THE WEYL'S CORRESPONDENCE - Part I

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Abstract. Weyl's correspondence is a rule for quantizing phase space functions; an equivalent and more direct rule was later given by Grossmann. Both rules provide operator expansions on continuous basis of operators. The latter allows a straightforward definition of the Wigner function for phase-space averaging in quantum mechanics.
The correspondence establishes an isomorphism between the Hilbert–Schmidt operator algebra and the space of square integrable functions on phase space equipped with the Moyal product. These properties are reviewed in a pedagogical way and explained simply in the frame of von Neumann's theorem.

It is well known that, due to the non–commutativity of fundamental operators, there are several ways to construct self–adjoint operators from real functions defined on the classical phase space, in the process of quantization. To remove this ambiguity, a rule of quantization must be added to the standard axioms of quantum mechanics. A successfull procedure is given by Hermann Weyl's correspondence, explained in his book *Gruppentheorie und Quantenmechanick* (1932), which I am here going to describe. In short, it states that operators have the same Fourier expansions as the corresponding classical functions, the exponential basis being replaced by Weyl operators, or shift operators in phase space. Among its many advantages are the preservation in a remarkable way of many formal properties of the classical description, and its staightforward relation with Wigner's function, introduced in an independent way and allowing the formulation of quantum mechanics in classical phase space.
The quantization procedure has been later improved by A. Grossmann (1976), who exploited the role of the parity operator, which in Weyl's quantization corresponds to a delta function in the origin of phase space. Grossmann's expansion of an operator reflects the expansion of classical functions in the basis of delta functions in phase space.
The purpose of this exposition is mainly pedagogical. A perhaps original contribution is the definition of two commuting representations of Heisenberg's commutation relations on the space of operators, which has a parallel on the space of classical functions. This establishes an interesting isomorphism and gives a natural interpretation of Weyl's correspondence.
For simplicity, only the one–dimensional case is analyzed, the extension to three or more dimensions being almost straightforward.

§1 Weyl's Commutation Relation.

The abstract Hilbert space $\mathcal{H}$ for a point spinless particle in 1d is defined by the requirement that it carries an irreducible representation of Weyl's commutation relations:

$$\hat{U}(t)\hat{V}(s) = \hat{V}(s)\hat{U}(t)\exp(i\hbar ts) \tag{1.1}$$

where $\hat{U}(t)$ and $\hat{V}(s)$ are two strongly continuous unitary groups, and t, s are real parameters. A fundamental theorem by von Neumann, from which we shall derive many consequences, states that all irreducible representations of Weyl's commutation relation are unitarily equivalent [ReSi].
Let $\hat{P}$, the "momentum" operator, and $\hat{Q}$, the "position" operator, be the generators of the two groups:

$$\hat{U}(t) = \exp(it\hat{P}) \quad ; \quad \hat{V}(s) = \exp(is\hat{Q}) \tag{1.2}$$

then there exists a common domain $\mathcal{D}$, dense in $\mathcal{H}$ and invariant under the action of the groups, such that the familiar Heisenberg's commutation relation holds in it:

$$\frac{1}{i\hbar}[\hat{P}, \hat{Q}] = -1 \tag{1.3}$$

A consequence of the commutation relation (1.3) is that $\hat{P}$ and $\hat{Q}$ cannot be both bounded operators [Gudd].
If we perform a parity transformation: $\hat{P}' = -\hat{P}$ and $\hat{Q}' = -\hat{Q}$ we get a new representation of Heisenberg's relation (1.3). According to von Neumann's theorem there exists a unitary operator $\hat{M}$, called Parity, such that:

$$\hat{M}^\dagger \hat{P} \hat{M} = -\hat{P} \quad , \quad \hat{M}^\dagger \hat{Q} \hat{M} = -\hat{Q} \tag{1.4}$$

This operator will play an essential role in the following. Since its square is unity, it may be chosen to be selfadjoint.

