WEYL COMPATIBLE TENSORS

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Abstract. The algebraic condition of Riemann compatibility for symmetric tensors generalizes the differential Codazzi condition, but preserves much of the geometric content. The compatibility condition can be extended to other curvature tensors. This paper is about Weyl compatible symmetric tensors and vectors. In particular it is shown that the existence of a Weyl compatible vector implies the Weyl tensor to be algebraically special, and it is a necessary and sufficient condition for the magnetic part to vanish. Some theorems (Derdziński and Shen, Hall) are extended to the broader hypothesis of Weyl or Riemann compatibility; Weyl compatibility includes conditions that were investigated in the literature of general relativity (as McIntosh et al.) Hypersurfaces of pseudo Euclidean spaces provide a simple example of Weyl compatible Ricci tensor.

1. Introduction

The geometry of Riemannian or pseudo-Riemannian manifolds of dimension $n \geq 3$ is intrinsically described by $\mathscr{N} = \frac{1}{12}n(n-1)(n-2)(n+3)$ algebraically independent scalar fields, constructed with the Riemann and the metric tensors. The same counting is provided by the Weyl, the Ricci and the metric tensors. The Weyl tensor bears the symmetries of the Riemann tensor, with the extra property of being traceless[1]:

$$C_{jkl}{}^m = R_{jkl}{}^m + \frac{1}{n-2}(\delta_{[j}{}^m R_{k]l} + R_{[j}{}^m g_{k]l}) - \frac{1}{(n-1)(n-2)} R\delta_{[j}{}^m g_{k]l}. \tag{1}$$

The trace condition $C_{jab}{}^j = 0$ reduces the parameters of the Riemann tensor by a number $\frac{1}{2}n(n+1)$ that is accounted for by considering the Ricci tensor as algebraically independent. The two tensors are linked by functional relations as the following one [17, 14]:

$$-\nabla_m C_{abc}{}^m = \frac{n-3}{n-2}\left[\nabla_{[a}R_{b]c} - \frac{1}{2(n-1)}\nabla_{[a}g_{b]c}R\right]. \tag{2}$$

In the coordinate frame that locally diagonalizes the Ricci and the metric tensors (the latter with diagonal elements ± 1), the parameters that survive are the components of the Weyl tensor and the n eigenvalues of the Ricci tensor, whose number is precisely $\mathscr{N}$ [32]. This choice of fundamental tensors offers advantages, as in the classification of manifolds and in general relativity.

Date: 6 dec 2012.

1991 *Mathematics Subject Classification.* 53B20, 53B21, 53B30, 53C50, 83C20.

Key words and phrases. Weyl tensor, Riemann compatibility, Petrov types.

[1]Conventions: $X_{[ab]} =: X_{ab} - X_{ba}$, $R_{ab} = R_{amb}{}^m$, $u^2 = u^a u_a$.

Debever and Penrose [26] proved that in four-dimensional space-time manifolds the equation

$$k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0 \tag{3}$$

always admits four null solutions (principal null directions). When two or more coincide, the Weyl tensor is named *algebraically special*, and the condition for degeneracy is

$$k_{[b}C_{a]rsq}k^r k^s = 0 \tag{4}$$

The degeneracies classify space-time manifolds in classes that coincide with the Petrov types, which are determined by the degeneracies of the eigenvalues of the self-dual part of the Weyl tensor [27]. Milson et al. showed in 2005 that also in $n > 4$ there are always n null (Weyl aligned) solutions [24], that provide the backbone for a classification of high dimensional space-times [25].

The Einstein equations of general relativity link the energy momentum tensor T_{ij} to the Ricci tensor and the curvature scalar, but not to the Weyl tensor:

$$R_{ij} - \frac{R}{2}\, g_{ij} = 8\pi\, T_{ij}. \tag{5}$$

The Weyl tensor in $n = 4$ may be replaced by two symmetric tensors, the electric and magnetic components, and the identity (2) for the Weyl tensor translates into Maxwell-like equations for the components [30, 2]. The construction was extended to $n > 4$ [15].

In the study of Derdziński and Shen's theorem [9, 3] on the restrictions imposed by a Codazzi tensor on the structure of the Riemann tensor, we introduced the new algebraic notion of *Riemann compatible tensors* [18]. This enabled us to extend the theorem in two directions: the replacement of the Codazzi condition with the milder hypothesis of Riemann compatibility together with a drastic simplification of the proof, the restatement of the theorem for curvature tensors other than Riemann's.

Riemann compatible tensors were investigated in [19]. Most of the statements valid for (pseudo) Riemannian manifolds equipped with a nontrivial Codazzi tensor, such as the vanishing of Pontryagin forms, were shown to persist in presence of a nontrivial Riemann compatible tensor. The application to geodesic mappings was then discussed.

This paper is mainly about Weyl compatibility of symmetric tensors, a property which is broader than Riemann compatibility. The restrictions on the structure of the Weyl tensor has consequences on the Petrov type of the manifold and the electric and magnetic components of the Weyl tensor.

Definitions and main properties of Riemann and Weyl compatibile symmetric tensors are reviewed in Sect. 2, with the subcase of Riemann and Weyl-*permutable* tensors. Tensors $u_i u_j$ naturally define Riemann and Weyl compatibility for vectors, which is discussed in Sect. 3 with various new results, as the extensions of Derdziński and Shen's theorem and of Hall's theorem. Riemann (Weyl) permutable vectors are considered in Sect. 4, where results by McIntosh and others for the special case $R_{ijkl}u^l = 0$ are reobtained.

