

## Periodic and Non-Periodic Band Random Matrices: Structure of Eigenstates

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**Abstract.** — The structure of eigenstates for the ensembles of standard and periodic Band Random Matrices (BRM) is analysed. The main attention is drawn to the scaling properties of the inverse participation ratio and other measures of localization length. Numerical data are compared with analytical results recently derived for standard BRMs of very large band size. The data for periodic and standard BRM allow us to exhibit the influence of boundary conditions on the properties of eigenstates.

### 1. Introduction

Recently, much attention has been drawn to the so-called band random matrices (BRM) (see, e.g. [1–19]). In contrast with the traditional Gaussian ensembles of full random matrices, which enjoy rotational invariance and have no free parameter, BRMs depend on a control parameter, related to their band structure. Since the typical localization length  $l$  of an eigenstate directly depends on the effective band size  $b$ , statistical properties both of spectra and eigenvectors change significantly with this parameter. Moreover, for finite matrices, the localization length can be compared to the matrix size  $N$  and the transition from localized to extended states can be studied.

The increasing interest in matrices of this kind is explained by their broad application. In particular, much progress has occurred in connection with “quantum chaos”, the subject that deals with dynamical systems exhibiting strong chaos in the classical limit (see, e.g. [20–23]). It was shown that, in the semiclassical limit, the generic structure of Hamiltonian matrices in some basis is banded and, if the corresponding classical motion is chaotic, matrix elements can be assumed pseudo-random, see details in [2, 21, 22].

Another important application is related to 1D and quasi-1D solid state models with random potentials. In the 1D geometry, the size of the band  $b \gg 1$  has the meaning of long range

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random hopping in the tight-binding representation. In the quasi-1D case, the parameter  $b$  plays the role of number of transverse channels; this analogy has been proved rigorously [7]. Other applications are currently investigated in the statistical approach to complex atoms [24, 25] and nuclei [26].

An ensemble of random matrices with non-zero matrix elements inside a band was introduced for the first time by Wigner [27]. In his model, the main diagonal of the matrix contains increasing elements, the other entries in the band being random and drawn according to the same probability distribution. After the pioneering work of Wigner, the model has been forgotten for a long time, mainly because of mathematical difficulties in getting rigorous results, in favour of the more accessible ensembles of full random matrices [29]. Matrices with a band structure have been considered only recently, in application to chaotic dynamical systems [1, 2, 17, 28].

After extensive numerical studies [3, 4, 6, 8–12, 14, 17, 18, 22], revealing the remarkable scaling properties of eigenvectors of BRMs, important progress has recently been made in the analytic approach, based on the powerful supersymmetric method [7, 13, 15, 16, 19]. Many new results about the structure of eigenstates were obtained, in the limit  $b, N \rightarrow \infty$  with finite ratio  $\lambda = b^2/N$ .

In this paper, we present numerical data for periodic band random matrices (PBRM), which are still open for analytical treatment. The structure of these matrices is slightly different from that of standard BRMs, and it relates to a periodic structure of all eigenstates. They appear in a natural way when considering periodic 1D or quasi-1D disordered models. In a descriptive language, if the model of standard BRM may be associated with a chain of  $N$  interacting sites, PBRMs correspond to the same chain of interacting sites joined into a ring.

An important application has been discussed in [30], where the relation between level curvature (second derivative of the energy with respect to the flux) and conductance is studied by varying the magnetic flux through a quasi-1D ring. In particular, it was found that the mean absolute value of curvature is proportional to the mean conductance both in localized and metallic regimes; other important conclusions were drawn for the distribution of the level curvature. The same model of PBRM has been recently considered [31] to analyze the velocity autocorrelation  $C(\phi)$  and to exhibit its singularity in  $\phi = 0$ . Since the sensitivity of levels to a small change of an external perturbation is related to the structure of eigenstates, it is important to compare the statistical properties of the eigenstates in the two versions of BRM.

This paper is organized as follows: in Section 2 we describe the model, refer to the known results on the scaling of the mean localization length, present new data for standard BRM and demonstrate the different scaling properties of PBRMs. Section 3 is devoted to analyze various measures characterizing fluctuations of localization lengths of individual eigenvectors. We study the distribution of inverse participation ratio (IPR) for both models of BRM. To better visualize the differences between the linear system (standard BRM) and the circular system (PBRM), we introduce the concept of *barycenter* of an eigenvector. It is a complex quantity. The distribution of the phase allows us to show the effect of boundary conditions, while the distribution of the modulus of the barycenter proves it to be an alternative measure of the localization length.

## 2. Scaling of the Mean Localization Length

The ensemble of standard real symmetric BRMs is specified by the band width  $b$  and by the matrix size  $N$ . Inside the band, matrix elements  $H_{nm}$  are distributed according to the Gaussian distribution with zero mean and fixed variance

$$\langle H_{nm} \rangle = 0, \quad \langle H_{nm}^2 \rangle = \frac{(N+1)(1+\delta_{nm})}{b(2N-b+1)} \quad (1)$$

With this normalization, eigenvalues (in the limit of large  $N$ ) are located in the interval  $(-2, +2)$ , independently of the value  $b$ . The delta-symbol is introduced to assure normalization of the Gaussian Orthogonal Ensemble (GOE) in the limit of full matrices ( $b = N$ ). Non-symmetrical hermitian BRMs have recently been studied in [32]. Since analytical results for BRMs were obtained in the limit  $N \gg b \gg 1$  [7,13,15,16], we numerically study large matrices, their size being only restricted by the computer.

