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THE JORDAN ALGEBRAS OF RIEMANN, WEYL AND CURVATURE COMPATIBLE TENSORS

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Abstract. Given the Riemann, or the Weyl, or a generalized curvature tensor K, a symmetric tensor b_{ij} is called *compatible* with the curvature tensor if $b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0$. In addition to establishing some known and some new properties of such tensors, we prove that they form a special Jordan algebra, i.e. the symmetrized product of K-compatible tensors is K-compatible.

1. Introduction. Let (M, g) be an n-dimensional Riemannian or pseudo-Riemannian manifold, and K_{jklm} a generalized curvature tensor (the Riemann tensor, the Weyl tensor, or any tensor with the algebraic properties of the Riemann tensor). In [14] we introduced this concept: a symmetric tensor b_{ij} is K-*compatible* if

$$b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0. \tag{1}$$

We call (K, b) a *compatible pair*. The motivation was the following theorem [14]: if b_{ij} is K-compatible with eigenvectors X, Y, Z and eigenvalues x, y, z with $z \neq x, y$, then

$$K_{ijlm} X^i Y^j Z^m = 0. \tag{2}$$

It extends a result by Derdziński and Shen [6] who proved the same for the Riemann tensor, under the hypothesis that b_{ij} is a Codazzi tensor, $\nabla_i b_{jk} = \nabla_j b_{ik}$. Despite the increased generality, the replacement of the Codazzi condition with the algebraic condition (1), allowed a much simpler proof of the new theorem.

Equation (1) with Riemann's tensor originally appeared in a paper by Roter on conformally symmetric spaces [20, Lemma 1]. Riemann and Weyl compatible tensors were studied in [15, 17, 7].

Examples of Riemann compatible tensors are the Codazzi tensors [14], the Ricci tensors of Robertson–Walker space-times or perfect-fluid generalized Robertson–Walker space-times [18], the second fundamental form and

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the Ricci tensor of a hypersurface embedded in a (pseudo)Riemannian manifold [17], the Ricci tensors of 'weakly Z-symmetric' manifolds ($\nabla_i Z_{jk} = A_i Z_{jk} + B_j Z_{ik} + D_k Z_{ij}$ with $Z_{ij} = R_{ij} + \varphi g_{ij}$, $A_k - B_k$ a closed 1-form) [16] that include 'weakly Ricci-symmetric' ones ($\varphi = 0$) [24] and others (see [3, 2]), or 'pseudosymmetric manifolds' [9] ($[\nabla_i, \nabla_j] R_{klmp} = L Q_{klmpij}$, where $L \neq -1/3$ is a scalar function and Q is the Tachibana tensor built with the Riemann and Ricci tensors).

A Riemann compatible tensor is also Weyl compatible, but not conversely. The Ricci tensors of Gödel [10, Th. 2] or pseudo-Z symmetric space times [19] are Weyl compatible.

In Sections 2 and 3 we review Riemann and Weyl compatible tensors, with some new results and examples, and their relation to known identities due to Lovelock. Then, in Sections 4, 5 and 6, we investigate the algebraic properties of generalized curvature tensors and K-compatible tensors. The main result is that the latter form a *special Jordan algebra*, i.e. the set of K-compatible tensors is closed under the symmetrized product.

2. Riemann compatible tensors. A symmetric tensor is *Riemann compatible* if

$$b_i{}^m R_{jklm} + b_j{}^m R_{kilm} + b_k{}^m R_{ijlm} = 0. \tag{3}$$

This relation may be written as $b_{(i}{}^m R_{jk)lm} = 0$, where (ijk) denotes the sum over cyclic permutations of the indices. Contraction with the metric tensor g^{jl} gives $R_{km} b_i^m - b_k{}^m R_{mi} = 0$, so that b commutes with the Ricci tensor. Contraction with b^{jl} gives $b_i{}^m R_{jklm} b^{jl} + b_k{}^m R_{ijlm} b^{jl} = 0$, and hence b commutes with the symmetric tensor $\hat{R}_{jm} = R_{jklm} b^{kl}$.