In Sect. 5 it is shown that the existence of a Weyl compatible vector is a sufficient

condition for the Weyl tensor to be special, with results regarding the Penrose-Debever classification of spacetimes. The electric and magnetic components of the Weyl tensor are considered in Sect. 6, with the statements that existence of a Weyl compatible vector is necessary and sufficient for the Weyl tensor to be purely electric, and that Weyl permutability implies a conformally flat space-time.
Sect. 7 is devoted to hypersurfaces; the Gauss and Codazzi equations specify the induced Riemann tensor of the hypersurface as a quadratic expression of a Codazzi tensor. It is shown that the corresponding Ricci tensor is Riemann and Weyl compatible.
Conformal maps obviously preserve Weyl compatibility, geodesic maps preserve Riemann compatibility [19] but not necessarily Weyl compatibility; a sufficient condition is presented in Sect. 8.

2. Riemann and Weyl compatible tensors

We briefly review the concept of compatibility for symmetric tensors, first introduced in [18], and investigated in [19]. Permutable tensors are then defined, as a special class.

Definition 2.1. A symmetric tensor b_{ij} is Riemann compatible if:

$$b_{am}R_{bcl}{}^m + b_{bm}R_{cal}{}^m + b_{cm}R_{abl}{}^m = 0. \tag{6}$$

The metric tensor is trivially Riemann compatible because of the first Bianchi identity.

Remark 2.2. *The definition has a natural origin. Consider the vector-valued 1-form $B_l = b_{kl}dx^k$, where b is a symmetric tensor. A covariant exterior derivative gives*

$$DB_l = \frac{1}{2}\mathscr{C}_{jkl}\, dx^j \wedge dx^k,$$

where $\mathscr{C}_{ijk} =: \nabla_i b_{jk} - \nabla_j b_{ik}$ is the "Codazzi deviation tensor", defined in [19]*. As it is well known, $DB_l = 0$ if and only if b_{kl} is a Codazzi tensor* [4, 20]*.*
If $DB_l \neq 0$, another derivative gives

$$D^2B_l = \frac{1}{3!}\left(\nabla_i\mathscr{C}_{jkl} + \nabla_j\mathscr{C}_{kil} + \nabla_k\mathscr{C}_{ijl}\right) dx^i \wedge dx^j \wedge dx^k.$$

The following identity links the Codazzi deviation to Riemann compatibility [19]*:*

$$\nabla_i\mathscr{C}_{jkl} + \nabla_j\mathscr{C}_{kil} + \nabla_k\mathscr{C}_{ijl} = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{7}$$

Therefore, $D^2B_l = 0$ (i.e. DB_l is closed) if and only if b is Riemann compatible. It follows that Codazzi tensors, $\nabla_i b_{jk} = \nabla_j b_{ik}$, are Riemann compatible.

As an example, consider the Ricci tensor. Its Codazzi deviation is $\mathscr{C}_{abc} =: \nabla_{[a}R_{b]c} = -\nabla_m R_{abc}{}^m$, by the contracted Bianchi identity. If the deviation is nonzero, one may consider the possibility that the Ricci tensor is Riemann compatible. The identity (7) implies another identity, by Lovelock [16]

$$\begin{aligned} -(\nabla_a\nabla_m R_{bcd}{}^m + \nabla_b\nabla_m R_{cad}{}^m + \nabla_c\nabla_m R_{abd}{}^m) = \\ R_{am}R_{bcd}{}^m + R_{bm}R_{cad}{}^m + R_{cm}R_{abd}{}^m \end{aligned}$$

Compatibility was extended to generalized curvature tensors K, i.e. tensors having the symmetries of the Riemann tensor for the exchange of indices, and the first Bianchi property. The Weyl tensor (1) is the most notable example. A symmetric tensor b_{ij} is *Weyl compatible* if:

$$b_{am}C_{bcl}{}^m + b_{bm}C_{cal}{}^m + b_{cm}C_{abl}{}^m = 0. \tag{8}$$

A Weyl compatible tensor poses strong restrictions on the Weyl tensor. In [18] we proved a broad generalization of Derdzinski and Shen's theorem that holds both in Riemannian and pseudo-Riemannian manifolds. For the Weyl tensor it reads:

Proposition 2.3. *On a pseudo-Riemannian manifold with a Weyl compatible tensor b, if X, Y and Z are eigenvectors of b with eigenvalues λ, μ, ν ($b^i{}_jX^j = \lambda X^i$, etc.) then:*

$$C_{abcd}X^aY^bZ^c = 0, \quad \nu \neq \lambda, \mu. \tag{9}$$

The following algebraic identity relates a symmetric tensor b_{ij} to the Weyl, the Riemann and the Ricci tensors [19]:

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m \tag{10}$$
$$+\frac{1}{n-2}\left[g_{kl}(b_{im}R_j{}^m - b_{jm}R_i{}^m) + g_{il}(b_{jm}R_k{}^m - b_{km}R_j{}^m) + g_{jl}(b_{km}R_i{}^m - b_{im}R_k{}^m)\right].$$

Any contraction with the metric tensor gives zero; the identity is trivial if b is the metric tensor. An immediate consequence is:

Theorem 2.4. *A symmetric tensor is Riemann compatible if and only if it is Weyl compatible and it commutes with the Ricci tensor.*

Proof. If b is Riemann compatible, contraction of (6) with g^{cl} gives $b_{am}R_b{}^m - b_{bm}R_a{}^m = 0$, i.e. b commutes with the Ricci tensor. Then b is Weyl compatible by identity (10). The converse is obvious, by the same identity. □

In particular, Riemann and Weyl compatibility are equivalent for the Ricci tensor, or any symmetric tensor that commutes with it.