For infinite BRMs ( $N \rightarrow \infty$ ) with a finite band  $b \gg 1$ , all eigenstates are found to be exponentially localized [4, 7, 8, 17, 18, 22]. This means that the amplitude  $\varphi_n$  of an eigenstate decays exponentially fast away from a center of localization,  $\varphi_n \sim \exp(-|n - n_0|/l_\infty)$ . The quantity  $l_\infty$ , known as the localization length, is of great importance in describing the global structure of eigenstates. Based on scaling arguments, in [4, 22] it was predicted that the localization length  $l_\infty$  is proportional to  $b^2$ . The exact expression was obtained in [7] and applies also to BRMs where the band structure is achieved in full matrices by decreasing the variance of matrix elements away from the main diagonal by a suitable envelope function:

$$l_\infty = c\rho^2(E)b^2, \quad (2)$$

where  $c$  is a constant of order 1 depending on the envelope function and  $\rho(E)$  is the density of states given by the semicircle law,

$$\rho(E) = \frac{2}{\pi R^2} \sqrt{R^2 - E^2}; \quad R^2 = 8b\langle H_{nm}^2 \rangle. \quad (3)$$

Our interest is in BRMs of finite but large size  $N$ ; such matrices can be associated with physical systems in a finite reference basis. To study the structure of eigenstates in the finite basis, we use a definition of localization length based on the information entropy  $\mathcal{H}_N$  of a normalized eigenstate of components  $\varphi_n$ ,

$$\mathcal{H}_N = - \sum_{n=1}^N |\varphi_n|^2 \ln(|\varphi_n|^2), \quad (4)$$

Since the maximal value of the information entropy is  $N$ , in references [22, 33] it was suggested to define the localization length  $l_N^{(1)}$  as

$$l_N^{(1)} = N \exp(\langle \mathcal{H}_N \rangle - \mathcal{H}_{\text{GOE}}). \quad (5)$$

In this definition, the localization length is normalized in such a way that for the most extended eigenstates, which appear in the limit  $b = N$ , the value of  $l_N^{(1)}$  is equal to the size of the basis,  $l_N^{(1)} = N$ . The quantity  $\mathcal{H}_{\text{GOE}}$  is the average entropy of random eigenstates of GOE and can be evaluated analytically

$$\mathcal{H}_{\text{GOE}} = \psi\left(\frac{N}{2} + 1\right) - \psi\left(\frac{3}{2}\right) \approx \ln\left(\frac{N}{2.07}\right), \quad (6)$$

$\psi(x)$  is the digamma function (see details in [18, 22]) In equation (5)  $\langle \dots \rangle$  has the meaning of ensemble average over eigenstates of the same energy. The entropy localization length  $l_N^{(1)}$  defined by equation (5) varies from 0 to  $N$  depending on the degree of localization.

The essential point of the previous studies [4, 6, 13, 15, 22] is that the statistical properties both of eigenstates and spectrum depend on the ratio of the localization length  $l_\infty$  to the size of the basis  $N$ . Since for a fixed energy the localization length  $l_\infty$  is proportional to  $b^2$ , the scaling parameter in the limit  $N \gg b \gg 1$  was suggested to be of the form

$$\lambda = \frac{b^2}{N}. \quad (7)$$

In particular, in [4] it was numerically found that the localization length  $l_N^{(1)}$  scales with  $\lambda$  as follows:

$$\beta_1 \equiv \frac{l_N^{(1)}}{N} = \frac{c_1 \lambda}{1 + c_1 \lambda}, \quad (8)$$

with the estimated parameter value  $c_1 \approx 1.4$ . A detailed analytical study [16] has shown that the above scaling expression is approximate, although very close to the correct one. For the comparison with the results of [13] it is convenient to use the variable

$$y_1 = \frac{\beta_1}{1 - \beta_1}, \quad (9)$$

in which the scaling (8) reads:

$$\ln y_1 = \ln \lambda + a_1; \quad a_1 = \ln c_1, \quad (10)$$

This formula is exact only asymptotically, with two different values for the coefficient  $a_1$  [15]:  $a_1^{(-)} \approx 1.58$  for strongly localized states ( $\lambda \ll 1$ ) and  $a_1^{(+)} \approx 1.79$  for extended states ( $\lambda \gg 1$ ). Though the absolute values of the asymptotic slopes  $a_1^{(\pm)}$  depend on the normalization used to define the scaling parameter, their difference  $\Delta a = a_1^{(+)} - a_1^{(-)} = 0.21$  can be directly compared with the numerical results. Due to its smallness, the shift  $\Delta a$  could hardly be detected in the first numerical data [4] and yet in Figure 1a, where the scaling is exhibited in the variables  $\ln y_1, \ln \lambda$  using new data of a more careful study of BRMs of type (1).

The existence of two limit parameters  $a_1^{(\pm)}$  can be settled from the same data of Figure 1a, using variables mapping the dependence (10) in the simplest form  $\ln(y_1/\lambda) = a_1$ . This is done in Figure 1b, where the two straight lines are fits yielding  $a_1^{(-)} \approx 0.18$  (lower line),  $a_1^{(+)} \approx 0.30$  (upper line), obtained from the data in the two regions. Although the data confirm the existence of two different limit values in the expression (10), the obtained shift  $\Delta a_{\text{exp}} \approx 0.12$  is about half the theoretical prediction. The discrepancy is not clear, and may be related to not sufficiently large/small values of  $\beta_1$  (we recall that the band size has to be large,  $b \gg 1$ , in order to expect a correct scaling).

The property of scaling for BRMs was derived analytically [15] under the assumption  $b \ll N$ . For the numerically accessible values of  $N$ , this condition is violated for large values of the parameter  $\lambda$ , and in Figure 1 one can observe deviations from the dependence (10) for  $\lambda \geq 3$  and an expected failure of scaling in this strongly delocalized regime.