Example 2.1. Codazzi tensors are Riemann compatible.

Proof. In the identity $[\nabla_i, \nabla_j] b_{kl} = -R_{ijl}{}^m b_{km} - R_{ijk}{}^m b_{ml}$, sum over cyclic permutations of ijk. The first Bianchi identity $R_{(ijk)}{}^m = 0$ gives

$$[\nabla_i, \nabla_j] b_{kl} + [\nabla_j, \nabla_k] b_{il} + [\nabla_k, \nabla_i] b_{jl} = -(b_i{}^m R_{jklm} + b_j{}^m R_{kilm} + b_k{}^m R_{ijlm}).$$

The left-hand side is zero for Codazzi tensors. ∎

Example 2.2. If $\nabla_j A_k = p_j A_k$, then $A_i A_j$ is Riemann compatible.

Proof. We have $A_i [\nabla_j, \nabla_k] A_l = A_i (\nabla_j p_k - \nabla_k p_j) A_l = A_l [\nabla_j, \nabla_k] A_i$. Then $A_i R_{jkl}{}^m A_m = A_l R_{jki}{}^m A_m$; the sum over cyclic permutations of ijk gives zero on the right-hand side. ∎

2.1. Codazzi deviation. In [15] we introduced the natural concept of *Codazzi deviation* of a symmetric tensor:

$$\mathscr{C}_{jkl} = \nabla_j b_{kl} - \nabla_k b_{jl}. \tag{4}$$

It satisfies $\mathscr{C}_{jkl} = -\mathscr{C}_{kjl}$, $\mathscr{C}_{jkl} + \mathscr{C}_{klj} + \mathscr{C}_{ljk} = 0$, and

$$\nabla_i \mathscr{C}_{jkl} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{ijl} = -(b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m). \tag{5}$$

Once again we see that a Codazzi tensor is Riemann compatible. By (5) the differential condition $\nabla_{(i} \mathscr{C}_{jk)l} = 0$ is equivalent to the algebraic formula (3). A Veblen-type identity holds:

$$\begin{aligned} \nabla_i \mathscr{C}_{jlk} + \nabla_j \mathscr{C}_{kil} + \nabla_k \mathscr{C}_{lji} + \nabla_l \mathscr{C}_{ikj} \\ = b_{im} R_{jlk}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{lji}{}^m + b_{lm} R_{ikj}{}^m. \end{aligned} \tag{6}$$

Example 2.3. For a concircular vector field X, with $\nabla_i X_j = \rho g_{ij}$, the tensor $X_i X_j$ is Riemann compatible.

Proof. One has $\mathscr{C}_{jkl} = (\nabla_j \rho) g_{kl} - (\nabla_k \rho) g_{jl}$ and $\nabla_i \mathscr{C}_{jkl} = (\nabla_i \nabla_j \rho) g_{kl} - (\nabla_i \nabla_k \rho) g_{jl}$. The left-hand side of (5) thus equals zero. ■

Note: the existence of a concircular time-like vector field is necessary and sufficient for a space-time to be generalized Robertson–Walker [5].

Example 2.4 (Lovelock's identities). 1. The Codazzi deviation of the Ricci tensor is $\mathscr{C}_{jkl} = \nabla_j R_{kl} - \nabla_k R_{jl} = -\nabla^m R_{jklm}$. Property (5) becomes Lovelock's identity for the Riemann tensor [13, p. 289]:

$$\nabla_i \nabla^m R_{jklm} + \nabla_j \nabla^m R_{kilm} + \nabla_k \nabla^m R_{ijlm} = -R^m{}_{(i} R_{jk)lm}. \tag{7}$$

2. The Codazzi deviation of Schouten's tensor (1) is $\mathscr{C}_{jkl} = -\frac{1}{n-3} \nabla^m C_{jklm}$. Property (5) reads $\nabla_{(i} \mathscr{C}_{jk)l} = -(n-3) S^m{}_{(i} R_{jk)lm}$. The term with the metric tensor in S_{ij} does not contribute (due to the Bianchi identity), and one is left with (see [15])

$$\nabla_i \nabla^m C_{jklm} + \nabla_j \nabla^m C_{kilm} + \nabla_k \nabla^m C_{ijlm} = -\frac{n-3}{n-2} R^m{}_{(i} R_{jk)lm}. \tag{8}$$

In particular, for $n > 3$, if $\nabla_m C_{jkl}{}^m = 0$ (conformally symmetric spaces, Roter [20]) then the Ricci tensor is Riemann compatible.