An example of a Riemann tensor with a Riemann compatible symmetric tensor is constructed, based on the Kulkarni-Nomizu product of two symmetric tensors [3, 4, 9]:

Proposition 2.5. *Suppose that the Riemann tensor has the form:*

$$R_{jklm} = b_{l[j}a_{k]m} + b_{m[k}a_{j]l} \tag{11}$$

with symmetric tensor fields a_{ij} and b_{ij}. Then, if a and b commute, $a_i{}^m b_{mj} = a_j{}^m b_{mi}$, the tensors a and b are Riemann compatible.

Proof. Evaluate: $b_i{}^m R_{jklm} = b_{lj}(ba)_{ik} - b_{lk}(ba)_{ij} + b^2_{ik}a_{jl} - b^2_{ij}a_{kl}$. The sum on cyclic permutations of ijk cancels all terms in the r.h.s. and b is Riemann compatible. Because of the symmetry of (11) in the exchange of a and b, also a is Riemann compatible. □

The Weyl tensors may be evaluated from the Riemann tensor (11), and both a and b will be Weyl compatible tensors.

The same Kulkarni-Nomizu product of two symmetric tensors can be used to construct a Weyl tensor:

Proposition 2.6. *Let a and b be commuting symmetric tensor fields, such that*

$$b^m{}_m a_{kl} + a^m{}_m b_{kl} - 2b_{km} a^m{}_l = 0, \tag{12}$$

then the tensor $C_{jklm} = b_{l[j} a_{k]m} + b_{m[k} a_{j]l}$ has the symmetries of the Weyl tensor, and a and b are Weyl compatible.

The additional equation (12) is required to enforce tracelessness; it can be solved to obtain the "potential" a that produces b.

Definition 2.7. A symmetric tensor b_{ij} is *Riemann permutable* if:

$$b_{im} R_{jkl}{}^m = b_{lm} R_{jki}{}^m \tag{13}$$

It is *Weyl permutable* if:

$$b_{im} C_{jkl}{}^m = b_{lm} C_{jki}{}^m \tag{14}$$

Proposition 2.8. *If a symmetric tensor is Riemann (Weyl) permutable then it is Riemann (Weyl) compatible.*

Proof. In the relation for Riemann compatibility use (13) for each term: $b_{im} R_{jkl}{}^m + b_{jm} R_{kil}{}^m + b_{km} R_{ijl}{}^m = b_{lm}(R_{jki}{}^m + R_{kij}{}^m + R_{ijk}{}^m) = 0$ by the first Bianchi identity. An analogous proof holds for Weyl permutable tensors. □

Note that Riemann permutability does not imply Weyl permutability. A Riemann permutable tensor (being Riemann compatible) commutes with the Ricci tensor.
Derdziński and Shen's theorem for the Riemann tensor, or theorem 2.3 for the Weyl tensor, become more stringent for permutable tensors:

Proposition 2.9. *If b is a symmetric tensor and X, Y are two eigenvectors, $b^i{}_j X^j = \lambda X^i$ and $b^i{}_j Y^j = \mu Y^i$, with $\lambda + \mu \neq 0$, then*
1) if b is Riemann permutable it is: $R_{jklm} X^l Y^m = 0$;
2) if b is Weyl permutable it is: $C_{jklm} X^l Y^m = 0$.

Proof. Contraction of (13) with $X^i Y^l$ gives $\lambda R_{jkl}{}^m X_m Y^l = \mu R_{jki}{}^m X^i Y_m$, then $0 = (\lambda + \mu) R_{jklm} X^l Y^m$. The proof for the Weyl tensor is analogous. □

3. Riemann and Weyl compatible vectors

The notion of K-compatible symmetric tensor includes vectors u_i in a natural way, through the symmetric tensor $u_i u_j$:

Definition 3.1. A vector field u_i is K-compatible (where K is the Riemann, the Weyl or a generalized tensor) if:

$$(u_i K_{jkl}{}^m + u_j K_{kil}{}^m + u_k K_{ijl}{}^m) u_m = 0 \tag{15}$$

Theorem 3.2. *A vector field u with $u^2 \neq 0$ is K-compatible if and only if there is a symmetric tensor D_{ij} such that:*

$$K_{abcm} u^m = D_{ac} u_b - D_{bc} u_a \tag{16}$$

Proof. Multiplication by u_d and cyclic summation on abd makes the r.h.s. vanish and K-compatibility is obtained.
If u is K-compatible then multiplication of (15) by u^i gives

$$(u^2 K_{jkl}{}^m - u_j u^i K_{ikl}{}^m + u_k u^i K_{ijl}{}^m) u_m = 0$$

where we read $D_{jl} = K_{ijlm} u^i u^m / u^2$. □

For the Weyl tensor, D will be identified with its electric component (see Sect.5). It can be easily shown that u is an eigenvector of the symmetric tensor D.