In order to improve statistics, we collected eigenvectors in an energy window at the center of the spectrum including half of all eigenvectors since, by relation (2), they have similar localization lengths. For each chosen pair of parameters  $b$  and  $N$ , an additional average was performed over an ensemble of 20 – 500 matrices.

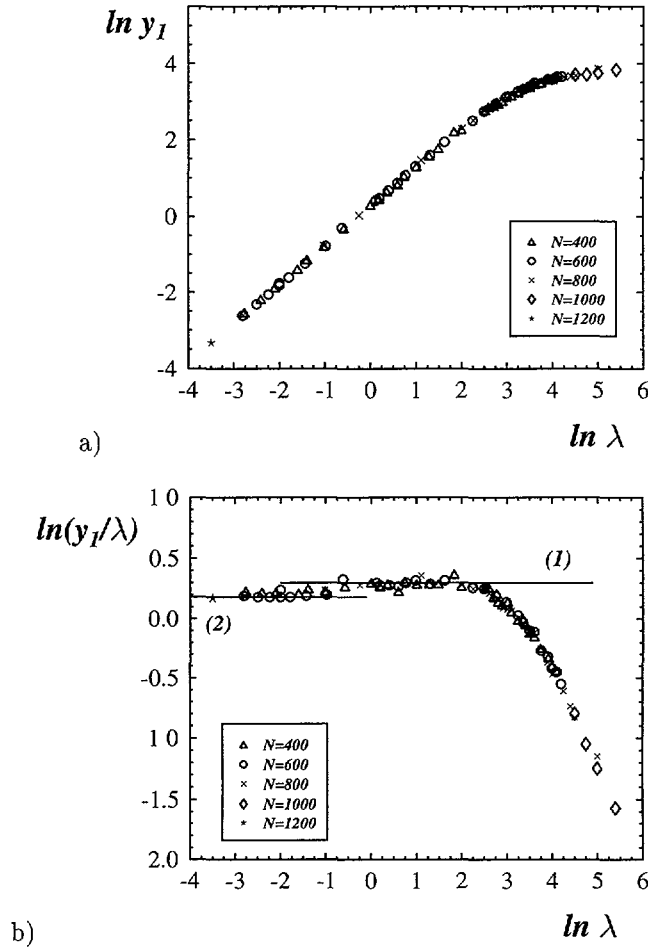
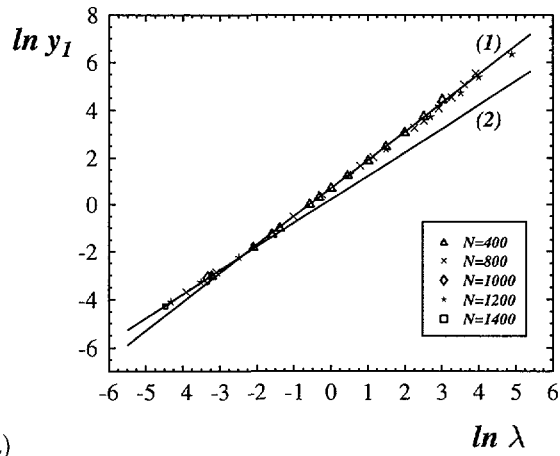


Fig. 1. — Scaling properties for standard BRM. a) Dependence of the entropy localization length  $l_N^{(1)}$  on the scaling parameter  $\lambda = b^2/N$ , using the variables in equation (10). b) The same data as in (a) in other variables, with evidence of the difference in values of  $a_1$ . The two straight lines are the fits (10); the upper line (1) gives  $a_1 = 0.30$  for the range  $0 \leq \ln \lambda \leq 2.5$  and the lower line (2) gives  $a_1 = 0.18$  for the range  $-3.0 \leq \ln \lambda \leq 0$ .

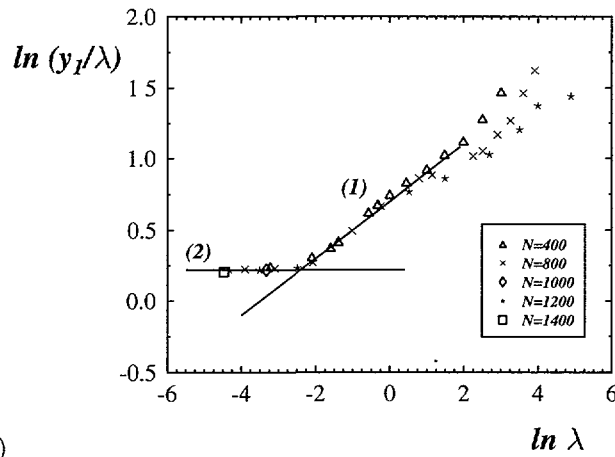
To study the influence of boundary conditions on eigenstates, we consider an ensemble of periodic BRMs. To have all eigenstates periodic,  $\varphi_{n+N} = \varphi_n$ , one adds non-zero matrix elements (which are assumed to be distributed in the same way as inside the band) to the lower left and upper right corners of band matrices in such a way that the number of non-zero elements in each row is constant and equal to  $2b - 1$  [30]. To obtain the same spectral radius, the Gaussian distribution of matrix elements has zero mean and variance

$$\langle H_{nm}^2 \rangle = \frac{(1 + \delta_{nm})}{2b - 1} \quad (11)$$

In this ensemble, the limit case of GOE is recovered for an odd value of  $N$  and  $b = (N + 1)/2$ . Due to the periodic structure of eigenstates, one can expect a different scaling behaviour of



a)



b)

Fig. 2. — As in Figure 1, for periodic BRMs, with fitting parameters  $d_1 = 0.22$ ,  $\alpha_1 = 1.0$  (line (1)) and  $d_1 = 0.70$ ,  $\alpha_1 = 1.2$  (line (2)).

the localization length  $l_N^{(1)}$ , which may be important in view of the application of PBRMs to quasi-1D disordered models with a magnetic flux through a ring [30,31].

