I. INTRODUCTION

In the appendix of their paper “Bound States in Quantum Field Theory,” Murray Gell-Mann and Francis Low\(^1\) proved a fundamental relation that bridges the ground states \(|\Psi_0\rangle\) and \(|\Psi\rangle\) of Hamiltonians \(H_0\) and \(H=H_0+gV\) by means of time propagators, and makes the transition of time-ordered correlators from the Heisenberg to the interaction picture possible:

\[
\langle \Psi | T\psi(1)\cdots\psi(n) | \Psi \rangle = \frac{\langle \Psi_0 | TS\psi(1)\cdots\psi(n) | \Psi_0 \rangle}{\langle \Psi_0 | S | \Psi_0 \rangle}.
\]

The single operator \(S=U_\epsilon(\infty, -\infty)\) contains all the effects of the interaction. The theorem borrows ideas from the scattering and the adiabatic theories and makes use of the concept of adiabatic switching of the interaction\(^2\) through the time-dependent operator

\[
H_\epsilon(t) = H_0 + e^{-i\epsilon t}gV
\]

that interpolates between the operators of interest, \(H\) at \(t=0\) and \(H_0\) at \(|t| \to \infty\). The adiabatic limit is obtained for \(\epsilon \to 0^+\). With the operator \(H_0\) singled out, the theorem requires the time propagator in the interaction picture,

\[
U_\epsilon(t,s) = e^{i\epsilon H_0 t}U(t,s)e^{-i\epsilon H_0 s},
\]

where \(U(t,s)\) is the full propagator.\(^3\) The statement of Gell-Mann and Low’s theorem is as follows.

*Theorem:* Let \(|\Psi_0\rangle\) be an eigenstate of \(H_0\) with eigenvalue \(E_0\), and consider the vectors

\[
|\Psi_\epsilon^{(\pm)}\rangle = \frac{U_\epsilon(0, \pm \infty)|\Psi_0\rangle}{\langle \Psi_0 | U_\epsilon(0, \pm \infty)|\Psi_0\rangle}
\]

If the limit vectors \(|\Psi_\epsilon^{(\pm)}\rangle\) for \(\epsilon \to 0^+\) exist, then they are eigenstates of \(H\).

The theorem is used to represent the ground state of an interacting system starting from a noninteracting one. For a time-dependent Hamiltonian, the eigenvalues evolve parametrically in time: if they do not cross and are not degenerate, eigenvectors can be traced univocally. According to adiabatic theory, the parametric evolution of eigenvectors is provided by time propagation and multiplication by a phase factor.\(^4\)\(^5\) Then Gell-Mann and Low’s theorem can be regarded as a
statement concerning asymptotic states where the phase factor is properly dealt with. Adiabatic evolution of degenerate states\(^9\) or with more general switching functions\(^7\) has been considered.

In many-body theory the adiabatic switch of the interaction is smooth for Fermi liquids and takes free fermions into renormalized quasiparticles. It fails when symmetry changes: these systems require appropriate tools. In nonequilibrium theory the interaction is switched on in the past only, and time ordering is defined along a time loop beginning and ending in the past.\(^8\) High energy physics emphasizes a scattering picture based on Lippmann-Schwinger equation.\(^7\) The covariant realization of the adiabatic switch of the interaction in Lagrangian formalism was achieved by Bogoliubov and Shirkov.\(^9\) The adiabatic switch is a tool to study interaction of quantum particles with time-periodic external fields \(gV(t+T)=gV(t)\), with \(\epsilon T \ll 1.\)\(^10,11\) The analytic properties in \(g\) of the quasienergy states become intricate as the size of the Hilbert space increases and avoided crossings coalesce.\(^12\) The property that adiabatic evolution takes the eigenspaces of \(H_0\) into eigenspaces of \(H\) is used in quantum field theory (QFT) to construct effective Hamiltonians for bound states in restricted Hilbert space.\(^13\)

Despite the validity of the theorem beyond perturbation theory, in the original paper\(^1\) and in textbooks\(^14-16\) the proof makes use of Dyson’s expansion of the interaction propagator, and is rather cumbersome. An elegant mathematical proof based on it was given by Hepp,\(^17\) for the case where \(H_0\) describes free particles and the interaction \(V\) is norm bounded. This ensures strong convergence of the Dyson series for the propagators \(U(t,0)\), as discussed by Lanford.\(^18\) Other mathematical proofs are based on versions of the adiabatic theorem.\(^19\) They generally apply to a portion of the spectrum of \(H(t)\) isolated from the rest at any time, but this gap condition can be relaxed.\(^20\)

In this paper a simple equation for the propagator is derived, without use of Dyson’s expansion. The equation can be used as intermediate nonperturbative result in the standard proof of Gell-Mann and Low’s formula given in textbooks. This is described in the conclusion, where a short derivation of Sucher’s formula is also given.