Suppose that in a pseudo-Riemannian manifold there is a concircular vector such that $\nabla_k u_l = A g_{kl} + B u_k u_l$, with constant A and B. The condition gives $R_{jkl}{}^m u_m = AB(u_j g_{kl} - u_k g_{jl})$, which has the form (16). Therefore, a concircular vector is both Riemann and Weyl compatible.

Some theorems and propositions valid for compatible tensors [19] become stronger, and new facts are presented. For example, the generalized Derdziński and Shen's theorem has now a surprisingly simple proof, with no need of auxiliary tensors:

Theorem 3.3. *If K is a generalized curvature tensor, u is a non null K-compatible vector ($u^2 \neq 0$) and v, w are vectors orthogonal to u ($u_a v^a = 0$, $u_a w^a = 0$), then:*

$$K_{abcd} v^a w^b u^c = 0 \tag{17}$$

Proof. The condition that u is K-compatible, $(u_a K_{bcde} + u_b K_{cade} + u_c K_{abde}) u^e = 0$, is contracted with $u^a v^b w^c$:

$$(u^a u_a) v^b w^c K_{bcde} u^e + u^a (u_b v^b) w^c K_{cade} u^e + u^a v^b (w^c u_c) K_{abde} u^e = 0.$$

The last two terms cancel because of orthogonality. □

Remark 3.4. *For the Riemann tensor, $R_{abcd} v^a w^b u^c$ is the vector obtained through parallel transport of u along a parallelogram with infinitesimal vectors v and w. It is known that, if $R_{abcd} v^a w^b u^c = 0$ for any v and w, then it is $R_{abcd} u^c = 0$. If u is Riemann compatible, then it has zero variations along infinitesimal parallelograms with directions orthogonal to it.*

Hereafter we concentrate on Riemann or Weyl compatible vectors. The identity (10) that relates the two properties for vectors is:

$$\begin{aligned}(u_a C_{bclm} + u_b C_{calm} + u_c C_{ablm}) u^m &= (u_a R_{bclm} + u_b R_{calm} + u_c R_{ablm}) u^m \\ &+ \frac{1}{n-2} \left[g_{cl} u_{[a} R_{b]m} + g_{al} u_{[b} R_{c]m} + g_{bl} u_{[c} R_{a]m} \right] u^m.\end{aligned} \tag{18}$$

A first consequence is the restatement of theorem 2.4 for vectors:

Proposition 3.5. *A vector field u is Riemann compatible if and only if it is Weyl compatible and $u_{[a} R_{b]}{}^m u_m = 0$.*

Remark 3.6. *The statement $u_{[a} R_{b]}{}^m u_m = 0$ is equivalent to the statement that u is an eigenvector of the Ricci tensor, $R_a{}^m u_m = \lambda u_a$. It follows that a null vector is Riemann compatible if and only if it is Weyl compatible.*

A second consequence is an extension of a theorem by Hall, which he proved for null vectors in $n = 4$ space-times [12]. It is valid in any dimension and metric signature, and for vectors not necessarily null:

Theorem 3.7. *Consider the following conditions on a vector field u:*

$$\begin{aligned} A)&\quad u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m = 0,\\ B)&\quad u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = 0,\\ C)&\quad u_{[a}R_{b]m}u^m = 0. \end{aligned}$$

Two conditions imply the third one. In particular, if $u^2 \neq 0$ the stronger statement holds: A is true if and only if B and C are true.

Proof. Eq.(18) is contracted with u^c,

$$u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m \tag{19}$$
$$+\frac{1}{n-2}\left[u_l u_{[a}R_{b]m}u^m + (g_{al}u_b - g_{bl}u_a)u^c u^m R_{cm} - u^2(g_{al}R_{bm} - g_{bl}R_{am})u^m\right].$$

If condition C is true, its contraction with u^b gives $(u_a u^b R_{bm} - u^2 R_{am})u^m = 0$ and (19) becomes $u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}k^m = u_{[a}R_{b]clm}u^c u^m + u^2 R_{ablm}u^m$. Therefore B and C imply A, or A and C imply B.

Suppose now that A is true; contraction of condition A by g^{al} gives $u^2 R_{bm}u^m - u_b(u^c u^m R_{cm}) = 0$, and (19) becomes:

$$u_{[a}C_{b]clm}u^c u^m + u^2 C_{ablm}u^m = \frac{1}{n-2}u_l u_{[a}R_{b]m}u^m \tag{20}$$

Validity of A and B imply that $u_l u_{[a}R_{b]m}u^m = 0$ i.e. C is true.

A stronger result holds if $u^2 \neq 0$. Contraction of (20) by u^l makes the left-hand-side vanish and condition C is true. Then, the same equation (20) states that also B is true, i.e. A implies B and C. □

Remark 3.8. *Condition C is met in Einstein spaces, defined by $R_{ab} - \frac{1}{n} R\, g_{ab} = 0$.*

Remark 3.9. *Condition B plays a special role in the classification of manifolds. Some cases where it holds are listed:*

1) $u^m R_{abcm} = 0$, [22, 30]*;*

2) k is a recurrent null vector, $\nabla_a k_b = \lambda_a k_b$, with closed λ_a ($\nabla_{[a}\lambda_{b]} = 0$), [31] *page 69;*

3) Manifold with constant curvature, [31] *page 101:*

$$R_{bclm} = \frac{R}{n(n-1)}(g_{bl}g_{cm} - g_{cl}g_{bm})$$

Proof. 1) The relation implies $R_{am}u^m = 0$ and then the whole r.h.s. of (18) is zero; then $(u_a C_{bclm} + u_b C_{calm} + u_c C_{ablm})u^m = 0$. Multiply by u^c and obtain A.