Proposition 2.5. *If $u_i u_j$ is Riemann compatible, and $u^k u_k \neq 0$, then u_i is an eigenvector of the Ricci tensor.*

Proof. Since $u_i u_j$ is Riemann compatible, it commutes with the Ricci tensor: $R_{ij} u^j u_k = R_{kj} u^j u_i$. Contraction with u^k gives

$$R_{ij} u^j (u_k u^k) = (R_{kj} u^j u^k) u_i = 0. \quad ■$$

We extrapolate a simple statement from [7, Proposition 5.1]. A direct proof is possible, by writing (3) for the Ricci tensor in warping coordinates:

(1) Schouten's tensor is $S_{ij} = \frac{1}{n-2}\left[R_{ij} - \frac{R}{2(n-1)} g_{ij}\right]$. It satisfies $\nabla_k S^k{}_j = \nabla_j S^k{}_k$, $\nabla^m C_{jklm} = (n-3)(\nabla_k S_{jl} - \nabla_j S_{kl})$.

Proposition 2.6. *In a warped-product spacetime*

$$ds^2 = \pm dt^2 + a(t)^2 g^*_{\mu\nu} dx^\mu dx^\nu$$

the Ricci tensor is Riemann compatible if and only if the Ricci tensor of the Riemannian submanifold (M^*, g^*) *is compatible with the Riemann tensor of the submanifold:*

$$R^*_{\mu\sigma} R^*_{\nu\rho\lambda}{}^\sigma + R^*_{\nu\sigma} R^*_{\rho\mu\lambda}{}^\sigma + R^*_{\rho\sigma} R^*_{\mu\nu\lambda}{}^\sigma = 0.$$

2.2. Geodesic maps. A map $(M, g) \to (M, \overline{g})$ is *geodesic* if every geodesic line is mapped to a geodesic line. For the identity mapping of M to be geodesic, it is necessary and sufficient that there exists a 1-form such that the Christoffel symbols are related by $\overline{\Gamma}^k_{ij} = \Gamma^k_{ij} + \delta_i{}^k X_j + X_i \delta^k{}_j$ (Levi-Civita, 1896). The relation between the Riemann tensors then is

$$\overline{R}_{jkl}{}^m = -\partial_j \overline{\Gamma}^m_{kl} + \partial_k \overline{\Gamma}^m_{jl} - \overline{\Gamma}^d_{kl} \overline{\Gamma}^m_{jd} + \overline{\Gamma}^d_{jl} \overline{\Gamma}^m_{kd} = R_{jkl}{}^m - \delta_k{}^m P_{jl} + \delta_j{}^m P_{kl},$$

where $P_{kl} = \nabla_k X_l - X_k X_l = P_{lk}$. One has $\overline{R}_{jl} = R_{jl} + (n-1) P_{jl}$.

Geodesic maps preserve the $(3,1)$ projective curvature tensor [23]: $\overline{P}_{jkl}{}^m = P_{jkl}{}^m$, where $P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl})$.

Proposition 2.7 ([15]). *For a geodesic map and a symmetric tensor* $b_{ij} = b_{ji}$*, the following identity holds:*

$$b_{im}\overline{R}_{jkl}{}^m + b_{jm}\overline{R}_{kil}{}^m + b_{km}\overline{R}_{ijl}{}^m = b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m. \quad (9)$$

Therefore, if (R, b) *is a compatible pair, also the pair* $(\overline{R}, b)$ *is compatible.*

