{D}g}{\text{Vol}(Diff)} e^{-\Lambda A - \gamma \chi} \, .$$

"Physically" this is a string propagating in no embedding space.

First step: discretize the worldsheet by means of a triangulation.

Divide the worldsheet surface in equilateral triangles with fixed unit area and sum over all possible triangulations (configurations and total number of triangles).



The discretized surface is not smooth and the metric as well, but they become smooth in the limit of large number of triangles.

With this method we can

- integrate over all possible deformations of a given genus surface (integrate over Feynman parameters for a given loop diagram),
- sum over all genera (sum over all loop diagrams).

The equivalent of the area element is $\frac{N_i}{3}$, where N_i is the number of triangles meeting at the vertex i. Indeed,

$$\int d^2\xi\sqrt{-g} \quad \longleftrightarrow \quad \sum_{i\in V(\mathcal{T})} \frac{N_i}{3} = n_F = \text{total area} := A(\mathcal{T})$$

The intrinsic curvature is concentrated at the vertices and is equal to the deficit angle $2\pi\frac{6-N_i}{N_i}$. Indeed,

in a flat space $N_i = 6 \implies$ null curvature

$$\int d^2\xi\sqrt{-g}\, R \quad \longleftrightarrow \quad \sum_{i\in V(\mathcal{T})} \frac{N_i}{3}\, 2\pi\frac{6-N_i}{N_i} = 2\pi(2n_V - n_F) = 4\pi\chi$$

Recalling the pure gravity partition function

$$Z_{string} = \sum_{\text{genera}} \int \frac{\mathcal{D}g}{\text{Vol}(Diff)} e^{-\Lambda A - \gamma\chi} .$$

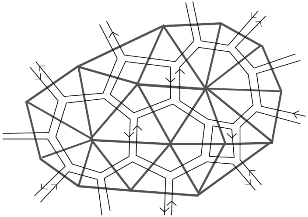
the (discretized) string partition function is

$$Z^{\mathrm{d}}_{string} = \sum_{h=0}^{+\infty} \lambda^h \sum_{\mathcal{T}_h} \frac{1}{|G(\mathcal{T}_h)|} e^{-\Lambda A(\mathcal{T}_h)}$$

- $\mathcal{T}_h$ is one of the possible triangulations for fixed genus h.
- $G(\mathcal{T})$ is the (discrete) group of symmetries of the triangulation $\mathcal{T}$, analog of the isometry group of a continuum manifold.
- $|G(\mathcal{T})|$ is the order of $G(\mathcal{T})$.

Second step: dualize every triangulation by means of a graph.

Dualize the triangulations to graphs with 3-point vertices and double-line links and generate them as Feynman diagrams from a 1-matrix model. The matrix integral will behave as a generating functional for random triangulations.



$$e^{Z_{matrix}} = \int dM \, e^{-N\beta \, \mathrm{tr} \, V(M)} \, , \qquad V(M) = \frac{1}{2} M^2 - g M^3 \, ,$$

where M is a $N \times N$ Hermitian matrix.

Remarks:

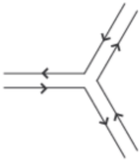
- Continuous graphs (smooth surfaces) are recovered in the $N \to \infty$ limit.
- We want Z_{matrix} to count only the connected diagrams, whereas the integral takes into account the disconnected too.
- Real symmetric matrices bring indistinguishable indices, thus an ensemble of both orientable and non-orientable surfaces.

We know how to deal with this kind of expressions: expand the "interacting part" of the exponential and use Feynam diagrams!

$$e^{Z_{matrix}} = \int dM \, e^{-N\frac{\beta}{2}\operatorname{tr} M^2} \sum_{n=0}^{+\infty} \left[\frac{1}{n!} (N\beta g)^n \left(\operatorname{tr} M^3\right)^n \right]$$

Double-line diagram with each line carrying one of the indices of the matrix *(Cvitanović, '76)*.

$(N\beta)^{-1}$



$N\beta g$

every loop

N

The contribution to the partition function of a diagram with p propagators, v vertices and l loops is

$$Z_{matrix}(v) \sim (N\beta)^{-p}(N\beta g)^v N^l$$

Considering this diagram as a graph ($p \leftrightarrow L$, $v \leftrightarrow V$, $l \leftrightarrow F$)

$$Z_{matrix}(V) \sim N^{V-L+F}\beta^{V-L}g^V = N^\chi \beta^{-V/2} g^V \ ,$$

since in a closed graph with cubic vertices $2L = 3V$.

The partition function is obtained summing over all the possible graphs, i.e. number of vertices and configurations for a given number of vertices,

$$Z_{matrix} = \sum_\chi N^\chi \sum_{\{V_h\}} \frac{1}{n_V} \beta^{-V/2} g^V$$

Factors $1/n_V$ take into account a possible incomplete cancellation of the $1/n!$'s due to some symmetry of the diagrams.