2) $[\nabla_a, \nabla_b]u_c = R_{abc}{}^m u_m$; because of recurrency and closedness, the l.h.s. is $\nabla_a(\lambda_b u_c) - \nabla_b(\lambda_a u_c) = (\lambda_b \nabla_a - \lambda_a \nabla_b)u_c = 0$. Then case 1) is obtained.

3) Contraction with g^{cm} shows that the manifold is Einstein, then condition $C1$ holds. If u is a vector, obtain $u_a R_{bclm}u^c u^m = \frac{R}{n(n-1)}u_a(g_{bl}u^2 - u_b u_l)$; then $u_{[a,}R_{b]clm}u^m = u^2\frac{R}{n(n-1)}(u_a g_{bl} - u_b g_{al}) = -u^2 R_{ablm}u^m$ i.e. condition B is true, and B and C imply A. □

In cases 1,2 the vector is Riemann compatible.

4. Permutable vectors

In the same way that compatibility is defined for vectors, permutability of a vector is defined by permutability of the tensor $u_i u_j$:

Definition 4.1. A vector is Riemann (Weyl) permutable if $R_{kl[i}{}^m u_{j]} u_m = 0$ ($C_{kl[i}{}^m u_{j]} u_m = 0$).

Remark 4.2. *If u is Riemann (Weyl) permutable and $u^2 \neq 0$, it follows that $R_{kljm} u^m = 0$ ($C_{kljm} u^m = 0$).*

A special class of Riemann permutable vectors is:

$$R_{abc}{}^m u_m = 0. \tag{21}$$

Null vectors of this sort describe gravitational waves in Einstein's linearized theory (see [30] page 244). A complete classification of space times that satisfy (21) is given in Theorem 1.1 of ref.[22].
Eq.(21) arises as the integrability condition for the equation $\nabla_a u_b + \nabla_b u_a = 2\lambda g_{ab}$ with constant λ and the constraint $\nabla_a u_b - \nabla_b u_a = 0$ (then u is a *homothetic* vector, see [31] pp. 69, 564).
Vectors that fullfill (21) also arise in the symmetric solution of the equation

$$R_{abc}{}^m x_{dm} + R_{abd}{}^m x_{cm} = 0 \tag{22}$$

which, by the Ricci identity, is equivalent to $[\nabla_a, \nabla_b] x_{cd} = 0$. The equation has a trivial solution $x_{ab} = \phi g_{ab}$ (ϕ a scalar). McIntosh and Halford [21] investigated spacetimes whose Riemann tensor admits a non trivial solution, such as Einstein spaces, Gödel metric, Bertotti-Robinson metric. It was then proven by McIntosh and Hall [13] that the only nontrivial solution is $x_{ab} = \alpha u_a u_b$, where u has the property (21) and α is a scalar field.
Besides the uniqueness stated above in $n = 4$, we prove in general:

Theorem 4.3. *If x_{ab} is a symmetric tensor that fulfills (22), X and Y are two eigenvectors, $X^m x_{cm} = \lambda X_c$ and $Y^m x_{cm} = \mu Y_c$ with $\lambda \neq \mu$, then:*
1) x_{ab} is Riemann (and Weyl) compatible;
2) $R_{abcm} X^c Y^m = 0$.

Proof. Summation on cyclic permutations of indices abc in (22) gives a vanishing term (first Bianchi identity) and Riemann compatibility. Property 2 is proven exactly as in Prop.2.9. □

5. Petrov types and Weyl compatible vectors

In 1954 Petrov classified $n = 4$ space-times according to the degeneracy of the eigenvalues of the self-dual part of the Weyl tensor. The eigenvalues solve an equation of degree four [27]. In type I spaces they are distinct, in type II spaces two are coincident and two are distint, in type D spaces they are pairwise coincident, type III spaces have three equal eigenvalues, and finally in type N spaces all eigenvalues coincide [30]. Type O spaces are conformally flat. The same types arise in the classification by Bel and Debever [1, 7], which is based on null vectors that solve increasingly restricted equations:

(23)	type I	$k_{[b}C_{a]rs[q}k_{n]}k^r k^s = 0$
(24)	type II, D	$k_{[b}C_{a]rsq}k^r k^s = 0$
(25)	type III	$k_{[b}C_{a]rsq}k^r = 0$
(26)	type N	$C_{arsq}k^r = 0$
(27)	type O	$C_{arsq} = 0$

When at least two vectors k are degenerate, i.e. k meets condition (24), the Weyl tensor is named *algebraically special* [28, 30].
The classification was generalized to $n > 4$ and contains the above relations as necessary conditions [5, 6, 15].

Let's consider the above classification in the perspective of Weyl compatibility. According to the general definition (15), a vector is Weyl compatible if

$$(u_i C_{jklm} + u_j C_{kilm} + u_k C_{ijlm})u^m = 0. \tag{28}$$

Theorem 5.1. *If a null vector k is Weyl compatible (or Riemann compatible), then the Weyl tensor is algebraically special.*

Proof. Multiply (28) by k^c and use the antisymmetry of Weyl's tensor:
$0 = (k_a C_{bcd}{}^m + k_b C_{cad}{}^m)k^c k_m = k_{[a}C_{b]cd}{}^m k^c k_m = -k_{[a}C_{b]cmd}k^c k^m$. □

For example, if a space-time admits a null concircular vector such that $\nabla_k u_l = A g_{kl} + B u_k u_l$, then the Weyl tensor is algebraically special.
The theorem extends Theorem 1.1 in [22], which holds for null vectors such that $R_{ijk}{}^m k_m = 0$.