Numerical data for the rescaled localization length  $l_N^{(1)}$  for the ensemble (11) of PBRMs are presented in Figure 2a-b. From these figures one can conclude that the scaling of the form (8) holds only in the region of strongly localized states ( $\ln \lambda \leq -2$ ). For larger values of the scaling parameter  $\lambda$ , the scaling has a different form: the fit of the data for  $\ln \lambda \geq -2$  gives the expression

$$\ln y_1 = \alpha_1 \ln \lambda + d_1, \quad (12)$$

with  $\alpha_1 \approx 1.2$  and  $d_1 \approx 0.70$ . In the variables  $\beta_1$ ,  $\lambda$  this leads to the law

$$\beta_1 = \frac{D\lambda^\alpha}{1 + D\lambda^\alpha}, \quad D = \exp(d_1), \quad (13)$$

which still waits for a theoretical explanation. For large values of  $\ln \lambda$ , the fluctuations are extremely large; this does not allow for any phenomenological expression.

In the above study, we used a definition of localization length based on information entropy (4). Another definition, often used in solid state applications, is based on the inverse participation ratio  $P_2^{-1}$ , where

$$P_2 = \sum_{n=1}^N |\varphi_n|^4 ; \quad (14)$$

By normalizing this localization length with reference to GOE, one introduces the quantity  $\beta_2$  [11, 15] :

$$\beta_2 \equiv \frac{l_N^{(2)}}{N} = \left( \frac{\langle P_2 \rangle}{P_2(\text{GOE})} \right)^{-1} = \frac{3/N}{\langle P_2 \rangle} \quad (15)$$

where  $P_2(\text{GOE})$  is the participation ratio for GOE random matrices and tends to  $N/3$  in the limit of large matrices.

In reference [15] the quantity  $\beta_2$  for standard BRM was found to scale in the same way as  $\beta_1$ :

$$\beta_2 = \frac{c_2 \lambda}{1 + c_2 \lambda} \quad (16)$$

or, in the variables  $y_2 \lambda$ ,

$$\ln y_2 = \alpha_2 \ln \lambda + d_2 , \quad y_2 = \frac{\beta_2}{1 - \beta_2}. \quad (17)$$

In contrast to  $\beta_1$ , the expression (16) is exact in the whole range of  $\lambda$ . Numerical data for the scaling of  $\beta_2$  are presented in Figure 3 for standard and periodic BRMs. Figure 3a shows a quite good correspondence to the predicted scaling (16) for standard BRMs; the fit gives the slope one and the value  $d_2 \equiv \ln c_2 \approx -0.15$ . One should remind that the parameter  $c_2$  depends on the shape of the envelope function for the variance of matrix elements (for general expressions see ([7])).

The main result of the numerical study of periodic BRMs is that the scaling of  $\beta_2$  seems to have the same law as for  $\beta_1$  (see Fig. 3b). Another important observation is that the fluctuations for the localization length  $l_N^{(2)}$  are much stronger compared to those for  $l_N^{(1)}$ . This fact does not allow to obtain a more accurate estimate for the factor  $\alpha$ ; the data show that, approximately,  $\alpha_2 \approx \alpha_1 \approx 1.2$ .

### 3. Statistical Properties of Eigenstates

It is well known that conductance of finite samples generically shows anomalous sample-to-sample fluctuations. In particular, in the metallic regime the so-called universal conductance fluctuations have been discovered, with the remarkable property that the ratio of the variance of conductance to its mean value does not depend on the size of the sample. In this regime the fluctuations of the conductance are Gaussian. In the other limit of strong localization, the fluctuations are of very specific form, namely, the log-normal distribution for the conductance holds if the localization length  $l_\infty$  is much less than the size of the sample. Since these properties of conductance are directly related to the structure of eigenstates, it is important to investigate the statistical properties of eigenstates.

A first study of the distribution of the entropy localization length in eigenvectors of standard BRM is described in reference [10]. In papers [16, 19] other characteristics of eigenstates are studied for standard BRMs both in the localized  $\lambda \ll 1$  and delocalized  $\lambda \gg 1$  regimes. One should note that, in the description as a quasi-1D disordered model, the size of the band  $b$  can be