II. AN EQUATION FOR THE PROPAGATOR

**Lemma:** If \(U(t,s)\) is the time propagator for \(H(t)\) then, for all positive \(e\), the following relations hold:

\[
\begin{align*}
\frac{i\hbar}{\epsilon} g \frac{\partial}{\partial g} U(t,s) &= H(t) U(t,s) - U(t,s) H(s) \quad \text{if} \quad 0 \leq t \leq s, \\
&= - H(t) U(t,s) + U(t,s) H(s) \quad \text{if} \quad t \geq s \geq 0. 
\end{align*}
\]

**Proof:** The trick is to make the \(g\)-dependence of the propagator explicit into the time dependence of some related propagator. Schrödinger’s equation

\[
\frac{i\hbar}{\epsilon} \partial_s U(t,s) = H(t) U(t,s), \quad U(s,s) = 1
\]

corresponds to the integral one, where we put \(g = e^{\epsilon t}\):

\[
U(t,s) = I + \frac{1}{i \hbar} \int_s^t dt' (H_0 + e^{\epsilon (t-t')} V) U(t',s).
\]

Consider the \(g\)-independent operators \(H(t) = H_0 + e^{\epsilon t} V\), with corresponding propagators \(U(t,s)\). For \(0 \leq t \leq s\), a time translation in Eq. (8) gives

\[
U(t,s) = I + \frac{1}{i \hbar} \int_{s+\theta}^{t+\theta} dt' H(t') U(t' - \theta,s).
\]

Comparison with the equation for \(U(t+\theta,s+\theta)\)
\[ U^{(+)}(t+\theta,s+\theta) = I + \frac{1}{i\hbar} \int_{s+\theta}^{t+\theta} dt' H^{(+)}(t')U^{(+)}(t',s+\theta) \]

and unicity of the solution imply the identification

\[ U_d(t,s) = U^{(+)}(t+\theta,s+\theta). \]  

Since \( \theta \) enters in the operator \( U^{(+)}(t+\theta,s+\theta) \) only in its temporal variables, we obtain

\[ \partial_\theta U_d(t,s) = \partial_\theta U_d(t,s) + \partial_s U_d(t,s). \]  

By using Eq. (7) and its adjoint, the first identity is proven.

If \( t \geq s \geq 0 \), the same procedure gives \( U_d(t,s) = U^{(+)}(t-\theta,s-\theta) \) and therefore \( \partial_\theta U_d(t,s) = -\partial_s U_d(t,s) - \partial_\theta U_d(t,s) \), which leads to the identity (6). An identity for \( t \geq 0 \geq s \) can be obtained by writing \( U_d(t,s) = U_d(t,0)U_d(0,s) \).

In the interaction picture, Eq. (3), the identities transform straightforwardly into the following ones:

\[ i\hbar \varepsilon g \frac{\partial}{\partial \varepsilon} U_{\varepsilon}(t,s) = H_{\varepsilon}(t)U_{\varepsilon}(t,s) - U_{\varepsilon}(t,s)H_{\varepsilon}(s) \quad \text{if} \quad 0 \geq t \geq s, \]

\[ = -H_{\varepsilon}(t)U_{\varepsilon}(t,s) + U_{\varepsilon}(t,s)H_{\varepsilon}(s) \quad \text{if} \quad t \geq s \geq 0, \]

where \( H_{\varepsilon}(t) = e^{i\varepsilon \hbar t}H(t)e^{-i\varepsilon \hbar t} \).

By applying Eqs. (12) with \( s = -\infty \) or \( t = \infty \) to an eigenstate \( |\Psi_0\rangle \) of \( H_0 \), we obtain

\[ \left( H - E_0 \pm i\hbar \varepsilon g \frac{\partial}{\partial \varepsilon} \right) U_{\varepsilon}(0, \pm \infty)|\Psi_0\rangle = 0. \]

This same equation is proven in the literature by direct use of Dyson’s expansion. From now on, the proof of Gell-Mann and Low’s theorem proceeds in the standard path, and is sketched for completeness in the next section.

