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WEAKLY Z-SYMMETRIC MANIFOLDS

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Abstract. We introduce a new kind of Riemannian manifold that includes weakly-, pseudo- and pseudo projective Ricci symmetric manifolds. The manifold is defined through a generalization of the so called Z tensor; it is named *weakly Z-symmetric* and is denoted by $(\mathrm{WZS})_n$. If the Z tensor is singular we give conditions for the existence of a proper concircular vector. For non singular Z tensors, we study the closedness property of the associated covectors and give sufficient conditions for the existence of a proper concircular vector in the conformally harmonic case, and the general form of the Ricci tensor. For conformally flat $(\mathrm{WZS})_n$ manifolds, we derive the local form of the metric tensor.

1. Introduction

Tamássy and Binh [31] introduced and studied a Riemannian manifold whose Ricci tensor satisfies the equation

$$\nabla_k R_{jl} = A_k R_{jl} + B_j R_{kl} + D_l R_{kj}. \tag{1}$$

The manifold is called *weakly Ricci symmetric* and is denoted by $(\mathrm{WRS})_n$. The Ricci tensor and the scalar curvature are $R_{kl} = -R_{mkl}{}^m$ and $R = g^{ij} R_{ij}$. ∇_k is the covariant derivative with reference to the metric g_{kl}. We also put $\|\eta\| = \sqrt{\eta^k \eta_k}$. The covectors A_k, B_k and D_k are the *associated* 1*-forms.* The same manifold with the 1-form A_k replaced by $2A_k$ was studied by Chaki and Koley [6], and called *generalized pseudo Ricci symmetric.* The two structures extend *pseudo Ricci symmetric* manifolds, $(\mathrm{PRS})_n$, introduced by Chaki [4], where $\nabla_k R_{jl} = 2A_k R_{jl} + A_j R_{kl} + A_l R_{kj}$ (this definition differs from that of R. Deszcz [17]).

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Later on, other authors studied the manifolds [10,12,20]; in [12] some global properties of $(\mathrm{WRS})_n$ were obtained, and the form of the Ricci tensor was found. In [10] generalized pseudo Ricci symmetric manifolds were considered, where the conformal curvature tensor

$$C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl} + R_j{}^m g_{kl} - R_k{}^m g_{jl}) - \frac{R}{(n-1)(n-2)}(\delta_j{}^m g_{kl} - \delta_k{}^m g_{jl}) \tag{2}$$

vanishes (for $n=3$, $C_{jkl}{}^m = 0$ holds identically [27]) and the existence of a proper concircular vector was proven. In [20] a quasi conformally flat $(\mathrm{WRS})_n$ was studied, and again the existence of a proper concircular vector was proven.

In [2] $(\mathrm{PRS})_n$ with harmonic curvature tensor (i.e. $\nabla_m R_{jkl}{}^m = 0$) or with harmonic conformal curvature tensor (i.e. $\nabla_m C_{jkl}{}^m = 0$) were considered.

Chaki and Saha [18] considered the projective Ricci tensor P_{kl}, obtained by a contraction of the projective curvature tensor $P_{jkl}{}^m$:

$$P_{kl} = \frac{n}{n-1}\left(R_{kl} - \frac{R}{n} g_{kl}\right), \tag{3}$$

and generalized $(\mathrm{PRS})_n$ to manifolds such that

$$\nabla_k P_{jl} = 2A_k P_{jl} + A_j P_{kl} + A_l P_{kj}. \tag{4}$$

The manifold is called *pseudo projective Ricci symmetric* and is denoted by $(\mathrm{PWRS})_n$ [8]. Recently another generalization of a $(\mathrm{PRS})_n$ was considered in [5] and [11], whose Ricci tensor satisfies the condition

$$\nabla_k R_{jl} = (A_k + B_k) R_{jl} + A_j R_{kl} + A_l R_{kj}. \tag{5}$$

The manifold is called *almost pseudo Ricci symmetric* and is denoted by $\mathrm{A}(\mathrm{PRS})_n$. In [11] the properties of conformally flat $\mathrm{A}(\mathrm{PRS})_n$ were studied, pointing out their importance in the theory of general relativity.

It seems worthwhile to introduce and study a new manifold structure that includes $(\mathrm{WRS})_n$, $(\mathrm{PRS})_n$ and $(\mathrm{PWRS})_n$ as special cases.

Definition 1.1. A $(0,2)$ symmetric tensor is a *generalized Z tensor* if

$$Z_{kl} = R_{kl} + \phi g_{kl}, \tag{6}$$

where ϕ is an arbitrary scalar function. The Z scalar is $Z = g^{kl} Z_{kl} = R + n\phi$.

The classical Z tensor is obtained with the choice $\phi = -\frac{1}{n}R$. Hereafter we refer to the generalized Z tensor simply as the Z tensor.

The Z tensor allows us to reinterpret several well known structures on Riemannian manifolds.

1) If $Z_{kl} = 0$ the (Z-flat) manifold is an Einstein space, $R_{ij} = (R/n)g_{ij}$ [3].