A graph with 3-point vertices is dual to a triangulation and the number of vertices of a graph corresponds to the number of triangles in the related triangulation.

$$Z_{matrix} \longleftrightarrow \sum_{\chi} N^{\chi} \sum_{\mathcal{T}_h} \frac{1}{|G(\mathcal{T}_h)|} \left(\frac{g}{\sqrt{\beta}}\right)^{A(\mathcal{T})}$$

It is a topological expansion in N^{χ}. In the large N limit only the first term survives ($\chi = 2,\ h = 0$), obtaining the so-called *planar approximation*.

We recall for comparison the (discretized) string partition function

$$Z^{\mathsf{d}}_{string} = \sum_{h=0}^{+\infty} \lambda^{h} \sum_{\mathcal{T}_h} \frac{1}{|G(\mathcal{T}_h)|} e^{-\Lambda A(\mathcal{T}_h)}$$

Computing the partition function

The volume element for a $N \times N$ Hermitian matrix is

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\,\mathrm{Re}\,M_{ij} d\,\mathrm{Im}\,M_{ij}$$

Every Hermitian matrix has N real eigenvalues $\{\lambda_i\}$ and can be diagonalized as $M = UXU^\dagger$. With a bit of computation

$$dM = \Delta(\lambda)^2 \prod_{k=1}^{N} d\lambda_k dU \,,$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant. The integration over dU is trivial and the final result is

$$e^{Z_{matrix}} = \int \prod_{k=1}^{N} d\lambda_k \, \Delta(\lambda)^2 e^{-N\beta \sum_i V(\lambda_i)}$$

There are many ways to solve the 1-matrix model, like using the *orthogonal polynomials*, the *saddle point method*, the *Stieltjes method*... We will follow the first one *(Bessis, '79)*.

1 We introduce an infinite set of monic polynomials of order n $\{P_n(\lambda)\}$ orthogonal with respect to the measure

$$\int_{-\infty}^{+\infty} d\lambda\, e^{-N\beta V(\lambda)} P_n(\lambda) P_m(\lambda) = h_n \delta_{nm}\,.$$

2 We apply the properties of the determinant in order to have $\Delta(\lambda) = \det P_{n-1}(\lambda_i)$.

3 With some more computation we arrive at

$$e^{Z_{matrix}} = N! \prod_{n=0}^{N-1} h_n = N!\, h_0^N \prod_{n=1}^{N-1} r_k^{N-n}\,, \qquad r_n := h_n / h_{n-1}\,.$$

In order to determine the h_n's we rely on the recursive relation

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + s_n P_n(\lambda) + r_n P_{n-1}(\lambda)$$

and the two *string equations*

$$\int d\lambda\, e^{-N\beta V(\lambda)} V'(\lambda) P_n(\lambda) P_{n-1}(\lambda) = \frac{n}{N\beta} h_{n-1}$$

$$\int d\lambda\, e^{-N\beta V(\lambda)} V'(\lambda) P_n(\lambda)^2 = 0$$

For our cubic potential

$$r_n[1-3g(s_n+s_{n-1})]=\frac{n}{N\beta}\,,\qquad s_n-3g(r_{n+1}+r_n+s_n^2)=0$$

In the large N limit n/N becomes a continuous variable ξ, $r_n\to r(\xi)$ and $r_{n\pm1}\to r(\xi\pm\varepsilon)=r(\xi)\pm r'(\xi)\,\varepsilon$, with $\varepsilon=1/N$. At first order in ε ($T=1/\beta$)

$$r[1-3g(2s+s'\varepsilon)]=\xi T\,,\qquad s-3g(2r+r'\varepsilon+s^2)=0$$

At zero-th order

$$\xi T=r\sqrt{1-72g^2r}=:W(r)=T_c+\frac{1}{2}W''(r_c)(r-r_c)^2+O(r-r_c)^3$$

where $W'(r_c)=0$ and $T_c=W(r_c)$.

$$r_c=\frac{1}{108g^2}\,,\qquad T_c=\frac{1}{108\sqrt{3}g^2}$$

From the string equations we get $r - r_c \sim (T_c - \xi T)^{1/2}$, hence

$$Z_{matrix} = N^2 \sum_{n=1}^{N-1} \frac{1}{N}\left(1 - \frac{n}{N}\right) \ln r_n + const \overset{N\to\infty}{\sim} N^2 \int_0^1 d\xi (1-\xi) \ln r(\xi)$$
$$\overset{T\to T_c}{\sim} N^2 \int_0^1 d\xi (1-\xi)(T_c - \xi T)^{1/2} \sim N^2 (T_c - T)^{5/2}$$

In order to properly perform the large N limit, $N \to \infty$ and $T \to T_c$ in a way such that $N^2(T_c - T)^{5/2}$ is finite.

Higher genus terms in the expansion of Z_{matrix} can be obtained taking into account first order contributions in the string equations.