Space-times with a null Weyl-compatible vector are Petrov type II or D. Are they more special than II or D? In general the answer is no. In a type III space-time there are three coincident principal directions, i.e. there is a null vector such that $k_{[b}C_{a]rsq}k^r = 0$. This means that the null k is Weyl-permutable, a property that implies Weyl compatibility (see def. 4.1):

Proposition 5.2. *If a null k solves* (25)*, which corresponds to $n = 4$ space-times of Petrov type III, then it is Weyl-permutable.*

Proposition 5.3. *If $u_k u_l$ is a Codazzi tensor and u_j is a closed 1-form, then:*
1) if u is a null vector, the Weyl tensor is algebraically special;
2) if u is a not null vector, the integral curves of u are geodesic lines.

Proof. 1) Codazzi tensors are Riemann compatibile, and thus Weyl compatible. If moreover $u^2 = 0$, prop. 5.1 applies. 2) From the Codazzi condition $\nabla_a(u_b u_c) = \nabla_b(u_a u_c)$, and closedness $\nabla_a u_b = \nabla_b u_a$, it follows that $u_b \nabla_a u_c - u_a \nabla_b u_c = 0$. Exchange a with c and subtract to obtain: $u_c \nabla_b u_a - u_a \nabla_b u_c = 0$. Multiply by $u^c u^b$:

$$(u^b \nabla_b) u_a = \left[\frac{u^c u^b \nabla_b u_c}{u^2}\right] u_a \tag{29}$$

i.e. the integral curves of u are geodesics (see [8], eqs. 2.9.4 and 2.9.5). □

6. Electric and magnetic tensors

In a $n = 4$ space-time the Weyl tensor has 10 independent components that can be accounted for by two symmetric tensors. Given a vector u with $u^a u_a = -1$, the electric and magnetic components of the Weyl tensor are:

$$E_{ab} = u^j u^m C_{jabm}\,, \qquad H_{ab} = u^j u^m \tilde{C}_{jabm} \tag{30}$$

where $\tilde{C}_{abcd} = \frac{1}{2}\epsilon_{abrs}C^{rs}{}_{cd}$ is the dual tensor. The two tensors are symmetric, traceless, and satisfy $E_{ab}u^b = 0$, $H_{ab}u^b = 0$. Then they each have 5 independent components, and completely describe the Weyl tensor.
If the electric and magnetic components are proportional, $\nu E = \mu H$ for some scalar fields μ and ν (including the case when one of them is zero), the space is type I, D or O [31] (page 73). We prove:

Theorem 6.1. *A vector u is Weyl-compatible if and only if $H = 0$.*

Proof. Consider the following chain of identities:

$$\begin{aligned} H^a{}_b &= u_j u^m \tilde{C}^{ja}{}_{bm} = \frac{1}{2} u_j u^m C_{rsbm}\epsilon^{jars} \\ &= \frac{1}{6}[u_j C_{rsbm}\epsilon^{jars} + u_r C_{sjbm}\epsilon^{rasj} + u_s C_{jrbm}\epsilon^{sajr}]u^m \\ &= \frac{1}{6}[u_j C_{rsbm} + u_r C_{sjbm} + u_s C_{jrbm}]u^m \epsilon^{jars} \end{aligned} \tag{31}$$

The equality shows that $H = 0$ is equivalent to Weyl compatibility. □

It then follows that a $n = 4$ space-time with a Weyl compatible time-like vector is type I, D or O. This extends Theorem 1.1 in [22].

If a spacetime admits a time-like concircular vector $\nabla_k u_l = A g_{kl} + B u_k u_l$, with constant A and B, then the magnetic part vanishes.

Theorem 6.2. *A $n = 4$ space-time with a non-null Weyl permutable vector is conformally flat, $C_{jkl}{}^m = 0$.*

Proof. Let E and H be the electric and magnetic components evaluated with u. If u is Weyl permutable, then it is Weyl compatible and $H = 0$. Let's show that also E is zero. Multiply the relation (28) for Weyl compatibility by u^j: $u^2 C_{kilm}u^m = u_k E_{il} - u_i E_{kl}$. Because u is Weyl permutable, it is $C_{kilm}u^m = 0$ (see remark 4.2); then $0 = u_k E_{il} - u_i E_{kl}$. Multiply by u^k and use $u^k E_{kl} = 0$ to obtain $E_{il} = 0$. □

The definitions of electric and magnetic components of the Weyl tensor can be generalized by replacing the symmetric tensor $u^i u^j$ by a symmetric tensor T^{ij}:

$$E_{ab} = T^{jm} C_{jabm}, \qquad H_{ab} = T^{jm}\tilde{C}_{jabm}. \tag{32}$$

Proposition 6.3. *E and H are symmetric and traceless.*

Proof. The first statement follows from the symmetry of C_{ijkl} or $\tilde{C}_{ijkl}$ in the exchange of ij with kl, and symmetry of T. The second follows from tracelessness of the Weyl tensor and its dual. □

Proposition 6.4.
1) If T is Weyl-compatible then E commutes with T;
2) $H = 0$ if and only if T is Weyl compatible.