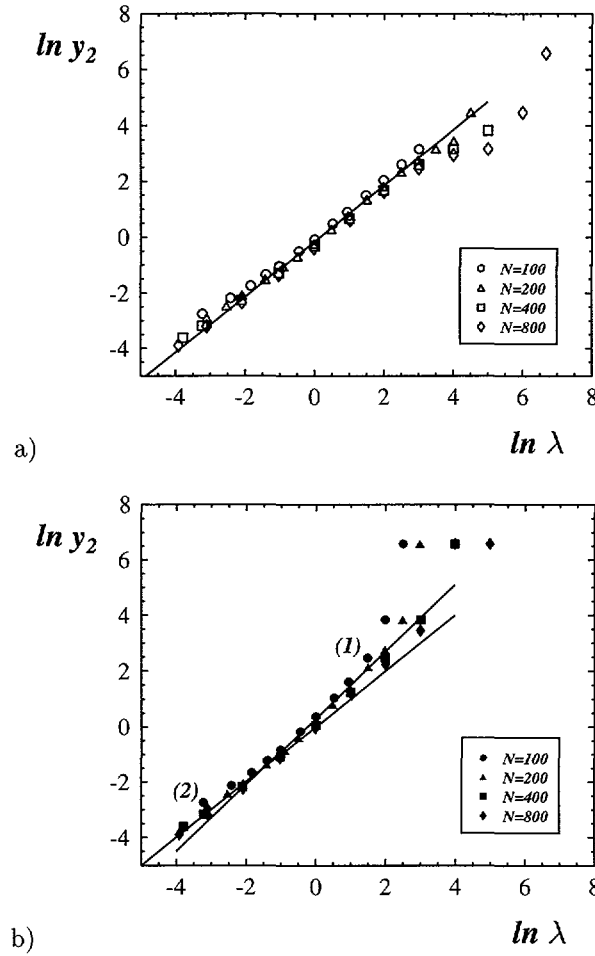


Fig. 3. — Scaling property for the localization length  $l_N^{(2)}$ , see (15), for standard and periodic BRMs: a) Standard BRMs, the straight line is the fit (17) with  $d_2 = -0.15$  and  $\alpha_2 = 1.0$ . b) Periodic BRMs, the lines (1,2) are given by  $d_2 = 0$ ,  $\alpha_2 = 1.0$  and  $d_2 = 0.3$ ,  $\alpha_2 = 1.2$ , correspondingly.

associated to the mean free path. Therefore, the metallic regime corresponds to the condition  $1 \ll \sqrt{N} \ll b \ll N$ . In what follows, we study fluctuations in the structure of eigenstates for standard and periodic BRMs, which may be relevant to the statistical properties of conductance in quasi-1D models (as well as in 1D-models with long range hoppings).

Since for standard BRMs the inverse participation ratio  $P_2$  has often been studied analytically in the literature, we investigated numerically fluctuations of  $P_2$  in different regimes. We start with a comparison of the IPR distributions  $W(P_2)$  for standard and periodic BRM. Figure 4a presents results obtained for both models for small values of the scaling parameter  $\lambda$ .

Differences between the distributions for the two models are clearly visible: fluctuations of  $P_2$  are smaller for periodic BRMs. Notice that the IPR distribution of standard BRM with  $b = 9$  ( $\lambda \approx 0.20$ ) corresponds to the data of PBRM obtained for  $b = 7$  (and therefore with  $\lambda \approx 0.12$ ).



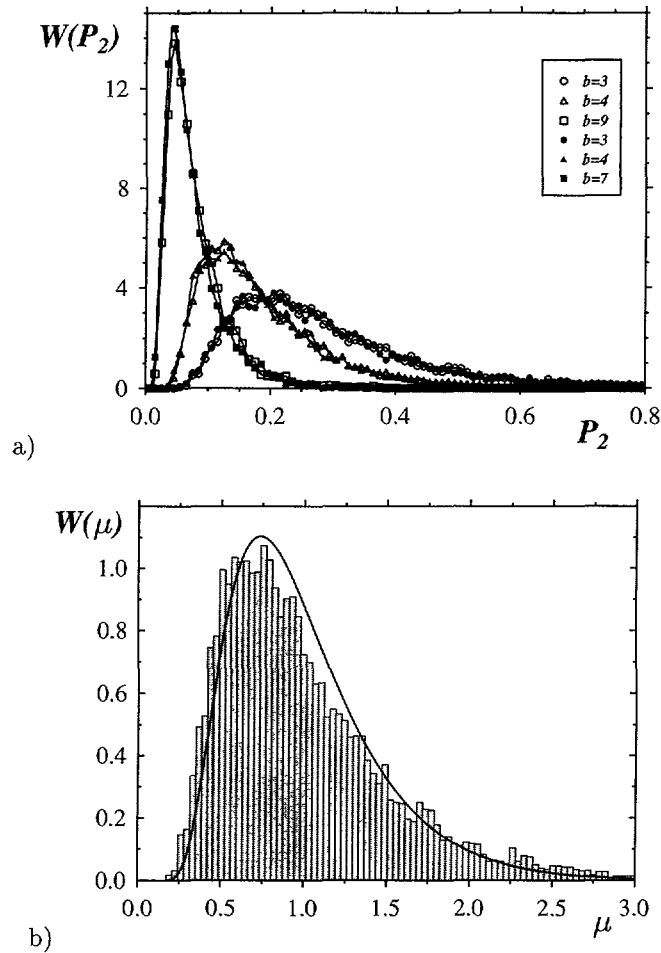


Fig. 4. — a) Distribution  $W(P_2)$  of the participation ratio  $P_2$  for  $N = 400$  and different values of  $b$ , for standard BRMs (open symbols) and for periodic ones (full symbols); b) Comparison of the scaled distribution of IPR  $W(\mu)$  obtained for standard BRM with  $b = 3$ ,  $N = 400$ , with the expression (18).

On the other hand, in the regime of strong localization  $\lambda \ll 1$  (in practice the band width  $b$  was set to 3 or to 4) the distribution  $W(P_2)$  does not differ for both ensembles. In this regime the probability distribution has the peculiar property of double scaling. First, it scales as other quantities characterizing band matrices, with the parameter  $\lambda = b^2/N$ . Second, in the case  $\lambda \ll 1$  the distribution becomes universal in the rescaled variable  $\mu = P_2/\langle P_2 \rangle$ . In Figure 4b a histogram of IPR in rescaled variables for standard BRM, with  $\lambda = 0.04$ , is shown. The solid line represents the distribution recently obtained by Fyodorov and Mirlin [19]:

$$W(\mu) = \sum_{k=1}^{\infty} \left( \frac{4}{9} k^4 \pi^4 \mu - 2k^2 \pi^2 \right) \exp \left( -\frac{\pi^2 k^2}{3} \mu \right). \tag{18}$$