2) If $\nabla_i Z_{kl} = \lambda_i Z_{kl}$, the ($Z$-recurrent) manifold is a generalized Ricci recurrent manifold [9,26]: the condition is equivalent to $\nabla_i R_{kl} = \lambda_i R_{kl} + (n-1)\mu_i g_{kl}$ where $(n-1)\mu_i \equiv (\lambda_i - \nabla_i)\phi$. If moreover $0 = (\lambda_i - \nabla_i)\phi$, a Ricci recurrent manifold is recovered.

3) If $\nabla_k Z_{jl} = \nabla_j Z_{kl}$ (i.e. Z is a Codazzi tensor, [16]) then $\nabla_k R_{jl} - \nabla_j R_{kl} = (g_{kl}\nabla_j - g_{jl}\nabla_k)\phi$. By transvecting with g^{jl} we get $\nabla_k [R + 2(n-1)\phi] = 0$ and, finally,

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R.$$

This condition defines a *nearly conformally symmetric* manifold, $(\mathrm{NCS})_n$. The condition was introduced and studied by Roter [29]. Conversely a $(\mathrm{NCS})_n$ has a Codazzi Z tensor if $\nabla_k [R + 2(n-1)\phi] = 0$.

4) Einstein's equations [14] with cosmological constant Λ and energy-stress tensor T_{kl} may be written as $Z_{kl} = kT_{kl}$, where $\phi = -\frac{1}{2}R + \Lambda$, and k is the gravitational constant. The Z tensor may be thought of as a generalized Einstein gravitational tensor with arbitrary scalar function ϕ.

Conditions on the energy-momentum tensor determine constraints on the Z tensor: the vacuum solution $Z = 0$ determines an Einstein space with $\Lambda = \frac{n-2}{2n}R$; conservation of total energy-momentum ($\nabla^l T_{kl} = 0$) gives $\nabla^l Z_{kl} = 0$ and $\nabla_k(\frac{1}{2}R + \phi) = 0$; the condition $\nabla_i Z_{kl} = 0$ describes a space-time with conserved energy-momentum density.

Several cases accommodate in a new kind of Riemannian manifold:

DEFINITION 1.2. A manifold is *weakly Z-symmetric,* and is denoted by $(\mathrm{WZS})_n$ if the generalized Z tensor satisfies the condition

$$\nabla_k Z_{jl} = A_k Z_{jl} + B_j Z_{kl} + D_l Z_{kj}. \tag{7}$$

If $\phi = 0$, we recover a $(\mathrm{WRS})_n$ and its particular case $(\mathrm{PRS})_n$. If $\phi = -R/n$ (classical Z tensor) and if A_k is replaced by $2A_k$, $B_k = D_k = A_k$, then $Z_{jl} = \frac{n-1}{n}P_{jl}$ and the space reduces to a $(\mathrm{PWRS})_n$.

Different properties follow from the Z tensor being singular or not. Z is singular if the matrix equation $Z_{ij}u^j = 0$ admits (locally) nontrivial solutions, i.e. Z cannot be inverted.

In Section 2 we obtain general properties of $(\mathrm{WZS})_n$ that descend directly from the definition and strongly depend on Z_{ij} being singular or not. The two cases are examined in Sections 3 and 4. In Section 3 we study $(\mathrm{WZS})_n$

that are conformally or quasi conformally harmonic with $B - D \neq 0$; we show that $B - D$, after normalization, is a proper concircular vector. Section 4 is devoted to $(\mathrm{WZS})_n$ with non-singular Z tensor, and gives conditions for the closedness of the 1-form $A - B$ that involve various generalized curvature tensors. In Section 5 we study conformally harmonic $(\mathrm{WZS})_n$ and obtain the explicit form of the Ricci tensor. In the conformally flat case we also give the local form of the metric.

2. General properties

From the definition of a $(\mathrm{WZS})_n$ and its symmetries we obtain

$$0 = \eta_j Z_{kl} - \eta_l Z_{kj}, \tag{8}$$

$$\nabla_k Z_{jl} - \nabla_j Z_{kl} = \omega_k Z_{jl} - \omega_j Z_{kl}, \tag{9}$$

with covectors

$$\eta = B - D, \qquad \omega = A - B \tag{10}$$

that will be used throughout.

Let us consider (8) first. It implies the following statements:

Proposition 2.1. *In a* $(\mathrm{WZS})_n$, *if the* Z *tensor is non-singular then* $\eta_k = 0$.

Proof. If the Z tensor is non singular, there exists a $(2,0)$ tensor Z^{-1} such that $\left(Z^{-1}\right)^{kh} Z_{kl} = \delta^h{}_l$. By transvecting (8) with $\left(Z^{-1}\right)^{kh}$ we obtain $\eta_j \delta_l{}^h = \eta_l \delta_j{}^h$; put $h = l$ and sum up to obtain $(n-1)\eta_j = 0$. □

Proposition 2.2. *If* $\eta_k \neq 0$ *and the scalar* $Z \neq 0$, *then the* Z *tensor has rank one*:

$$Z_{ij} = Z \frac{\eta_i \eta_j}{\eta^k \eta_k}. \tag{11}$$

Proof. Multiply (8) by η^j and sum up: $\eta^j \eta_j Z_{kl} = \eta_l \eta^j Z_{kj}$. Multiply (8) by g^{jk} and sum up: $\eta^k Z_{kl} = Z \eta_l$. The two results imply the assertion. □

The result translates to the Ricci tensor, whose expression is characteristic of *quasi Einstein* Riemannian manifolds [7], and generalizes the results of [12]:

Proposition 2.3. *A* $(\mathrm{WZS})_n$ *with* $\eta_k \neq 0$, *is a quasi Einstein manifold*:

$$R_{ij} = -\phi g_{ij} + Z T_i T_j, \qquad T_i = \frac{\eta_i}{\|\eta\|}, \tag{12}$$

Next we consider (9). If Z_{ij} is a Codazzi tensor, then the left hand side of the equation vanishes by definition, and the above discussion of (8) can be repeated. We merely state the result:

PROPOSITION 2.4. *In a* $(\mathrm{WZS})_n$ *with a Codazzi* Z *tensor, if* Z *is singular then* $\omega_k \neq 0$. *Conversely, if rank* $[Z_{kl}] > 1$ *then* $\omega_k = 0$.

3. Harmonic conformal or quasi conformal $(\mathrm{WZS})_n$ with $\eta \neq 0$

In this section we consider manifolds $(\mathrm{WZS})_n$ $(n > 3)$ with $\eta_k \neq 0$, and the property $\nabla_m C_{jkl}{}^m = 0$ (i.e. harmonic conformal curvature tensor [3]) or $\nabla_m W_{jkl}{}^m = 0$ (i.e. harmonic quasi conformal curvature tensor [34]). We provide sufficient conditions for $\eta/\|\eta\|$ to be a proper concircular vector [28,32].

We begin with the case of harmonic conformal tensor. From the expression for the divergence of the conformal tensor,

$$\nabla_m C_{jkl}{}^m = \frac{n-3}{n-2}\left[\nabla_k R_{jl} - \nabla_j R_{kl} + \frac{1}{2(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R\right] \tag{13}$$

we read the condition $\nabla_m C_{jkl}{}^m = 0$:

$$\nabla_k R_{jl} - \nabla_j R_{kl} = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)R. \tag{14}$$

We need the following theorem, whose proof given here is different from that in [13] (see also [10]):

THEOREM 3.1. *Let* M *be a* $n > 3$ *dimensional manifold, with harmonic conformal curvature tensor, and Ricci tensor* $R_{kl} = \alpha g_{kl} + \beta T_k T_l$, *where* α, β *are scalars, and* $T^k T_k = 1$. *If*

$$(T_j\nabla_k - T_k\nabla_j)\beta = 0, \tag{15}$$

then T_k *is a proper concircular vector.*

PROOF. Since M is conformally harmonic, (14) gives

$$\beta\left[\nabla_k(T_jT_l) - \nabla_j(T_kT_l)\right] = \frac{1}{2(n-1)}(g_{jl}\nabla_k - g_{kl}\nabla_j)S, \tag{16}$$

where $S = -(n-2)\alpha + \beta$, and condition (15) was used. The proof is in four steps.

1) We show that $T^l\nabla_l T_k = 0$: multiply (16) by g^{jl} to obtain: a) $-\beta\nabla^l(T_kT_l) = \frac{1}{2}\nabla_k S$. The result a) is multiplied by T^k to give: b) $-\beta\nabla_l T^l$

$= \frac{1}{2}T^l\nabla_l S$. a) and b) combine to give: c) $-\beta T^l\nabla_l T_k = \frac{1}{2}\left[\nabla_k - T_k T^l \nabla_l\right] S$. Finally multiply (16) by $T^k T^l$ and use the property $T^l\nabla_k T_l = 0$ to obtain

$$\beta T^k \nabla_k T_j = \frac{1}{2(n-1)}(T_j T^k \nabla_k - \nabla_j) S$$

which, compared to c) shows that d) $T^l\nabla_l T_k = 0$ and $(T_j T^k\nabla_k - \nabla_j)S = 0$.

2) We show that T is a closed 1-form: multiply (16) by T^l

$$\beta[\nabla_k T_j - \nabla_j T_k] = \frac{1}{2(n-1)}(T_j\nabla_k - T_k\nabla_j)S.$$

T is a closed form if the right hand side is null. This is proven by using identity a) to write

$$(T_j\nabla_k - T_k\nabla_j)S = -2\beta\left[T_j\nabla^l(T_k T_l) - T_k\nabla^l(T_j T_l)\right] = 0$$

by property d).

3) With condition d) in mind, transvect (16) with T^k and obtain

$$-\beta\nabla_j T_l = \frac{1}{2(n-1)}\left(g_{jl}T^k\nabla_k - T_l\nabla_j\right) S.$$

Use d) to replace $T_l\nabla_j S$ with $T_l T_j T^k \nabla_k S$. Then

$$\nabla_j T_l = f(T_j T_l - g_{jl}), \quad f \equiv \frac{T^k\nabla_k S}{2\beta(n-1)} \tag{17}$$

which means that T_k is a concircular vector.