3. Weyl compatible tensors. A symmetric tensor is *Weyl compatible* if

$$b_{im} C_{jkl}{}^m + b_{jm} C_{kil}{}^m + b_{km} C_{ijl}{}^m = 0. \quad (10)$$

The following identity holds for any symmetric tensor [15]:

$$\begin{aligned} b_{im} C_{jkl}{}^m + b_{jm} C_{kil}{}^m + b_{km} C_{ijl}{}^m &= b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m \\ + b_{km} R_{ijl}{}^m + \tfrac{1}{n-2}[g_{kl}(b_{im} R_j{}^m - b_{jm} R_i{}^m) &+ g_{il}(b_{jm} R_k{}^m - b_{km} R_j{}^m) \\ &+ g_{jl}(b_{km} R_i{}^m - b_{im} R_k{}^m)]. \end{aligned} \quad (11)$$

A simple consequence is obtained in dimension $n = 3$, where the Weyl tensor is zero (see [8], in a less simple manner):

Proposition 3.1. *In dimension* $n = 3$ *the Ricci tensor is Riemann compatible.*

If b_{ij} is Riemann compatible, then it commutes with the Ricci tensor. As a result, (11) shows that b_{ij} is also Weyl compatible. Therefore, Riemann compatibility is a stronger condition than Weyl compatibility. The identity

(11) can be rewritten in terms of the Codazzi deviation:

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = \nabla_i \mathscr{D}_{jkl} + \nabla_j \mathscr{D}_{kil} + \nabla_k \mathscr{D}_{ijl} - \tfrac{1}{n-2}\nabla^m(\mathscr{C}_{ijm}g_{kl} + \mathscr{C}_{jkm}g_{il} + \mathscr{C}_{kim}g_{jl}), \tag{12}$$

where $\mathscr{D}_{jkl} = \mathscr{C}_{jkl} - \frac{1}{n-2}(\mathscr{C}_{jm}{}^m g_{kl} - \mathscr{C}_{km}{}^m g_{jl})$.

EXAMPLE 3.2. If a vector field is *torqued* [4], i.e. $\nabla_i \tau_j = \rho g_{ij} + \alpha_i \tau_j$ with $\alpha_k \tau^k = 0$, then $\tau_i \tau_j$ is Weyl compatible.

Proof. One evaluates $\mathscr{C}_{jkl} = -\rho(\tau_j g_{kl} - \tau_k g_{jl})$ and $\mathscr{D}_{jkl} = -\frac{1}{n-2}\mathscr{C}_{jkl}$. It turns out that the right-hand side of (12) is zero. ∎

Note: the existence of a torqued time-like vector is necessary and sufficient for a space-time to be twisted [4].

PROPOSITION 3.3 (see [11, Remark 4.2]). *In a space-time of dimension $n = 4$, if $u_i u_j$ is a Weyl compatible and time-like unit ($u^k u_k = -1$) then the Weyl tensor is completely determined by the electric tensor $E_{kl} = C_{jklm}u^j u^m$:*

$$C_{abcd} = 2(u_a u_d E_{bc} - u_a u_c E_{bd} + u_b u_c E_{ad} - u_b u_d E_{ac}) + g_{ad}E_{bc} - g_{ac}E_{bd} + g_{bc}E_{ad} - g_{bd}E_{ac} \tag{13}$$

Proof. In $n = 4$ the following Lovelock identity holds [13, Ex. 4.9, p. 128]:

$$0 = g_{ar}C_{bcst} + g_{br}C_{cast} + g_{cr}C_{abst} + g_{at}C_{bcrs} + g_{bt}C_{cars} + g_{ct}C_{abrs} + g_{as}C_{bctr} + g_{bs}C_{catr} + g_{cs}C_{abtr}$$

Contraction with $u^a u^r$ gives

$$\begin{aligned} 0 = & -C_{bcst} + u_b u^r C_{crst} + u_c u^r C_{rbst} + u_t u^r C_{bcrs} + g_{bt}u^a u^r C_{cars} \\ & + g_{ct}u^a u^r C_{abrs} + u_s u^r C_{bctr} + g_{bs}u^a u^r C_{catr} + g_{cs}u^a u^r C_{abtr} \\ = & -C_{bcst} + u^r(u_b C_{stcr} + u_c C_{rbst} + u_t C_{cbsr} + u_s C_{bctr}) \\ & + g_{bt}E_{cs} - g_{ct}E_{bs} - g_{bs}E_{ct} + g_{cs}E_{bt}. \end{aligned}$$