Defining

- the new coordinate x: $T_c - \xi T = T_c a^2 x$,
- the new function $u(x)$: $r_c - r(\xi) = r_c a u(x)$,

with $\varepsilon = 1/N = a^{5/2}$, the string equations boil down to (at leading order in ε)

$$u''(x) = 6u(x)^2 - 4$$

This is the *Painlevé-I equation* and for large values of x the solution goes like

$$u(x) \sim x^{1/2}\left(1 - \sum_{k=1}^{+\infty} u_k x^{-5k/2}\right)$$

The partition function satisfies

$$Z''_{matrix}(z) \sim -u(z)$$

with $T_c - T = T_c a^2 z$, thus *(Bessis, Itzykson and Zuber, '80)*

$$Z_{matrix}(z) \sim \sum_{k=-1}^{+\infty} Z_k z^{-5k/2} = \sum_{h=0}^{+\infty} \tilde{Z}_h N^{2-2h} (T_c - T)^{5(1-h)/2}$$

In the double limit $N \to \infty$, $T \to T_c$ with $N^2 (T_c - T)^{5/2}$ finite all the genera contribute to the partition function.

The Liouville approach

The total action is invariant under diffeomorphisms and Weyl transformations.
$\Longrightarrow$ In 2 dimensions every metric g can be written as $e^{\phi} f(\hat{g}(\tau))$.

In order not to overcount equivalent metrics we make a gauge-fixing picking up a representative $\hat{g}$. This splits the integral over the g's into three integrals

- over the moduli space τ,
- over the so-called Liouville field ϕ,
- over the Faddeev-Popov ghosts b and c (BRST quantization related to the $Diff$-invariance).

The gauge-fixing introduces new pieces in the total action

- the Polyakov action (with the chosen metric $\hat{g}$),
- the ghosts action S_{gh},
- the Liouville action

$$S_L = \frac{1}{8\pi} \int_\Sigma d^2\xi \sqrt{-\hat{g}} \left(\hat{g}^{ab} \partial_a \phi \partial_b \phi + Q \hat{R} \phi - 8\pi \Lambda_c e^{\lambda \phi} \right)$$

Q can be fixed requiring that the total action does not depend on the choice of $\hat{g}$.
λ can be fixed requiring that the action is invariant under Weyl transformations.

$$Q = \sqrt{\frac{25 - D}{3}}, \qquad \lambda = \frac{\sqrt{25 - D} - \sqrt{1 - D}}{\sqrt{12}}$$

The partition function for fixed area and genus is

$$Z_{str}^{(h)}(A) = \int \mathcal{D}\phi \mathcal{D}X e^{-S[\phi,X]}\,\delta\left(\int_\Sigma d^2\xi\sqrt{-\hat{g}}\,e^{\lambda\phi} - A\right)$$

A shift in the Liouville field $\phi \to \phi + k/\lambda$ does not change the measure of integration.
The only contribution is

$$\frac{Q}{8\pi}\int_\Sigma d^2\xi\sqrt{-\hat{g}}\,\hat{R}\phi \to \frac{Q}{8\pi}\int_\Sigma d^2\xi\sqrt{-\hat{g}}\,\hat{R}\phi + \frac{Qk}{2\lambda}\chi$$

therefore *(Knizhnik, Polyakov and Zamolodchikov, '88 & David, '88)*

$$Z_{str}^{(h)}(A) = e^{-\frac{Qk}{2\lambda}\chi - k} Z_{str}^{(h)}(e^{-k}A) \overset{e^k=A}{\sim} A^{-\frac{Q}{2\lambda}\chi - 1} = A^{\gamma_{str}-3}$$

with the *string susceptibility*

$$\gamma_{str} := 2 - \frac{Q}{2\lambda}\chi = 2h + \gamma_0(1-h)\,, \quad \gamma_0 = \frac{D - 1 - \sqrt{(25-D)(1-D)}}{12}$$

The total partition function for fixed genus is

$$Z^{(h)}_{string} = \int_0^{+\infty} dA\, Z^{(h)}_{str}(A)\, e^{-(\Lambda-\Lambda_c)A} \sim (\Lambda - \Lambda_c)^{-\gamma_{str}+2}$$

For pure gravity we have

$$Z^{(h)}_{string} \sim (\Lambda - \Lambda_c)^{5(1-h)/2}$$

Comparing with

$$Z^{(h)}_{matrix} \sim (T_c - T)^{5(1-h)/2}$$

we see that

The critical exponents of string theory and 1-matrix model near their critical point match exactly order by order in the topological expansion.

The 1-matrix model results agree with 2d gravity order by order in perturbation theory.

However, whether this agreement holds even in the non-perturbative regime is still an open question.

In the perturbative/topological expansion, terms of large genus (large h) have a $(2h)!$ behavior. The perturbation series is divergent and non-Borel summable.

It is conjectured that the matrix model does not define the sum over topologies beyond perturbation theory and that the non-perturbative effects are expected to be strong.

Thank you for your attention