§2 Weyl Operators.

The fundamental building blocks for the process of quantization by Weyl, are the Weyl Operators. They are a family of unitary operators, parametrized by two real numbers:

$$\hat{W}(t, s) = \hat{U}(t)\hat{V}(s) \exp(-\frac{i}{2}\hbar ts) \tag{2.1}$$

They constitute a "ray group", or a group up to a phase factor. A ray is an equivalence class of vectors in Hilbert space, all having the same modulus. The product property is:

$$\hat{W}(t, s)\hat{W}(t', s') = \hat{W}(t + t', s + s') \exp\left[\frac{i}{2}\hbar(ts' - t's)\right] \tag{2.2}$$

This formula implies that, for fixed t and s, the operators $\hat{W}(\alpha t, \alpha s)$ form a group in the parameter α, with generator $t\hat{P} + s\hat{Q}$. Another consequence is the following useful relation

$$\hat{W}(t, s)\hat{W}(t', s') = \hat{W}(t', s')\hat{W}(t, s) \exp i\hbar(ts' - t's) \tag{2.3}$$

The matrix elements of a Weyl operator in the continuous basis of the position operator are:

$$\langle q'|\hat{W}(t, s)|q\rangle = \frac{1}{\hbar}\delta(t - \frac{q - q'}{\hbar})e^{is(q'+q)/2} \tag{2.4}$$

The inverse of a Weyl operator (which being unitary coincides with the adjoint) is $\hat{W}(t,s)^\dagger = \hat{W}(-t,-s)$ and it coincides also with $\hat{M}\hat{W}(t,s)\hat{M}$.

§3 Coherent States.

By introducing a complex phase-space coordinate

$$z = \frac{1}{\sqrt{2\hbar}}(q + ip) \tag{3.1}$$

and its complex conjugate, we may accordingly define the operators

$$\hat{a} = \frac{1}{\sqrt{2\hbar}}(\hat{Q} + i\hat{P}) \quad , \quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar}}(\hat{Q} - i\hat{P}) \quad , \quad \hat{N} = \hat{a}^*\hat{a} \tag{3.2}$$

respectively called annihilation, creation and number operators. The latter has a spectrum given by integers $n = 0, 1, 2 \ldots$ with eigenvectors $|n\rangle$. The basic formulae are:

$$[\hat{a}, \hat{a}^\dagger] = 1 \quad , \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \quad , \quad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle \tag{3.3}$$

A special class of vectors in $\mathcal{H}$ is given by Coherent States, which are in one-to-one correspondence with the points of the complex z-plane. They are defined as the solutions of the eigenvalue equation for the annihilation operator:

$$\hat{a}|z\rangle = z|z\rangle \tag{3.4}$$

The meaning of the parameter z is that of mean position in the complex plane, since $\langle z|\hat{Q}|z\rangle = q$ and $\langle z|\hat{P}|z\rangle = p$. An important feature of coherent states is that of being minimal uncertainty vectors: $\Delta Q = \Delta P = \sqrt{\hbar/2}$. Their expansion in the basis $|n\rangle$ is:

$$|z\rangle = \exp(-\frac{|z|^2}{2}) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle \tag{3.5}$$

In particular, the ground state $|0\rangle$ of the Number operator is a coherent state, centered in the origin of phase space. All coherent states may be generated from it by means of unitary operators, which is a Weyl Operators:

$$|z\rangle = \exp(z\hat{a}^\dagger - z^*\hat{a})|0\rangle = \hat{W}(-q/\hbar, p/\hbar)|0\rangle \tag{3.6}$$

Coherent states are never orthogonal to each other:

$$\langle z|w\rangle = \exp\left(-\frac{|z|^2}{2} - \frac{|w|^2}{2} + z^*w\right) \tag{3.7}$$

although the overlap is significant for distances less than $\sqrt{\hbar}$. They form an overcomplete set of vectors in $\mathcal{H}$

$$\int \frac{d^2z}{\pi}|z\rangle\langle z| = \hat{I} \tag{3.8}$$

where the measure is $d^2z = dpdq/2\hbar$. Not surprisingly, the subset labelled by points of a lattice in phase space with cell area not grater than $\hbar$ is also overcomplete [Barg]. The use of coherent states as a basis represents vectors of the Hilbert space as functions of complex variable $\psi(z) = \langle z|\psi\rangle$, that live in a Bargmann space (see Part II).