Proof. The proof is based on the following identities:

$$E_{ab}T^b{}_c - T_a{}^b E_{bc} = -[T_{cb}C_{jam}{}^b + T_{ab}C_{cjm}{}^b + T_{jb}C_{acm}{}^b]T^{jm} \tag{33}$$

$$H^a{}_b = \frac{1}{6}[T_j{}^m C_{rsbm} + T_r{}^m C_{sjbm} + T_s{}^m C_{jrbm}]\epsilon^{jars} \tag{34}$$

The second identity is proven along the same line as (31). The first equation is proven here:

$$\begin{aligned} E_{ab}T^b{}_c - T_a{}^b E_{bc} =& [C_{jabm}T^b{}_c - C_{jbcm}T_a{}^b]T^{jm} \\ =& -[T^b{}_c C_{jamb} + T^b{}_a C_{cjmb} + T^b{}_j C_{acmb}]T^{jm} \\ =& -[T_{cb}C_{jam}{}^b + T_{ab}C_{cjm}{}^b + T_{jb}C_{acm}{}^b]T^{jm} \end{aligned}$$

where the last term added in the second line is identically zero. □

7. Hypersurfaces

Let $\mathscr{M}_n$ be a hypersurface in a pseudo Riemannian manifold $(\mathbb{V}_{n+1}, \tilde{g})$. The metric tensor (first fundamental form) is $g_{hj} = \tilde{g}(B_h, B_j)$, where $B_1 \dots B_n$ are the tangent vectors. If N is the vector normal to the hypersurface it is $\tilde{g}(B_h, N) = 0$. The Riemann tensor is given by the Gauss equation:

$$R_{jklm} = \tilde{R}_{\mu\nu\rho\sigma}B^\mu{}_j B^\nu{}_k B^\rho{}_l B^\sigma{}_m \pm (\Omega_{jl}\Omega_{km} - \Omega_{jm}\Omega_{kl}) \tag{35}$$

with a symmetric tensor Ω_{ij} (second fundamental form) constrained by the Codazzi equation:

$$\nabla_k \Omega_{jl} - \nabla_j \Omega_{kl} = N^\mu \tilde{R}_{\nu\mu\rho\sigma} B^\nu{}_j B^\rho{}_l B^\sigma{}_k \tag{36}$$

If $\mathbb{V}_{n+1}$ is a constant curvature manifold, the Gauss and Codazzi equations simplify:

$$R_{jklm} = \frac{\tilde{R}}{n(n+1)}(g_{jl}g_{km} - g_{jm}g_{kl}) \pm (\Omega_{jl}\Omega_{km} - \Omega_{jm}\Omega_{kl}), \tag{37}$$

$$\nabla_k \Omega_{jl} - \nabla_j \Omega_{kl} = 0, \tag{38}$$

If $\mathbb{V}_{n+1}$ is (pseudo)-Euclidean, the terms proportional to the scalar curvature $\tilde{R}$ vanish [31]. For this case Stephani proved that (37),(38) are sufficient conditions for a manifold $\mathscr{M}_4$ to have an embedding in $\mathbb{V}_5$, [31] (page 587). A general theorem by Goenner, restricted to pseudo-Euclidean manifolds $\mathbb{V}_{n+1}$, states that if Ω_{kl} is invertible, then it is a Codazzi tensor ([31], page 587). A simple proof of the same fact is here given, for the case of constant curvature $\mathbb{V}_{n+1}$:

Theorem 7.1. *If R_{jklm} has the form* (37) *and Ω is invertible, then Ω is a Codazzi tensor.*

Proof. The second Bianchi identity for the Riemann tensor is

$$\begin{aligned} \Omega_{mk}(\nabla_i\Omega_{jl} - \nabla_j\Omega_{li}) + \Omega_{mi}(\nabla_j\Omega_{kl} - \nabla_k\Omega_{lj}) + \Omega_{mj}(\nabla_k\Omega_{il} - \nabla_i\Omega_{lk}) + \\ + \Omega_{jl}(\nabla_i\Omega_{mk} - \nabla_k\Omega_{im}) + \Omega_{kl}(\nabla_j\Omega_{mi} - \nabla_i\Omega_{jm}) + \Omega_{il}(\nabla_k\Omega_{mj} - \nabla_j\Omega_{km}) = 0 \end{aligned}$$

Moltiplication by $(\Omega^{-1})^{km}$ gives:

$$\begin{aligned} &(n-3)(\nabla_i\Omega_{jl} - \nabla_j\Omega_{li}) \\ &= -(\Omega^{-1})^{km}[\Omega_{jl}(\nabla_i\Omega_{mk} - \nabla_k\Omega_{im}) + \Omega_{il}(\nabla_k\Omega_{mj} - \nabla_j\Omega_{km})]. \end{aligned}$$

Multiplication by $(\Omega^{-1})^{lj}$ gives: $2(n-2)(\Omega^{-1})^{lj}(\nabla_i\Omega_{jl}-\nabla_j\Omega_{li})=0$. This result is used to simplify the previous equation: $(n-3)(\nabla_i\Omega_{jl}-\nabla_j\Omega_{li})=0$, which for $n>3$ is the Codazzi property. □

Theorem 7.2. *Let $\mathscr{M}_n$ be a hypersurface isometrically embedded in a pseudo-Riemannian space $\mathbb{V}_{n+1}$ with constant curvature. Then:*
1) Ω is Weyl compatible;
2) the eigenvectors of Ω are Weyl compatible;
3) the Ricci tensor is Weyl compatible.