This gives the Weyl tensor in terms of its single and double contractions with u^i. If $u_i u_j$ is Weyl compatible, the single contraction is $C_{jklr}u^r = u_k E_{jl} - u_j E_{kl}$, and the result follows. For an extension to $n > 4$ see [11]. ∎

3.1. Conformal maps. The identity map $(M, g) \to (M, \hat{g})$ is *conformal* if $\hat{g}_{kl} = e^{2\sigma}g_{kl}$ for some function σ. The Christoffel symbols transform according to $\hat{\Gamma}^m_{ij} = \Gamma^m_{ij} + \delta^m{}_i X_j + X_i \delta^m{}_j - g_{ij}X^m$, where $X_i = \nabla_i \sigma$. A conformal map leaves the $(3,1)$ Weyl tensor unchanged: $\hat{C}_{jkl}{}^m = C_{jkl}{}^m$. Therefore, Weyl compatibility is a property invariant under conformal maps.

4. *K*-compatible tensors. Riemann and Weyl compatibility may be generalized to K-compatibility, where K is a generalized curvature tensor (GCT), i.e. a tensor with the algebraic properties of the Riemann tensor under permutations of indices [12]:

$$K_{jklm} = -K_{kjlm} = -K_{jkml}, \tag{14}$$

$$K_{jklm} + K_{kljm} + K_{ljkm} = 0, \tag{15}$$

$$K_{jklm} = K_{lmjk}. \tag{16}$$

In analogy with the Riemann tensor, one shows that (14) and (15) imply the symmetry (16), and the identity $K_{j(klm)} = 0$. The tensor $K_{jl} = K_{jml}{}^m$ is symmetric.

A symmetric tensor b_{ij} is K-compatible if

$$b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} = 0, \tag{17}$$

and (K, b) is then called a *compatible pair*. This can be written as $b^m{}_{(i}K_{jk)lm} = 0$.

The metric tensor is K-compatible, by the Bianchi property (15). The tensors b_{ij} and K_{ij} commute: $b_i{}^m K_{mk} - K_{im} b^m{}_k = 0$ (contract (17) with g^{jl} and use symmetry).

Examples of K-compatible tensors were obtained by Shaikh et al. (see for example [22, 21]) starting from specific metrics. Bourguignon [1] proved that if b_{ij} is a Codazzi tensor then $\mathring{R}_{jklm} = R_{jkrs} b^r{}_l b^s{}_m$ is a GCT. We prove a more general statement:

PROPOSITION 4.1. *If a_{ij} and b_{ij} are K-compatible, then $\mathring{K}_{jklm} = K_{jkrs}(a^r{}_l b^s{}_m + b^r{}_l a^s{}_m)$ is a* GCT.

Proof. The properties (14) and (16) are obvious; the Bianchi property (15) completes the proof: $\mathring{K}_{(jkl)m} = a^r{}_{(l}K_{jk)rs}\, b^s{}_m + b^r{}_{(l}K_{jk)rs}\, a^s{}_m = 0$ because each term is zero, both a and b being K-compatible. ∎

4.1. Properties of K-compatible tensors. A linear combination of K-compatible tensors obviously is K-compatible. Now we prove:

THEOREM 4.2. *If a and b are K-compatible, then $\frac{1}{2}(ab + ba)$ is K-compatible.*