§4 Weyl's correspondence.

Weyl's correspondence is a procedure to define in a unique way the operators corresponding to functions on the classical phase space in the process of quantization. According to it, the quantum operator $\hat{F}$ which corresponds to a phase function $f(p,q)$ is built through the general prescription which goes through the Fourier transform of f

$$\begin{aligned} f(p,q) =& \frac{1}{2\pi}\int dxdy\tilde{f}(x,y)\exp i(xp+yq) \\ \hat{F} =& \frac{1}{2\pi}\int dxdy\tilde{f}(x,y)\hat{W}(x,y) \end{aligned} \tag{4.1}$$

where $\hat{W}$ is a Weyl operator. In both cases, one has a continuos expansion in two fundamental basis, exponential functions and Weyl operators, with the same weight function $\tilde{f}$. Shortly speaking, Weyl's rule states

$$\exp i(xp+yq) \quad \to \quad \exp i(x\hat{P}+y\hat{Q}) \tag{4.2}$$

The operator corresponding to monomials $p^m q^n$ is the sum of all possible products of m operators $\hat{P}$ and n operators $\hat{Q}$, divided by the total number of terms [Agarw1].

It has then been shown by Grossmann in 1976 [Gros] that the correspondence may be rephrased without the need of a Fourier transform, but directly relating the operator to the function as follows:

$$\hat{F} = \frac{1}{\pi\hbar}\int dpdqf(p,q)\hat{W}(-\frac{2q}{\hbar},\frac{2p}{\hbar})\hat{M} \tag{4.3}$$

The operator $\hat{M}$ is the parity operator. Again, we may summarize Weyl's quantization rule in the equivalent way:

$$\delta(x-p)\delta(y-q) \quad \to \quad \frac{1}{\pi\hbar}\hat{W}(-\frac{2q}{\hbar},\frac{2p}{\hbar})\hat{M} \tag{4.4}$$

The equivalence of (4.1) and (4.3) may be checked by inserting the expression of $\tilde{f}$ in terms of f into the expansion (4.1) for the operator $\hat{F}$:

$$\hat{F} = \frac{1}{\pi\hbar}\int dpdqf(p,q)\left[\frac{\hbar}{4\pi}\int dxdy\hat{W}(x,y)\exp -i(xp+yq)\right]$$

One then shows that the operator in square brackets has the same matrix elements between two position eigenstates as the operator in the right side of (4.4).

§5 Grossmann Operators and Moyal product.

The special role of the product of a Weyl Operator with Parity justifies the definition of Grossmann Operators:

$$\hat{G}(p,q) = \hat{W}\left(-\frac{2q}{\hbar}, \frac{2p}{\hbar}\right)\hat{M} \tag{5.1}$$

They share the same properties of the parity operators: they are unitary, self–adjoint, and so their square is unity:

$$\hat{G}(p,q) = \hat{G}(p,q)^* \quad , \quad \hat{G}(p,q)^2 = 1 \tag{5.2}$$

The product of an even number of Grossmann operators is a Weyl operator, and the product of a Weyl with a Grossmann operator is of Grossmann type. An important formula, which is easily evaluated by taking the trace in the position basis, is the following:

$$\left(\frac{1}{\pi\hbar}\right)^2 \mathrm{Tr}(\hat{G}(p_1,q_1)\hat{G}(p_2,q_2)) = \frac{1}{2\pi\hbar}\delta(p_1-p_2)\delta(q_1-q_2) \tag{5.3}$$