Proof. 1) For a hypersurface that is isometrically embedded in a constant curvature space, Ω_{ij} is a Codazzi tensor, and then it is both Riemann and Weyl compatible.
2) Given the form (37) of the Riemann tensor, if $\Omega_{km}u^m=\lambda u_k$ then:

$$u_iu^mR_{jklm}=ku_i(u_kg_{jl}-u_jg_{kl})\pm\lambda u_i(\Omega_{jl}u_k-u_j\Omega_{kl})$$

where, for shortness, $k=\tilde{R}/n(n+1)$. Summation on cyclic permutations of i,j,k cancels all terms in the right-hand-side, and one is left with Riemann compatibility: $u_iu^mR_{jklm}+u_ju^mR_{kilm}+u_ku^mR_{ijlm}=0$.
3) The Ricci tensor for a hypersurface isometrically embedded in a costant curvature space is $R_{kl}=\pm(\Omega^2_{kl}-\Omega_p{}^p\Omega_{kl})+k(n-1)g_{kl}$. Let us first show that Ω^2 is Riemann compatible. Evaluate the expression $(\Omega^2)_{im}R_{jkl}{}^m+(\Omega^2)_{jm}R_{kil}{}^m+(\Omega^2)_{km}R_{ijl}{}^m$ with the Riemann tensor (37). The first term is:

$$\Omega^2_{im}R_{jkl}{}^m=k\left(g_{jl}\Omega^2_{ik}-g_{kl}\Omega^2_{ij}\right)\pm\left(\Omega_{jl}\Omega^3_{ik}-\Omega_{kl}\Omega^3_{ij}\right)$$

While summing on cyclic permutations of ijk, all terms in the r.h.s cancel. Therefore the tensor Ω^2 is Riemann compatible and thus Weyl compatible. Since the Ricci tensor is the sum of Riemann - compatible terms, it is itself Riemann compatible, and thus Weyl compatible. □

By considering Einstein's equation (5) one also has:

Corollary 7.3. *In a space-time that is isometrically embedded as a hypersurface is a pseudo Riemannian space $\mathbb{V}_{n+1}$ with constant curvature, the energy momentum tensor is Weyl compatible.*
If the energy momentum tensor has the form $T_{kl}=au_ku_l+bg_{kl}$ with $u^iu_i=-1$, then the Weyl tensor is purely electric.

8. Geodesic maps

Let $(\mathscr{M},g)$ be a pseudo-Riemannian manifold. A geodesic map $\mathscr{M}\to\mathscr{M}$ induces a pseudo-Riemannian structure $(\mathscr{M},\tilde{g})$ with Christoffel symbols $\tilde{\Gamma}^k_{ij}=\Gamma^k_{ij}+\delta_i{}^kX_j+\delta_j{}^kX_i$, where X is a closed 1-form [11, 23, 29]. Accordingly, the new Riemann tensor is

$$\tilde{R}_{jkl}{}^m=R_{jkl}{}^m+\delta_j{}^mP_{kl}-\delta_k{}^mP_{jl}$$

with *deformation tensor* $P_{kl}=\nabla_kX_l-X_kX_l$. Since X is closed, the deformation tensor is symmetric. The new Ricci tensor is $\tilde{R}_{kl}=R_{kl}-(n-1)P_{kl}$.

In [19] we showed that for geodesic maps the following identity holds for any symmetric tensor:

$$b_{im}\tilde{R}_{jkl}{}^m+b_{jm}\tilde{R}_{kil}{}^m+b_{km}\tilde{R}_{ijl}{}^m=b_{im}R_{jkl}{}^m+b_{jm}R_{kil}{}^m+b_{km}R_{ijl}{}^m; \tag{39}$$

As a consequence, the property of Riemann compatibility is conserved. What about Weyl compatibility? For general symmetric tensors the answer is difficult by the fact that the expression (1) for the new Weyl tensor contains $\tilde{g}$, which is not simply related to g. This is a sufficient condition:

Proposition 8.1. *If a symmetric tensor b commutes with the Ricci and the deformation tensors, then*

$$b_{im}\tilde{C}_{jkl}{}^m + b_{jm}\tilde{C}_{kil}{}^m + b_{km}\tilde{C}_{ijl}{}^m = b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m \tag{40}$$

Proof. If b commutes with the Ricci and the deformation tensors, then it commutes with $\tilde{R}_{ij}$. With these conditions, (10) implies that

$$b_{im}C_{jkl}{}^m + b_{jm}C_{kil}{}^m + b_{km}C_{ijl}{}^m = b_{im}R_{jkl}{}^m + b_{jm}R_{kil}{}^m + b_{km}R_{ijl}{}^m$$

and the same relation with tensors $\tilde{C}_{jkl}{}^m$ and $\tilde{R}_{jkl}{}^m$. Since (39) holds for geodesic maps, (40) follows. □

A simplification occurs for *special geodesic maps*, defined by the property $P_{kl} = \gamma g_{kl}$, i.e. X is a *concircular vector*: $\nabla_k X_l - X_k X_l = \gamma\, g_{kl}$ [10].

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