Though it was derived for complex Hermitian BRMs, it also describes the case of real symmetric random matrices discussed in this work. The analytical form (18) gives a satisfactory

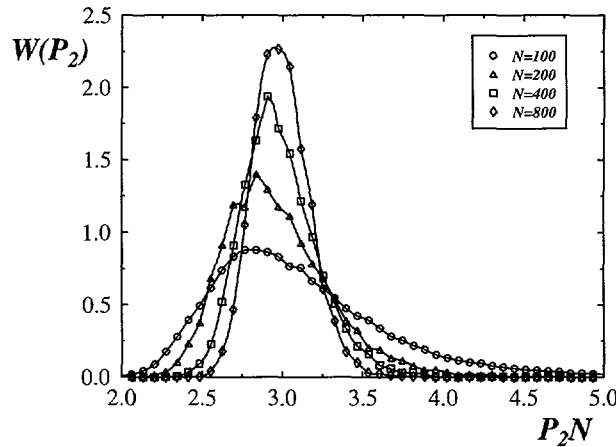


Fig. 5. — Distribution of the inverse participation ratio  $P_2$  for full random matrices ( $b = N$ ).

correspondence to the numerical data. Discrepancies may be explained by a not sufficiently large matrix size  $N$  since the analytical results were obtained [19] in the limit  $b, N \rightarrow \infty$  with  $\lambda = b^2/N \rightarrow 0$ .

With an increase of  $b$ , the distribution  $W(P_2)$  becomes very narrow around the mean value  $\langle P_2 \rangle \approx 3/N$ . Numerical data obtained in this case are presented in Figure 5. The width of the fluctuations of IPR around the mean decreases as  $1/N$  [19]. The finite size effect of asymmetry of the distribution  $W(P_2)$  is clearly visible.

A scaling property of the inverse participation ratio can be seen in the dependence of the mean square deviation  $\delta = (\langle P_2^2 \rangle - \langle P_2 \rangle^2)^{1/2}$  on the mean value  $\langle P_2 \rangle$ . In the theory of disordered solids, such a relation is important to characterize the type of fluctuations. In [16, 19] it was shown that, for non-symmetrical Hermitian BRMs and in the limit of strong localization, the ratio  $\delta/\langle P_2 \rangle$  approaches the constant value  $1/\sqrt{5}$ . The dependence of  $\delta$  on the mean value  $\langle P_2 \rangle$  for standard and periodic symmetric BRMs is given in Figure 6a. It was obtained by direct computations on matrices of types (1) and (11), with sizes  $N = 100, 200, 400, 800$  and different values of band size  $b$ . The scaling holds only in the very localized regime, for  $\langle P_2 \rangle \geq 0.1$ . Unexpectedly, for smaller values of the mean participation ratio ( $0.03 \leq \langle P_2 \rangle \leq 0.1$ ), there is no difference between standard and periodic BRMs. The ratio  $\delta/\langle P_2 \rangle$  being constant for localized states reflects extremely large fluctuations in the width of eigenstates (see also, [16, 19]); such strong fluctuations have also been found in the dynamical localization of eigenstates of the kicked rotator model, in momentum space [22].

The behaviour of the ratio  $\delta/\langle P_2 \rangle$  was studied analytically in [19]. In particular, it was shown that the convergence to the limit value  $1/\sqrt{5}$  is not monotonic and very slow. These peculiarities are reflected in Figure 6b where the same data are plotted in other variables. Only for extremely large values  $\lambda^{-1} \geq 20$ , the ratio  $\delta/\langle P_2 \rangle$  approaches the limit value (again, for small values of  $b$  there is clear deviation). A comparison of the ratio  $\delta/\langle P_2 \rangle$  for standard and periodic BRMs shows that in both limits of strongly localized or extended states, the differences appear to be small. The distribution of IPR is different in the two models in the intermediate case where the localization length is of the order of the sample size (see Fig. 6b).