4) We prove that T_k is a proper concircular vector, i.e. fT_k is a closed 1-form: from d) by a covariant derivative we obtain

$$\nabla_j\nabla_k S = (\nabla_j T_k)\left(T^l\nabla_l S\right) + T_k\nabla_j\left(T^l\nabla_l S\right);$$

subtract same equation with indices k and j exchanged. Since T_k is a closed 1-form we obtain $T_k\nabla_j\left(T^l\nabla_l S\right) = T_j\nabla_k\left(T^l\nabla_l S\right)$. Multiply by T^k:

$$\left(T_j T^k\nabla_k - \nabla_j\right)\left(T^l\nabla_l S\right) = 0.$$

From the relation (15), one obtains $\left(T_k T^l\nabla_l - \nabla_k\right)\beta = 0$. It follows that the scalar function f has the property $\nabla_j f = \mu T_j$ where μ is a scalar function. Then the 1-form fT_k is closed. □

With the identifications $\alpha = -\phi$ and $\beta = Z$, $T_i = \eta_i/\|\eta\|$ (see Proposition 2.3) the condition (15) is $(\eta_j\nabla_k - \eta_k\nabla_j)Z = 0$. Since $Z = S - (n-2)\phi$ and $(\eta_j\nabla_k - \eta_k\nabla_j)S = 0$, the condition can be rewritten as $(\eta_j\nabla_k - \eta_k\nabla_j)\phi = 0$. Thus we can state the following:

THEOREM 3.2. *In a* $(\mathrm{WZS})_n$ *manifold with* $\eta_k \neq 0$ *and harmonic conformal curvature tensor, if*

$$(\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0 \tag{18}$$

then $\eta_i/\|\eta\|$ *is a proper concircular vector.*

REMARK 1. If $\phi = 0$ or $\nabla_k \phi = 0$, the condition (18) is fulfilled automatically. In the case $\phi = 0$ we recover a $(\mathrm{WRS})_n$ manifold (and the results of [10,12]).

Now we consider the case of a $(\mathrm{WZS})_n$ manifold with harmonic quasi conformal curvature tensor. Yano and Sawaki [34] defined and studied a tensor $W_{jkl}{}^m$ on a Riemannian manifold of dimension $n > 3$, which includes as particular cases the conformal curvature tensor $C_{jkl}{}^m$, (2), and the concircular curvature tensor

$$\tilde{C}_{jkl}{}^m = R_{jkl}{}^m + \frac{R}{n(n-1)}(\delta_j{}^m g_{kl} - \delta_k{}^m g_{jl}). \tag{19}$$

The tensor is known as the *quasi conformal* curvature tensor:

$$W_{jkl}{}^m = -(n-2)bC_{jkl}{}^m + \left[a + (n-2)b\right]\tilde{C}_{jkl}{}^m; \tag{20}$$

a and b are nonzero constants. From the expressions (13) and (33) we evaluate

$$\nabla_m W_{jkl}{}^m = (a+b)\nabla_m R_{jkl}{}^m + \frac{2a - b(n-1)(n-4)}{2n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R. \tag{21}$$

A manifold is *quasi conformally harmonic* if $\nabla_m W_{jkl}{}^m = 0$. By transvecting the condition with g^{jk} we get

$$(1 - 2/n)\left[a + b(n-2)\right]\nabla_j R = 0, \tag{22}$$

which means that either $a + b(n-2) = 0$ or $\nabla_j R = 0$. The first condition implies $W = C$, and gives back the harmonic conformal case. If $\nabla_j R = 0$ then $\nabla_m R_{jkl}{}^m = 0$ by (21), and the equations in the proof of Theorem 3.1 simplify and we can state the following (analogous to Theorem 3.2):

THEOREM 3.3. *Let* $(\mathrm{WZS})_n$ *be a quasi conformally harmonic manifold with* $\eta_k \neq 0$. *If* $(\eta_j \nabla_k - \eta_k \nabla_j)\phi = 0$, *then* $\eta/\|\eta\|$ *is a proper concircular vector.*

4. $(\mathrm{WZS})_n$ with non-singular Z tensor: conditions for closed ω

In this section we investigate in a $(\mathrm{WZS})_n$ $(n > 3)$ the conditions the 1-form ω_k to be closed: $\nabla_i \omega_j - \nabla_j \omega_i = 0$. We need

LEMMA 4.1 (Lovelock's differential identity, [23,24]). *In a Riemannian manifold the following identity is true*:

$$\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m = -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m. \tag{23}$$

Also the contracted second Bianchi identity is of the form

$$\nabla_m R_{jkl}{}^m = \nabla_k Z_{jl} - \nabla_j Z_{kl} + (g_{kl} \nabla_j - g_{jl} \nabla_k)\phi. \tag{24}$$

Now we prove the relevant theorem (see also [24]):

THEOREM 4.2. *In a* $(\mathrm{WZS})_n$ $(n > 3)$ *with non singular* Z *tensor*, ω_k *is a closed* 1*-form if and only if*

$$R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m = 0. \tag{25}$$

PROOF. The covariant derivative of (24) and (9) give

$$\nabla_i \nabla_m R_{jkl}{}^m = (\nabla_i \omega_k) Z_{jl} + \omega_k (\nabla_i Z_{jl}) - (\nabla_i \omega_j) Z_{kl} - \omega_j (\nabla_i Z_{kl}) + (g_{kl} \nabla_i \nabla_j \phi - g_{jl} \nabla_i \nabla_k \phi).$$

Cyclic permutations of the indices i, j, k are made, and the resulting three equations are added:

$$\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m$$
$$= (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_i - \nabla_i \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il}$$
$$+ \omega_j (\nabla_k Z_{il} - \nabla_i Z_{kl}) + \omega_k (\nabla_i Z_{jl} - \nabla_j Z_{il}) + \omega_i (\nabla_j Z_{kl} - \nabla_k Z_{jl}).$$