It implies that

$$\mathrm{Tr}(\hat{A}^*\hat{B}) = \frac{1}{2\pi\hbar}\int dpdq a^*(p,q)b(p,q) \tag{5.4}$$

in other words, provided that we define the Hilbert–Schmidt inner product between two operators with a prefactor equal to Planck's constant, which has the same dimensions as the measure in phase space $dpdq$

$$(\hat{A},\hat{B})_{HS} = 2\pi\hbar\mathrm{Tr}(\hat{A}^*\hat{B}) \tag{5.5}$$

Weyl's Correspondence is a unitary isomorphism between the Hilbert space of Hilbert–Schmidt Operators and the Hilbert space $L^2(\mathrm{R}^2)$ of phase space functions.

The isomorphism can be pushed further, at the algebraic level, once we define a *-product in the space of functions [Pool] as follows. Let us compute the function of phase space that corresponds to the product of two operators. Since only the product of an odd number of Grossmann operators is a Grossmann operators, it is convenient to multiply the Grossmann expansions of $\hat{F}\hat{G}\hat{I}$, where $\hat{I}$ is the identity operator. One finds

$$\hat{F}\hat{G} = \frac{1}{\pi\hbar}\int dpdq(f*g)(p,q)\hat{G}(p,q) \tag{5.6}$$

where:

$$(f*g)(p,q) = \left(\frac{1}{\pi\hbar}\right)^2\int dp_1dq_1dp_2dq_2 f(p_1+p,q_1+q)g(p_2+p,q_2+q)e^{\frac{2i}{\hbar}(q_1p_2-p_1q_2)} \tag{5.7}$$

defines a noncommutative product on phase space functions. The space $L^2(\mathrm{R})$ is closed for this product. Let us rewrite the formula in the following way:

$$(f*g)(p,q) = \left(\frac{1}{\pi\hbar}\right)^2\int dp_1dq_1dp_2dq_2 \exp\left[\frac{2i}{\hbar}(q_1p_2-p_1q_2)\right]$$
$$\exp\left[\left(p_1\frac{\partial}{\partial p'}+q_1\frac{\partial}{\partial q'}\right)+\left(p_2\frac{\partial}{\partial p''}+q_2\frac{\partial}{\partial q''}\right)\right]f(p',q')g(p'',q'')$$

where at the end of the computations: $q' = q'' = q$ and $p' = p'' = p$. The integrals can be done formally and we end with:

$$(f * g)(p,q) = \exp\left[i\frac{\hbar}{2}\left(\frac{\partial}{\partial q'}\frac{\partial}{\partial p''} - \frac{\partial}{\partial p'}\frac{\partial}{\partial q''}\right)\right] f(p',q')g(p'',q'') \tag{5.8}$$

The commutator $\{f,g\}_M = (\pi\hbar)^{-1}(f*g - g*f)$, that corresponds to $(i\hbar)^{-1}[\hat{F},\hat{G}]$, is known as the Moyal bracket between f and g [Moya]. In the limit $\hbar \to 0$ it identifies with the Poisson bracket $\{f,g\}_P$.
We have the following expansions for the Weyl correspondence for commutators and anti-commutators:

$$\frac{1}{i\hbar}[\hat{F},\hat{G}] \to \{f,g\}_P - \frac{\hbar^2}{24}\left[\frac{\partial^3 f}{\partial p^3}\frac{\partial^3 g}{\partial q^3} - 3\frac{\partial^3 f}{\partial p^2 \partial q}\frac{\partial^3 g}{\partial q^2 \partial p} - 3\frac{\partial^3 f}{\partial p \partial q^2}\frac{\partial^3 g}{\partial p^2 \partial q} + \frac{\partial^3 f}{\partial q^3}\frac{\partial^3 g}{\partial p^3}\right] + \dots \tag{5.9}$$

$$\frac{1}{2}(\hat{F}\hat{G} + \hat{G}\hat{F}) \to fg - \frac{\hbar^2}{8}\left[\frac{\partial^2 f}{\partial p^2}\frac{\partial^2 g}{\partial q^2} - 2\frac{\partial^2 f}{\partial p \partial q}\frac{\partial^2 g}{\partial p \partial q} + \frac{\partial^2 f}{\partial q^2}\frac{\partial^2 g}{\partial p^2}\right] + \dots \tag{5.10}$$