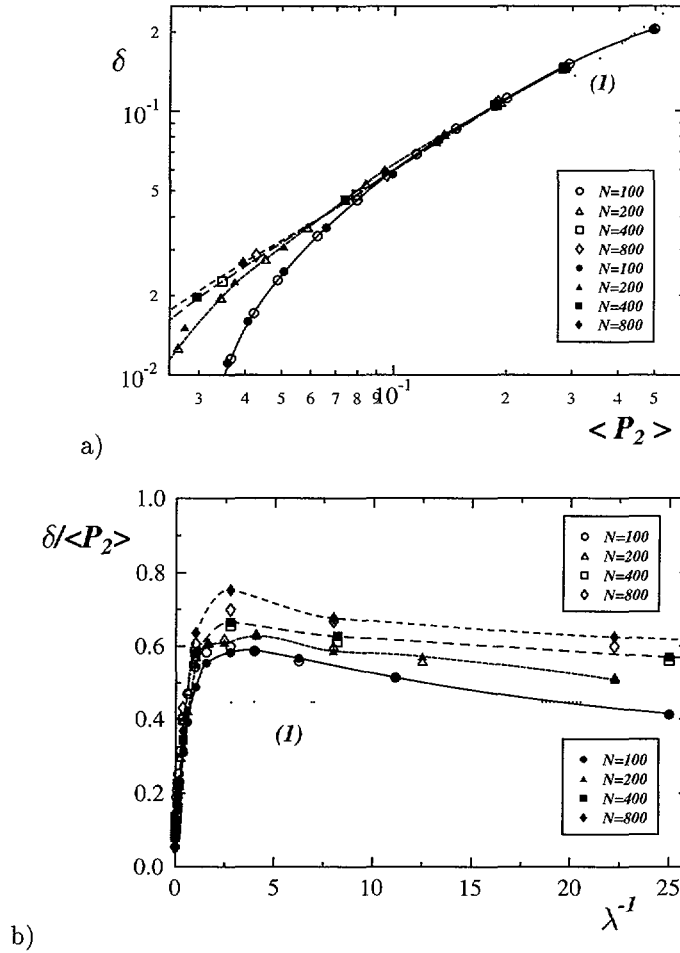


Fig. 6. — a) Mean deviation  $\delta$  versus mean value  $\langle P_2 \rangle$  of the inverse participation ratio. Open symbols correspond to standard BRMs, full symbols the periodic ones. The dotted straight line (1) is the theoretical limit  $\delta = \frac{1}{\sqrt{5}} \langle P_2 \rangle$  for very localized states. b) Dependence of the ratio  $\delta / \langle P_2 \rangle$  on the scaling parameter  $\lambda^{-1}$  extracted from the same data.

This indicates that the large fluctuations of the inverse participation ratio for standard BRMs (read, fluctuations of the localization length) are increased by the eigenstates localized in the vicinity of the edges (see discussion in [19]).

In order to study the difference in the structure of standard and periodic BRMs in more detail, we introduce the concept of barycenter  $B$  of a normalized eigenstate of components  $\varphi_n$  in a given basis

$$B = \rho \exp(i\phi) = \sum_{n=1}^N |\varphi_n|^2 \exp\left(i \frac{2\pi n}{N}\right). \tag{19}$$

It is a complex number in the unit circle. Its radius  $\rho$  is related to the localization length: for strongly localized eigenstates it is of order 1, and for extended states it is very small, of order  $1/N$ . The phase  $\phi$  can be associated with the position of the center of the eigenstate in the

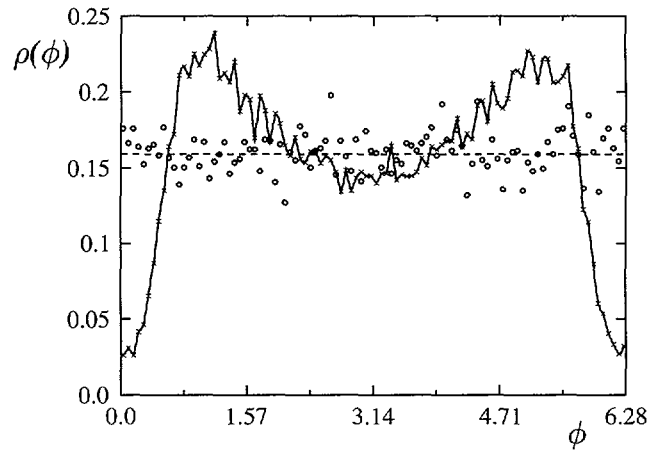


Fig. 7. — Dependence  $\rho(\phi)$  for standard BRM (crosses and broken line) and periodic BRM (open circles) with  $b = 10$ ,  $N = 100$ . The dashed line represents the mean value of  $\rho$ .

given basis. Therefore the barycenter contains information both about localization length and location of eigenstates. This parameter is convenient for the description of global properties of periodic eigenstates.

In Figure 7 the dependence  $\rho(\phi)$  is shown for the intermediate case of moderate value  $\lambda = 1$ , for standard and periodic BRMs of size  $N = 100$  and band size  $b = 10$ . In particular, one can see how the localization length (vertical axis) of an eigenstate depends on the position of the center of localization (horizontal axis). The data show that for standard BRM there is a strong decrease of localization length for states which are localized close to the edge. Contrary, for periodic BRM the centers of localization are uniformly distributed.

As above indicated, the quantity  $\rho$  may be treated as an alternative definition of localization length. In Figure 8a one can see how the distribution  $P(\rho)$  depends on the scaling parameter  $\lambda$  for standard BRMs of size  $N = 400$ . In particular, the most remarkable property of eigenstates, namely, extremely large fluctuations of the localization length is well reflected. The distribution  $P(\rho)$  turns out to be useful for the comparison of standard and periodic BRMs. The main difference in the structure of eigenstates can be easily extracted from Figure 8b where the distribution  $P(\rho)$  is given for standard and periodic matrices with same values of  $b$  and  $N$  (compare with Fig. 7). The two representations (Figs. 6 and 8b) together provide relevant comparative information about the structure of eigenstates of the same matrix.

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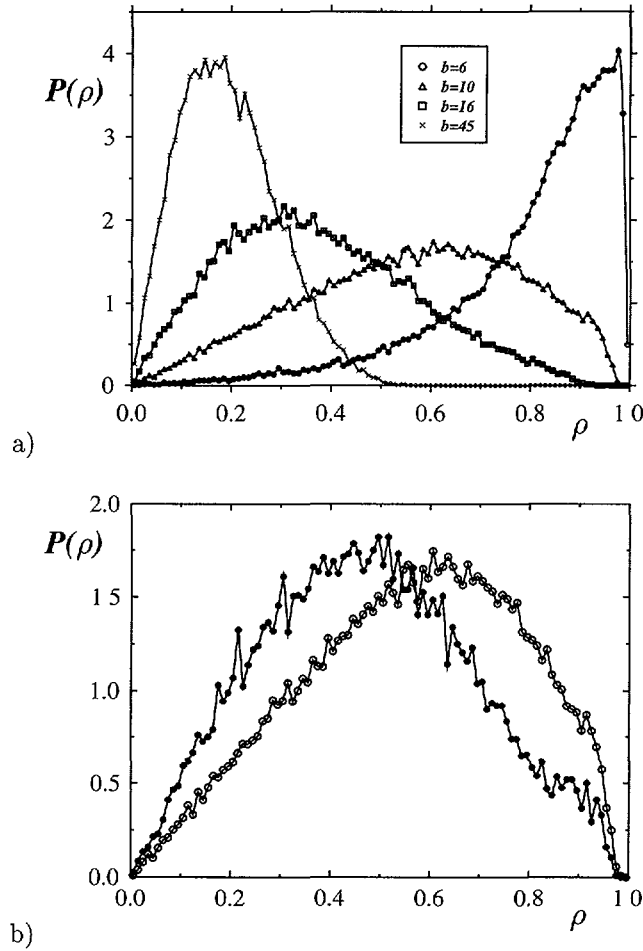