Cancellations occur by (9). By Lemma 4.1, one obtains

$$-R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m$$
$$= (\nabla_i \omega_k - \nabla_k \omega_i) Z_{jl} + (\nabla_j \omega_i - \nabla_i \omega_j) Z_{kl} + (\nabla_k \omega_j - \nabla_j \omega_k) Z_{il}.$$

If ω_k is a closed 1-form then (25) is fulfilled. Conversely, suppose that (25) holds: if the Z tensor is non singular, there is a $(2, 0)$ tensor such that $Z_{kl} \left(Z^{-1} \right)^{km} = \delta_l{}^m$. Multiply the last equation by $\left(Z^{-1} \right)^{hl}$:

$$(\nabla_i \omega_k - \nabla_k \omega_i) \delta_j{}^h + (\nabla_j \omega_i - \nabla_i \omega_j) \delta_k{}^h + (\nabla_k \omega_j - \nabla_j \omega_k) \delta_i{}^h = 0.$$

Set $h = j$ and sum up: $(n - 2)(\nabla_i \omega_k - \nabla_k \omega_i) = 0$. Since $n > 2$, ω_k is a closed 1-form. $\square$

Remark 2. By Lovelock's identity, the condition (25) is obviously true if $\nabla_m R_{ijk}{}^m = 0$, i.e. the $(\mathrm{WZS})_n$ is a harmonic manifold. However, we have shown in [24] that there is a broad class of generalized curvature tensors for which the case $\nabla_m K_{ijk}{}^m = 0$ implies the same condition. This class includes several well known curvature tensors, and is the main subject of this section.

Definition 4.3. A tensor $K_{jkl}{}^m$ is a *generalized curvature tensor*[1] if
1) $K_{jkl}{}^m = -K_{kjl}{}^m$,
2) $K_{jkl}{}^m + K_{klj}{}^m + K_{ljk}{}^m = 0$.

The second Bianchi identity does not hold in general, and is modified by a tensor source $B_{ijkl}{}^m$ that depends on the specific form of the curvature tensor:

$$\nabla_i K_{jkl}{}^m + \nabla_j K_{kil}{}^m + \nabla_k K_{ijl}{}^m = B_{ijkl}{}^m. \tag{26}$$

Proposition 4.4 [24]. *If $K_{jkl}{}^m$ is a generalized curvature tensor such that*

$$\nabla_m K_{jkl}{}^m = A\nabla_m R_{jkl}{}^m + B(a_{lk}\nabla_j - a_{lj}\nabla_k)\psi, \tag{27}$$

where $A \neq 0$, B are constants, ψ is a scalar field, and a_{ij} is a symmetric $(0,2)$ Codazzi tensor (i.e. $\nabla_i a_{kl} = \nabla_k a_{il}$), then the following relation holds:

$$\nabla_i \nabla_m K_{jkl}{}^m + \nabla_j \nabla_m K_{kil}{}^m + \nabla_k \nabla_m K_{ijl}{}^m \tag{28}$$

$$= -A\big(R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m\big).$$

Remark 3. In [16] it is proven that any smooth manifold carries a metric such that (M,g) admits a non trivial Codazzi tensor (i.e. proportional to the metric tensor) and the deep consequences on the structure of the curvature operator are presented (see also [25]).

Given a Codazzi tensor it is possible to exhibit a K tensor that satisfies the condition (27):

$$K_{jkl}{}^m = AR_{jkl}{}^m + B\psi\big(\delta_j{}^m a_{kl} - \delta_k{}^m a_{jl}\big). \tag{29}$$

Its trace is $K_{kl} = -K_{mkl}{}^m = AR_{kl} - B(n-1)\psi a_{kl}$. Note that for $a_{kl} = g_{kl}$ the tensor K_{kl} is up to a factor a Z tensor. Thus Z tensors arise naturally from the invariance of Lovelock's identity.

[1] The notion was introduced by Kobayashi and Nomizu [22], but with the further antisymmetry in the last pair of indices.

Remark 4. In the literature one meets generalized curvature tensors whose divergence has the form (27), with trivial Codazzi tensor:

$$\nabla_m K_{jkl}{}^m = A\nabla_m R_{jkl}{}^m + B(g_{kl}\nabla_j - g_{jl}\nabla_k)R. \tag{30}$$

They are the projective curvature tensor $P_{jkl}{}^m$ [18], the conformal curvature tensor $C_{jkl}{}^m$ [27], the concircular tensor $\tilde{C}_{jkl}{}^m$ [28,32], the conharmonic tensor $N_{jkl}{}^m$ [26,30] and the quasi conformal tensor $W_{jkl}{}^m$ [34].

Definition 4.5. A manifold is K-harmonic if $\nabla_m K_{jkl}{}^m = 0$.