§6 Heisenberg's rules in $HS(\mathcal{H})$

The isomorphism defined through Weyl's Correspondence and the nature of Weyl's and Grossmann's expansions of operators are best explained by giving in the space of operators a representation of Heisenberg's rules suited for a phase space description.
The space $HS(\mathcal{H})$ of Hilbert-Schmidt operators on $\mathcal{H}$ consists of compact operators such that $\mathrm{Tr}(\hat{A}^\dagger \hat{A}) < \infty$. It is a Hilbert space with the inner product (5.5). It is also a C^*–Algebra, with *-conjugation given by the adjoint operation and the ordinary operator product.
An important dense subspace is that of "Trace Class operators"; they are the dual space of the Banach space $\mathcal{B}(\mathcal{H})$ of bounded operators.
Let us recall some basic definitions, that will be very useful in the following.

a) The product of two self–adjoint operators is generally not self–adjoint, one therefore defines a simmetrized operator product, called <u>Jordan product</u>,

$$\hat{A} \odot \hat{B} = \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}) \tag{6.1}$$

with the properties of being commutative, but not associative. Other properties are:
$\hat{A} \odot (\hat{B} \odot \hat{C}) - (\hat{A} \odot \hat{B}) \odot \hat{C} = \frac{1}{4}[[\hat{A},\hat{C}],\hat{B}]$
$(\hat{A} \odot \hat{B})^\dagger = \hat{A}^\dagger \odot \hat{B}^\dagger$

b) An important concept is that of <u>Derivation</u>: if $\hat{G} = \hat{G}^\dagger$ is a bounded operator, it is possible to define a derivation super-operator of $HS(\mathcal{H})$ into itself

$$\mathcal{D}_G : \hat{F} \to [\hat{G},\hat{F}] \tag{6.2}$$

The term derivation is due to the fact that formal properties of derivation are satisfied, like linearity and the Leibnitz rule.

$\mathcal{D}_G(\hat{A} \odot \hat{B}) = \mathcal{D}_G(\hat{A}) \odot \hat{B} + \hat{A} \odot \mathcal{D}_G(\hat{B})$
Two distinct derivations $\mathcal{D}_{G_1}$ and $\mathcal{D}_{G_2}$ commute if $[\hat{G}_1, \hat{G}_2]$ is proportional to the Identity.
The derivation is self-adjoint in the inner product of Hilbert–Schmidt.

c) If $\hat{U}$ is a unitary operator on $\mathcal{H}$, the left or right multiplications by $\hat{U}$ define a super-operator on $HS(\mathcal{H})$ which is unitary in the norm defined by the inner product of Hilbert-Schmidt.

Given a unitary representation of Weyl's rule on $\mathcal{H}$, it is easy to define a couple of commuting representations of Weyl's relation with unitary super-operators on $HS(\mathcal{H})$. Their action on a generic operator $\hat{A}$ is the following:

$$\mathcal{U}_t(\hat{A}) = \hat{V}^\dagger(t)\hat{A}\hat{V}(t) \quad ; \quad \mathcal{V}_s(\hat{A}) = \hat{U}^\dagger(s/2)\hat{A}\hat{U}(s/2) \tag{6.3}$$

These two super-operators satisfy

$$\mathcal{U}_t\mathcal{V}_s = \mathcal{V}_s\mathcal{U}_t e^{i\hbar ts} \tag{6.4}$$