III. CONCLUSION

The mathematical properties of the operators \( U_{\varepsilon}(0, \pm \infty) \) were studied first by Dollard\(^{21}\) for the case \( H_0 = -\Delta_x \) and square integrable or locally square integrable and asymptotically bounded potential \( V(\vec{x}) \), and extended to the many-particle Schrödinger equation. He showed that the operators are unitary and the Hamiltonians \( H(t) \) do not have proper eigenstates. In the adiabatic limit, under further restrictions on the potential, they yield isometric Möller operators \( \Omega = \lim_{\varepsilon \to 0} U_{\varepsilon}(0, \pm \infty) \). The intertwining property \( \Omega^* = \Omega^* H_0 \) implies that for scattering states the \( g \)-derivative term in Eq. (13) is zero. The emergence of a bound state from the adiabatic evolution of the unbounded states of \( H_0 \) was investigated by Suura et al.\(^{22}\) Through the study of the potential \( V(x) = -\Delta(x) \), that allows for a single bound state, they conjectured that bound states are associated with nonanalytic behavior in \( \varepsilon \) of the Dyson series for \( U_{\varepsilon}(0, \pm \infty)|\Psi_0\rangle \) when \( E_0 < \varepsilon \). A bound state requires a nontrivial adiabatic limit of Eq. (13) where the vector \( U_{\varepsilon}(0, \pm \infty)|\Psi_0\rangle \) develops a phase proportional to \( 1/\varepsilon \); this has been checked in diagrammatic expansion.\(^{23,24}\) The singular phase is responsible of the energy shift and is precisely removed by the denominator in the definition of the vectors \( |\Psi^{(a)}_\varepsilon\rangle \), before the limit is taken.

The standard steps of the proof are as follows.

1. For finite \( \varepsilon \), the two identities, Eq. (13), are projected on the vector \( |\Psi_0\rangle \), and yield a formula for the energy shift, where

\[ E^{(a)}_\varepsilon = \langle \Psi_0 | H | \Psi^{(a)}_\varepsilon \rangle, \]

\[ = i\hbar \varepsilon g \frac{\partial}{\partial \varepsilon} \log(\langle \Psi_0 | U_{\varepsilon}(0, \pm \infty)|\Psi_0\rangle) = E^{(a)}_\varepsilon - E_0. \]
(2) By eliminating $E_0$ in Eq. (13) with the aid of Eq. (14), with simple steps one obtains

$$\left( H - E_0^{(\pm)} \pm i\hbar \epsilon \frac{\partial}{\partial g} \right) |\Psi_0^{(\pm)}\rangle = 0.$$  \hfill (15)

The adiabatic limit $\epsilon \rightarrow 0^+$ is now taken, and the limit vectors $|\Psi^{(\pm)}\rangle$ obtained by pulling onward or backward in time the same asymptotic eigenstate $|\Psi_0\rangle$ are eigenvectors of $H = H_0 + gV$ with eigenvalues $E^{(\pm)}$.

(3) The time-reversal operator has the action $T^\dagger U_s(t,s)T = U_s(-t,-s)$. If $H_0$ commutes with $T$ the relation extends to the interaction propagator and $T^\dagger U_{el}(0,\infty)T = U_{el}(0,-\infty)$. If $|\Psi_0\rangle$ is also an eigenstate of $T$, it follows that $T^\dagger |\Psi^{(\pm)}\rangle$ is parallel to $|\Psi^{(\mp)}\rangle$ and $E^{(\pm)} = E^{(\mp)}$. The proportionality factor equals 1, since $(E_0 |\Psi^{(\pm)}\rangle = (E_0 |\Psi^{(\mp)}\rangle$.

The formula for the energy shift, Eq. (14), can be recast in a form involving the $S$-operator. From Eqs. (12), the following relation follows:

$$- i\hbar \epsilon g \frac{\partial S}{\partial g} = H_0S_e + S_eH_0 - 2U_{el}(\infty,0)HU_{el}(0,-\infty).$$

The expectation value on the eigenstate $|\Psi_0\rangle$ and use of the theorem give Sucher’s formula$^{25}$

$$E - E_0 = \lim_{\epsilon \rightarrow 0} \frac{i\hbar \epsilon}{2} g \frac{\partial}{\partial g} \log \langle \Psi_0 | U_{el}(\infty, -\infty) | \Psi_0 \rangle.$$  \hfill (16)

$^1$M. Gell-Mann and F. Low, Phys. Rev. 84, 350 (1951).


$^3$M. Reed and B. Simon, Methods of Mathematical Physics (Academic, New York, 1975), Vol. II.