Proposition 4.6. *In a K-harmonic manifold, if K is of type* (30) *and $A \neq 2(n-1)B$, then $\nabla_j R = 0$.*

Proof. By transvecting (30) with g^{kl} and by the second contracted Bianchi identity, we obtain $\frac{1}{2}\left[A - 2(n-1)B\right]\nabla_j R = 0$. □

Hereafter, we specialize to $(\mathrm{WZS})_n$ manifolds with non singular Z tensor, and with a generalized curvature tensor of the type (30). From (24) and (9) we obtain

$$\nabla_m K_{jkl}{}^m = A(\omega_k Z_{jl} - \omega_j Z_{kl}) + (g_{kl}\nabla_j - g_{jl}\nabla_k)(A\phi + BR). \tag{31}$$

Then, the manifold is K-harmonic if

$$A(\omega_k Z_{jl} - \omega_j Z_{kl}) = (g_{jl}\nabla_k - g_{kl}\nabla_j)(A\phi + BR). \tag{32}$$

Lemma 4.7. *In a K-harmonic $(\mathrm{WZS})_n$ with non singular Z tensor*
1) $\omega_k = 0$ *if and only if* $\nabla_k(A\phi + BR) = 0$; *and*
2) *if* $A \neq 2(n-1)B$, *then* $\omega_k = 0$ *if and only if* $\nabla_k\phi = 0$.

Proof. If $\nabla_k(A\phi + BR) = 0$ then $\omega_k Z_{jl} = \omega_j Z_{kl}$: if the Z tensor is non singular, by transvecting with $\left(Z^{-1}\right)^{lh}$ we obtain $\omega_j \delta^h{}_k = \omega_k \delta^h{}_j$. Now put $h = j$ and sum up to obtain $(n-1)\omega_k = 0$. On the other hand if $\omega_k = 0$, (32) gives $[g_{jl}\nabla_k - g_{kl}\nabla_j](A\phi + BR) = 0$ and transvecting with g^{kl} we get the result.
If $A \neq 2B(n-1)$ then $\nabla_k R = 0$ and part 1) applies. □

Theorem 4.8. *In a K-harmonic $(\mathrm{WZS})_n$ with non-singular Z tensor and K of type* (30), *if $\omega \neq 0$ then ω is a closed* 1*-form.*

This theorem extends Theorem 4.2 (where $K = R$), and has interesting corollaries according to the various choices $K = C, W, P, \tilde{C}, N$.

Corollary 4.9. *Let $(\mathrm{WZS})_n$ have non singular Z tensor and $\omega \neq 0$. If $\nabla_m K_{jkl}{}^m = 0$, and $K = P, \tilde{C}, N$, then ω is a closed* 1*-form.*

Proof. 1) Harmonic conformal curvature: $\nabla_m C_{jkl}{}^m = 0$. Note that in this case $A = 2B(n-1)$ and Theorem 4.8 applies.

2) Harmonic quasi conformal curvature: $\nabla_m W_{jkl}{}^m = 0$: (22) gives either $\nabla_j R = 0$ or $a + b(n-2) = 0$. If $\nabla_j R = 0$ then $\nabla_m R_{jkl}{}^m = 0$ and Theorem 4.2 applies. If $a + b(n-2) = 0$ it is $\nabla_m C_{jkl}{}^m = 0$ and Case 1) applies.

3) Harmonic projective curvature: $\nabla_m P_{jkl}{}^m = 0$. The components of the projective curvature tensor are [18,30]

$$P_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-1}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl}).$$

One evaluates $\nabla_m P_{jkl}{}^m = \frac{n-2}{n-1}\nabla_m R_{jkl}{}^m$, and Theorem 4.2 applies.

4) Harmonic concircular curvature: $\nabla_m \tilde{C}_{jkl}{}^m = 0$. The concircular curvature tensor is given in (19), [28,32]. Its divergence is

$$\nabla_m \tilde{C}_{jkl}{}^m = \nabla_m R^m_{jkl} + \frac{1}{n(n-1)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R. \tag{33}$$

Theorem 4.8 applies.

5) Harmonic conharmonic curvature: $\nabla_m N_{jkl}{}^m = 0$. The conharmonic curvature tensor [26,30] is

$$N_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_j{}^m R_{kl} - \delta_k{}^m R_{jl} + R^m_j g_{kl} - R^m_k g_{jl}).$$

A covariant derivative and the second contracted Bianchi identity give

$$\nabla_m N_{jkl}{}^m = \frac{n-3}{n-2}\nabla_m R_{jkl}{}^m + \frac{1}{2(n-2)}(g_{kl}\nabla_j - g_{jl}\nabla_k)R.$$

Theorems 4.8 applies. □

There are other cases where the 1-form ω_k is closed for a $(\mathrm{WZS})_n$ manifold.

Definition 4.10 [21,24]. An n-dimensional Riemannian manifold is *K-recurrent*, $(\mathrm{KR})_n$, if the generalized curvature tensor is recurrent, $\nabla_i K_{jkl}{}^m = \lambda_i K_{jkl}{}^m$, for some non zero covector λ_i.

Theorem 4.11 [24]. *In a* $(\mathrm{KR})_n$, *if* λ_i *is closed then*

$$R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = -\frac{1}{A}\nabla_m B_{ijkl}{}^m, \tag{34}$$

where B is the source tensor in (26). *In particular, for* $K = C, P, \tilde{C}, N, W$ *the tensor* $\nabla_m B_{ijkl}{}^m$ *either vanishes or is proportional to the left hand side of* (34).