The other pair, which commutes with the above one, is

$$\tilde{\mathcal{U}}_t(\hat{A}) = \hat{U}(t)\hat{A}\hat{U}^\dagger(t) \quad ; \quad \tilde{\mathcal{V}}_s(\hat{A}) = \hat{V}(s/2)\hat{A}\hat{V}(s/2) \tag{6.5}$$

and satisfies

$$\tilde{\mathcal{U}}_t\tilde{\mathcal{V}}_s = \tilde{\mathcal{V}}_s\tilde{\mathcal{U}}_t e^{i\hbar ts} \tag{6.6}$$

The generators of the two pairs of unitary groups of super-operator are self-adjoint superoperators (in the Hilbert Schmidt inner product), sharing a domain dense in $HS(\mathcal{H})$. They are respectively:

<u>Derivation and multiplication</u> by $\hat{Q}$

$$\mathcal{P}_q(\hat{A}) = [\hat{P}, \hat{A}] \quad ; \quad \mathcal{Q}_q(\hat{A}) = \hat{Q} \odot \hat{A} \tag{6.7}$$

<u>Derivation and multiplication</u> by $\hat{P}$

$$\mathcal{P}_p(\hat{A}) = [\hat{A}, \hat{Q}] \quad ; \quad \mathcal{Q}_p(\hat{A}) = \hat{P} \odot \hat{A} \tag{6.8}$$

The superoperators satisfy a representation of Heisenberg's relation "on phase space"

$$[\mathcal{P}_q, \mathcal{Q}_q] = -i\hbar \quad [\mathcal{P}_p, \mathcal{Q}_p] = -i\hbar \quad [\mathcal{P}_q, \mathcal{P}_p] = 0 \quad [\mathcal{Q}_q, \mathcal{Q}_p] = 0 \tag{6.9}$$

By the Theorem of von Neumann, there exists an isomorfism

$$\tau : HS(\mathcal{H}) \to \mathrm{L}^2(\mathrm{R}^2) \tag{6.10}$$

such that to any operator $\hat{F}$ there corresponds a function $f(p,q)$ with the following properties:

$$\frac{1}{i\hbar}[\hat{P}, \hat{F}] \to \{p, f\}_P = \frac{\partial f}{\partial q} \quad ; \quad \hat{Q} \odot \hat{F} \to qf \tag{6.11a}$$

$$\frac{1}{i\hbar}[\hat{F},\hat{Q}] \rightarrow \{f,q\}_P = \frac{\partial f}{\partial p} \quad ; \quad \hat{P} \odot \hat{F} \rightarrow pf \tag{6.11b}$$

The isomorphism is precisely Weyl's correspondence, which maps the algebra of classical phase space functions $L^2(\mathrm{R})$ on the space of Hilbert Schmidt quantum operators. The map can be given explicitly through the correspondence between elements of complete sets of proper or generalized vectors of both spaces, like in relations (4.2) and (4.4). This correspondence is discussed in the next section.

§7 Weyl's Correspondence revisited

Like in quantum mechanics, with formulae (6.11a and b) we have introduced in the space $L^2(\mathrm{R})$ of phase–space functions $f(p,q)$ the operators of multiplication by p and q, as well as derivations by p and q. The generalized eigenfunctions common to derivations or multiplications are respectively exponential functions and delta functions:

$$-i\frac{\partial}{\partial q}e^{i(xp+yq)} = ye^{i(xp+yq)} \quad , \quad -i\frac{\partial}{\partial p}e^{i(xp+yq)} = xe^{i(xp+yq)} \tag{7.1}$$

$$q\delta(q-a)\delta(p-b) = a\delta(q-a)\delta(p-b) \quad , \quad p\delta(q-a)\delta(p-b) = b\delta(q-a)\delta(p-b) \tag{7.2}$$