Proof. Let $c_{ij} = a_i{}^k b_{kj} + b_i{}^k a_{kj}$. Then

$$\begin{aligned}
c^m{}_{(i}K_{jk)rm} &= a_i{}^s b_s{}^m K_{jkrm} + a_j{}^s b_s{}^m K_{kirm} + a_k{}^s b_s{}^m K_{ijrm} + a \leftrightarrows b \\
&= -a_i{}^s(b_j{}^m K_{ksrm} + b_k{}^m K_{sjrm}) - a_j{}^s(b_k{}^m K_{isrm} + b_i{}^m K_{skrm}) \\
&\quad - a_k{}^s(b_i{}^m K_{jsrm} + b_j{}^m K_{sirm}) + a \leftrightarrows b \\
&= -(a_i{}^s b_j{}^m - a_j{}^s b_i{}^m)K_{ksrm} - (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m)K_{isrm} \\
&\quad - (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m)K_{jsrm} + a \leftrightarrows b \\
&= -(a_i{}^s b_j{}^m - a_j{}^s b_i{}^m)(K_{ksrm} - K_{kmrs}) \\
&\quad - (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m)(K_{isrm} - K_{imrs}) \\
&\quad - (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m)(K_{jsrm} - K_{jmrs})
\end{aligned}$$

$$
\begin{aligned}
&= (a_i{}^s b_j{}^m - a_j{}^s b_i{}^m) K_{krsm} + (a_j{}^s b_k{}^m - a_k{}^s b_j{}^m) K_{irsm} \\
&\quad + (a_k{}^s b_i{}^m - a_i{}^s b_k{}^m) K_{jrsm} \\
&= (a_i{}^s b_j{}^m + b_i{}^s a_j{}^m) K_{krsm} + (a_j{}^s b_k{}^m + b_j{}^s a_k{}^m) K_{irsm} \\
&\quad + (a_k{}^s b_i{}^m + b_k{}^s a_i{}^m) K_{jrsm} \\
&= \mathring{K}_{krij} + \mathring{K}_{irjk} + \mathring{K}_{jrki} = \mathring{K}_{(kri)j} = 0
\end{aligned}
$$

because $\mathring{K}$ is a GCT by Proposition 4.1. ∎

Therefore, the linear space of K-compatible tensors is a special Jordan algebra.

In particular, the powers of b are K-compatible (powers with exponents $n, n+1, \dots$ are linear combinations of lower powers by the Cayley–Hamilton theorem). In particular (by the exchange of indices) the tensor $(b^2)_j{}^s (b^2)_k{}^r K_{rslm}$ is a GCT. This enables us to come up with the simple proof of the theorem in [14], so short that we reproduce it here:

THEOREM 4.3 (Extended Derdziński–Shen theorem). *Let b_{ij} be K-compatible, and let X^i, Y^i, Z^i be eigenvectors of $b_i{}^m$ with eigenvalues x, y, z. If $x \neq z$ and $y \neq z$ then*

$$K_{ijkl} X^i Y^j Z^k = 0. \tag{18}$$

Proof. Consider the identities

$$g^m{}_{(i} K_{jk)lm} = 0, \qquad b^m{}_{(i} K_{jk)lm} = 0, \qquad (b^2)^m{}_{(i} K_{jk)lm} = 0$$

and contract them with $X^i Y^j Z^k$. The three algebraic relations are put in matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{bmatrix} \begin{bmatrix} K_{jkli} X^i Y^j Z^k \\ K_{kilj} X^i Y^j Z^k \\ K_{ijlk} X^i Y^j Z^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the matrix is $(x-y)(x-z)(z-y)$. If the eigenvalues are all different then $K_{ijkl} X^i Y^j Z^k = 0$ (with contraction of any three indices). If $x = y \neq z$, the reduced system of equations still implies $K_{ijkl} X^i Y^j Z^k = 0$. ∎

PROPOSITION 4.4. *If b is K-compatible and invertible, then b^{-1} is K-compatible:*

$$(b^{-1})^j{}_{(s} K_{rl)kj} = 0. \tag{19}$$

Proof. Multiply (17) by $(b^{-1})^i{}_r (b^{-1})^j{}_s$ to obtain the identity $(b^{-1})^j{}_s K_{jklr} + (b^{-1})^i{}_r K_{kils} + (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} = 0$. Rewrite it as

$$(b^{-1})^j{}_{(s} K_{rl)kj} - (b^{-1})^j{}_l K_{srkj} + (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} = 0.$$