COROLLARY 4.12. *Let* $(\mathrm{WZS})_n$ *have non singular* Z *tensor, and be* K *recurrent with closed* λ_i. *If* $K = C, P, \tilde{C}, N, W$, *then* ω *is a closed* 1*-form.*

DEFINITION 4.13. A Riemannian manifold is *pseudosymmetric in the sense of R. Deszcz* [17] if the following condition holds:

(35)
$$(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{jklm} = L_R\big(g_{js}R_{iklm} - g_{ji}R_{sklm} + g_{ks}R_{jilm} - g_{ki}R_{jslm} + g_{ls}R_{jkim} - g_{li}R_{jksm} + g_{ms}R_{jkli} - g_{mi}R_{jkls}\big),$$

where L_R is a non null scalar function.

In [24] the following theorem is proven:

THEOREM 4.14. *In a Riemannian manifold which is pseudosymmetric in the sense of R. Deszcz,* $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$.

Then we can state the following:

PROPOSITION 4.15. *In a* $(\mathrm{WZS})_n$ *which is pseudosymmetric in the sense of R. Deszcz, if the* Z *tensor is non-singular then* ω_k *is a closed* 1*-form.*

DEFINITION 4.16. A Riemannian manifold is *generalized Ricci pseudosymmetric in the sense of R. Deszcz*, [15], if the following condition holds:

(36)
$$(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{jklm} = L_S\big(R_{js}R_{iklm} - R_{ji}R_{sklm} + R_{ks}R_{jilm} - R_{ki}R_{jslm} + R_{ls}R_{jkim} - R_{li}R_{jksm} + R_{ms}R_{jkli} - R_{mi}R_{jkls}\big),$$

where L_S is a non null scalar function.

THEOREM 4.17. *In a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, either* $L_S = -\frac{1}{3}$, *or* $R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m = 0$.

PROOF. Equation (36) is transvected with g^{mj} to obtain

$$(\nabla_s\nabla_i - \nabla_i\nabla_s)R_{kl} = L_S\big[R_{im}(R_{skl}{}^m + R_{slk}{}^m) - R_{sm}(R_{ikl}{}^m + R_{ilk}{}^m)\big].$$

Then

$$(\nabla_i\nabla_k - \nabla_k\nabla_i)R_{jl} + (\nabla_j\nabla_i - \nabla_i\nabla_j)R_{kl} + (\nabla_k\nabla_j - \nabla_j\nabla_k)R_{il} = 3L_S\big(R_{im}R_{jkl}{}^m + R_{jm}R_{kil}{}^m + R_{km}R_{ijl}{}^m\big).$$

By Lovelock's identity (4.1), the left hand side of the previous equation is

$$\nabla_i \nabla_m R_{jkl}{}^m + \nabla_j \nabla_m R_{kil}{}^m + \nabla_k \nabla_m R_{ijl}{}^m$$

$$= (\nabla_i \nabla_k - \nabla_k \nabla_i) R_{jl} + (\nabla_j \nabla_i - \nabla_i \nabla_j) R_{kl} + (\nabla_k \nabla_j - \nabla_j \nabla_k) R_{il}$$

$$= -R_{im} R_{jkl}{}^m - R_{jm} R_{kil}{}^m - R_{km} R_{ijl}{}^m.$$

Compare the two results and conclude that either $L_S = -\frac{1}{3}$, or $R_{im} R_{jkl}{}^m + R_{jm} R_{kil}{}^m + R_{km} R_{ijl}{}^m = 0$. □

Finally we state:

PROPOSITION 4.18. *In a* $(\mathrm{WZS})_n$ *which is also a generalized Ricci pseudosymmetric manifold in the sense of R. Deszcz, if the Z tensor is non-singular and* $L_S \neq -\frac{1}{3}$, *then* ω_k *is a closed* 1*-form.*

5. Conformally harmonic $(\mathrm{WZS})_n$: form of the Ricci tensor

In this section we study conformally harmonic $(\mathrm{WZS})_n$ in depth. We show the existence of a proper concircular vector in such manifolds, and obtain the form of the Ricci tensor. The proof only requires the Z tensor to be non singular. For the conformally flat case, in particular, we give the explicit local form of the metric tensor.

The condition $\nabla_m C_{jkl}{}^m = 0$ is (14) which, by using $R_{ij} = Z_{ij} - g_{ij}\phi$ and the property (9), becomes

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{2(n-1)} (g_{jl} \nabla_k - g_{kl} \nabla_j) \left[R + 2(n-1)\phi \right]. \tag{37}$$

This is the starting point for the proofs. By Proposition 4.7, since Z is non singular, $\omega_k \neq 0$ if and only if $\nabla_k [R + 2(n-1)\phi] \neq 0$.

REMARK 5. 1) The condition $\nabla_m C_{jkl}{}^m = 0$ implies that the manifold is a $(\mathrm{NCS})_n$.

2) If $\nabla_k [R + 2(n-1)\phi] = 0$ the Z tensor is a Codazzi tensor.