Fig. 8. — a) Distribution  $P(\rho)$  for standard BRMs of size  $N = 100$  and different band sizes  $b$ . The change in the distribution is clearly seen when passing from localized ( $\lambda = b^2/N \ll 1$ ) to delocalized ( $\lambda \gg 1$ ) eigenstates. b) Comparison of the distribution  $P(\rho)$  for standard (open circles) and periodic (full circles) BRMs; all parameters as in Figure 7.

## References

- [1] Seligman T.H., Verbaarschot J.J.M. and Zirnbauer M.R., *Phys. Rev. Lett.* **53** (1985) 215.
- [2] Feingold M., Leitner D.M. and Piro O., *Phys. Rev. A* **39** (1989) 6507.
- [3] Cheon T., *Phys. Rev. Lett.* **65** (1990) 529.
- [4] Casati G., Molinari L. and Izrailev F.M., *Phys. Rev. Lett.* **64** (1990) 1851.
- [5] Kuś M., Lewenstein M. and Haake F., *Phys. Rev. A* **44** (1991) 2800; Casati G. and Girko V.L., *Rand. Oper. Stoch. Eqs.* **1** (1992) 1; Feingold M., *Europhys. Lett.* **17** (1992) 97; Molchanov S.A., Pastur L.A. and Khorunzhy A.M., *Teor. Mat. Fiz.* **90** (1992) 163.
- [6] Casati G., Izrailev F.M. and Molinari L., *J. Phys. A* **24** (1991) 4755.
- [7] Fyodorov Y.V. and Mirlin A.D., *Phys. Rev. Lett.* **67** (1991) 2405.
- [8] Wilkinson M., Feingold M. and Leitner D.M., *J. Phys. A* **24** (1991) 175.

- [9] Feingold M., Leitner D. and Wilkinson M., *Phys. Rev. Lett.* **66** (1991) 986.
- [10] Życzkowski K., Lewenstein M., Kuś M. and Izrailev F.M., *Phys. Rev. A* **44** (1991) 73.
- [11] Evangelou S.N. and Economou E.N., *Phys. Lett. A* **151** (1990) 345.
- [12] Gioletta A., Feingold M., Izrailev F.M. and Molinari L., *Phys. Rev. Lett.* **70** (1993) 2936.
- [13] Fyodorov Y.V. and Mirlin A.D., *Phys. Rev. Lett.* **69** (1992) 1093.
- [14] Casati G., Chirikov B.V., Guarneri I. and Izrailev F.M., *Phys. Rev. E* **48** (1992) R1613.
- [15] Mirlin A.D. and Fyodorov Y.V., *J. Phys. A* **26** (1993) L551.
- [16] Fyodorov Y.V. and Mirlin A.D., *Phys. Rev. Lett.* **71** (1993) 412.
- [17] Feingold M., in "Quantum Chaos – Quantum Measurement", P. Cvitanovic, I. Percival and A. Wirzba Eds. (Kluwer, Dordrecht, 1992) p.167.
- [18] Izrailev F.M., in "Quantum Chaos – Quantum Measurement", P. Cvitanovic, I. Percival and A. Wirzba Eds. (Kluwer, Dordrecht, 1992) p. 89.
- [19] Fyodorov Y.V. and Mirlin A.D., *Int. J. Mod. Phys. B* **8** (1994) 3795.
- [20] Bohigas O. and Giannoni M.-J., *Lect. Notes Phys.* **209** (1984) 1.
- [21] Chirikov B.V., Izrailev F.M. and Shepelyansky D.L., *Physica D* **33** (1988) 77.
- [22] Izrailev F.M., *Phys. Rep.* **196** (1990) 299.
- [23] Haake F., *Quantum Signatures of Chaos* (Springer, Berlin, 1991).
- [24] Chirikov B.V., *Phys. Lett. A* **108** (1985) 68.
- [25] Flambaum V.V., Gribakina A.A., Gribakin G.F. and Kozlov M.G., *Phys. Rev. A* **50** (1994) 267.
- [26] Zelevinsky V., Horoi M. and Brown B.A., Preprint MSUCL-942 (1994); Horoi M., Brown B.A. and Zelevinsky V., Preprint MSUCL-953 (1994).
- [27] Wigner E., *Ann. Math.* **62** (1955) 548; **65** (1957) 203.
- [28] Prozen T. and Robnik M., *J. Phys. A* **26** (1994) 1105.
- [29] *Statistical Theories of Spectra: Fluctuations*, C.E. Porter Ed. (Academic Press, 1965).
- [30] Casati G., Guarneri I., Izrailev F.M., Molinari L. and Życzkowski K., *Phys. Rev. Lett.* **72** (1994) 2697.
- [31] Guarneri I., Życzkowski K., Zakrzewski J., Molinari L. and Casati G., *Phys. Rev. E* **52** (1995) 2220.
- [32] Życzkowski K. and Serwicki R., *Z. Phys. B* (1995) in press.
- [33] Izrailev F.M., *J. Phys. A* **22** (1989) 865.