The last two terms cancel, as shown by

$$(b^{-1})^j{}_l K_{srkj} = (b^{-1})^i{}_r (b^{-1})^j{}_s b^m{}_k K_{ijlm} \iff K_{srkb} b^r{}_a = b^i{}_b (b^{-1})^j{}_s b^m{}_k K_{ajlm}$$
$$\iff b^s{}_c K_{srkb} b^r{}_a = b^l{}_b b^m{}_k K_{aclm}$$
$$\iff \mathring{K}_{kbca} = \mathring{K}_{acbk},$$

which is true as $\mathring{K}$ is a GCT. ■

We prove a Veblen-type identity:

PROPOSITION 4.5. *If b_{ij} is K-compatible, then*

$$b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_k{}^m K_{iljm} - b_l{}^m K_{jkim} = 0. \tag{20}$$

Proof. We have

$$\begin{aligned} 0 &= b_i{}^m K_{jklm} + b_j{}^m K_{kilm} + b_k{}^m K_{ijlm} \\ &= b_i{}^m K_{jklm} - b_j{}^m (K_{ilkm} + K_{lkim}) + b_k{}^m K_{ijlm} \\ &= b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_l{}^m K_{kjim} + b_k{}^m K_{jlim} + b_k{}^m K_{ijlm} \\ &= b_i{}^m K_{jklm} - b_j{}^m K_{ilkm} + b_l{}^m K_{kjim} - b_k{}^m K_{lijm}. \end{aligned}$$ ■

4.2. More on generalized curvature tensors. A linear combination of GCTs is a GCT. Given two compatible pairs (K, a) and (K, b), a new GCT tensor is obtained in Proposition 4.1. In particular, if $a_{ij} = g_{ij}$ (the metric tensor), the following K' is a GCT:

$$K'_{jklm} = K_{jkrs}(\delta^r{}_l b^s{}_m + b^r{}_l \delta^s{}_m) = K_{jkls} b^s{}_m - K_{jkms} b^s{}_l. \tag{21}$$

PROPOSITION 4.6. *If b is K-compatible, then it is K'-compatible.*

Proof. The tensor $K'_{jklm} = K_{jklr} b^r{}_m - K_{jkmr} b^r{}_l$ is a GCT. Let us evaluate

$$b^m{}_i K'_{jklm} = b^m{}_i K_{jklr} b^r{}_m - b^m{}_i K_{jkmr} b^r{}_l = (b^2)^r{}_i K_{jklr} - \mathring{K}_{jkim}.$$

Both tensors yield zero if the cyclic sum (ijk) is taken. ■

PROPOSITION 4.7. *(K, b) is a compatible pair for every symmetric tensor b if and only if*

$$K_{ijlm} = \frac{K}{n(n-1)} (g_{il} g_{jm} - g_{im} g_{jl}) \tag{22}$$

where K is a scalar.

Proof. The symmetry of the tensor is made explicit by writing $b_{ij} = \frac{1}{2} b^{rs}(g_{ir} g_{js} + g_{is} g_{jr})$. The compatibility relation must hold for any b^{rs}, so

$$0 = g_{ir} K_{jkls} + g_{jr} K_{kils} + g_{kr} K_{ijls} + g_{is} K_{jklr} + g_{js} K_{kilr} + g_{ks} K_{ijlr}.$$

Contraction with g^{ks} gives $(n-1) K_{ijlr} = g_{jr} K_{il} - g_{ir} K_{jl}$; contraction with g^{il} gives $K_{jr} = \frac{1}{n} g_{jr} K^i{}_i$ and (22) follows. The converse, namely, that (22) implies (17), is shown by direct check. ■

A pseudo-Riemannian manifold of dimension $n > 2$ is an *Einstein manifold* if $R_{ij} = \frac{1}{n} R g_{ij}$ where R is the scalar curvature. Since $\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R$, the scalar curvature is constant. A manifold is a *constant curvature manifold* if the Riemann tensor has the form (22). Such manifolds are Einstein manifolds.

COROLLARY 4.8. *A manifold is a constant curvature manifold if and only if* $b_i{}^m R_{jklm} + b_j{}^m R_{kilm} + b_k{}^m R_{ijlm} = 0$ *for all symmetric tensors.*

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