The following theorem generalizes a result in [11] for $\mathrm{A(PRS)}_n$:

THEOREM 5.1. *In a conformally harmonic* $(\mathrm{WZS})_n$ *the* 1*-form* ω *is an eigenvector of the* Z *tensor.*

PROOF. By transvecting (37) with g^{kl} we obtain

$$\omega_j Z - \omega^m Z_{jm} = \frac{1}{2} \nabla_j \left[R + 2(n-1)\phi \right]; \tag{38}$$

the result is inserted back in (37),

$$\omega_k Z_{jl} - \omega_j Z_{kl} = \frac{1}{(n-1)}\left[\,(\omega_k Z - \omega^m Z_{km})g_{jl} - (\omega_j Z - \omega^m Z_{jm})g_{kl}\right],$$

and transvected with $\omega^j\omega^l$ to obtain $\omega_k(\omega^j\omega^l Z_{jl}) = (\omega_j\omega^j)\omega^l Z_{kl}$. The last equation can be rewritten as $Z_{kl}\omega^l = \zeta\omega_k$. $\square$

Now (38) simplifies to $\omega_j(\zeta - Z) = -\frac{1}{2}\nabla_j\left[\,R + 2(n-1)\phi\right]$. The result is a natural generalization of a similar one given in [11] for $\mathrm{A(PRS)}_n$.

THEOREM 5.2. *Let M be a conformally harmonic* $\mathrm{(WZS)}_n$. *Then*
1) *M is a quasi Einstein manifold*;
2) *if the Z tensor is non singular and if $(\omega_j\nabla_k - \omega_k\nabla_j)\phi = 0$, then*

$$(\omega_j\nabla_k - \omega_k\nabla_j)\left[\frac{n\zeta - Z}{n-1}\right] = 0, \tag{39}$$

and M admits a proper concircular vector.

PROOF. (37) is transvected with ω^j and Theorem 5.1 is used to show that

$$R_{kl} = \left[\frac{Z-\zeta}{n-1} - \phi\right]g_{kl} + \left[\frac{n\zeta - Z}{n-1}\right]\frac{\omega_k\omega_l}{\omega_j\omega^j},$$

i.e. R_{kl} has the structure $\alpha g_{kl} + \beta T_k T_l$ and the manifold is quasi Einstein [7]. By transvecting (9) with g^{jl} we obtain

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k\phi = \omega_k Z - \omega^l Z_{kl}.$$

This and theorem (5.1) imply

$$\frac{1}{2}\nabla_k Z + \frac{n-2}{2}\nabla_k\phi = \omega_k(Z - \zeta). \tag{40}$$

A covariant derivative gives

$$\frac{1}{2}\nabla_j\nabla_k Z + \frac{n-2}{2}\nabla_j\nabla_k\phi = \nabla_j\left[\,\omega_k(Z-\zeta)\right].$$

Subtract the equation with indices k and j exchanged:

$$(Z-\zeta)(\nabla_j\omega_k - \nabla_k\omega_j) + (\omega_k\nabla_j - \omega_j\nabla_k)(Z-\zeta) = 0.$$

According to Corollary 4.9, in a conformally harmonic $\mathrm{(WZS)}_n$ with non singular Z the 1-form ω_k is closed. Then

$$(\omega_k\nabla_j - \omega_j\nabla_k)(Z-\zeta) = 0. \tag{41}$$

Multiply (40) by ω_j and subtract from it the equation with indices k and j exchanged:

$$(\omega_j\nabla_k-\omega_k\nabla_j)Z+(n-2)(\omega_j\nabla_k-\omega_k\nabla_j)\phi=0.$$

Suppose that ω_k, besides being a closed 1-form, has the property $(\omega_j\nabla_k-\omega_k\nabla_j)\phi=0$, then one obtains

$$(\omega_k\nabla_j-\omega_j\nabla_k)Z=0. \tag{42}$$

(41), (42) imply the assertion (39). The existence of a proper concircular vector follows from Theorem 3.1. □

Let us specialize to the case $C_{ijk}{}^m=0$ (conformally flat $(\mathrm{WZS})_n$). It is well known [1] that if a conformally flat space admits a proper concircular vector, then the space is subprojective in the sense of Kagan. From Theorem 5.2 we state the following:

THEOREM 5.3. *Let* $(\mathrm{WZS})_n$ $(n>3)$ *be conformally flat with nonsingular* Z *tensor and* $(\omega_j\nabla_k-\omega_k\nabla_j)\phi=0$. *Then the manifold is a subprojective space.*

In [33] K. Yano proved that a necessary and sufficient condition for a Riemannian manifold to admit a concircular vector is that there is a coordinate system in which the first fundamental form may be written as

$$ds^2=\left(dx^1\right)^2+e^{q(x^1)}g^*_{\alpha\beta}(x^2,\dots,x^n)\,dx^\alpha\,dx^\beta, \tag{43}$$

where $\alpha,\beta=2,\dots,n$. Since a conformally flat $(\mathrm{WZS})_n$ with non singular Z tensor admits a proper concircular vector field, this space is the warped product $1\times e^qM^*$, where (M^*,g^*) is a $(n-1)-$dimensional Riemannian manifold. Gebarosky [19] proved that the warped product $1\times e^qM^*$ has the metric structure (43) if and only if M^* is Einstein. Thus the following theorem holds:

THEOREM 5.4. *Let* M *be an* n *dimensional conformally flat* $(\mathrm{WZS})_n$ $(n>3)$. *If* Z_{kl} *is non singular and* $(\omega_j\nabla_k-\omega_k\nabla_j)\phi=0$, *then* M *is the warped product* $1\times e^qM^*$, *where* M^* *is Einstein.*

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