In the same way, Weyl and Grossmann Operators may be viewed as 'eigen-operators' of the super-operators given in §6 that provide two commuting representations of Heisenberg's relations. More precisely, the Weyl Operators are a basis of operators common to the two derivations and correspond, through the isomorphism, to exponentials, the basis common to derivations on classical phase space functions:

$$\mathcal{P}_q(\hat{W}(x,y)) = [\hat{P},\hat{W}(x,y)] = \hbar x\hat{W}(x,y) \tag{7.3a}$$

$$\mathcal{P}_p(\hat{W}(x,y)) = [\hat{W}(x,y),\hat{Q}] = \hbar y\hat{W}(x,y) \tag{7.3b}$$

The Grossmann operators, or Weyl operators multiplied by parity, play a role analogous to that of delta functions centered on points of classical phase space:

$$\mathcal{Q}_q(\hat{G}(p,q)) = \hat{Q} \odot \hat{G}(p,q) = q\hat{G}(p,q) \tag{7.4a}$$

$$\mathcal{Q}_p(\hat{G}(p,q)) = \hat{P} \odot \hat{G}(p,q) = p\hat{G}(p,q) \tag{7.4b}$$

These two relations show that the operators $\hat{G}(p,q)$ are a basis of operators common to the position super-operators.

The equation (5.3) shows that the correct normalization in the Hilbert–Schmidt norm for Grossmann operators to form a continuous basis of operators is the prefactor $(i\hbar)^{-1}$. We then may write symbolically

$$\delta_2(\hat{P}-p,\hat{Q}-q) = \frac{1}{\pi\hbar}\hat{G}(p,q) \tag{7.5}$$

and obtain the continuous expansion for an operator in the form given by Grossmann:

$$\hat{F} = \int dpdqf(p,q)\delta_2(\hat{P}-p,\hat{Q}-q) \tag{7.6}$$

where

$$f(p,q) = \frac{1}{\pi\hbar}\left(\hat{G}(p,q),\hat{F}\right)_{HS} = 2\mathrm{Tr}\left[\hat{G}(p,q)\hat{F}\right] \tag{7.7}$$

Weyl's correspondence states that f is precisely the classical function (if $\hbar$-independent) that would quantize into the operator $\hat{F}$.

§8 Wigner's function.

The definition of Wigner's function is strictly related to Weyl's correspondence, although hystorically it has been introduced in an independent way [Wign]. It is a distribution in phase space associated to a density matrix that allows to perform quantum mechanical computations in phase space. Contrary to a classical distribution, it may take negative values. In the frame of Weyl's transform its definition is straightforward and follows from the formula by Grossmann. If $\hat{\rho}$ is a statistical operator and $\hat{F}$ is the quantum operator corresponding to the classical observable $f(p,q)$, the average value is given by

$$\langle F\rangle = \mathrm{Tr}[\hat{\rho}\hat{F}] = \frac{1}{\pi\hbar}\int dpdq f(p,q)\mathrm{Tr}[\hat{\rho}\hat{G}(p,q)] = \int dpdq f(p,q)\rho(p,q) \tag{8.1}$$

The function

$$\rho(p,q) = \frac{1}{\pi\hbar}\mathrm{Tr}[\hat{\rho}\hat{G}(p,q)] \tag{8.2}$$

is Wigner's function. Note that it does not coincide with the classical function that would correspond according to Weyl to the operator $\hat{\rho}$. In the case of a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ it takes a particularly simple form:

$$\rho_\psi(p,q) = \frac{1}{\pi\hbar}\langle\psi|\hat{G}(p,q)|\psi\rangle \tag{8.3}$$

The following properties, which are to be expected from any phase–space density associated to a pure state, are easily verified:

$$\int \rho_\psi(p,q)dp = |\psi(q)|^2 \quad ; \quad \int \rho_\psi(p,q)dq = |\tilde{\psi}(p)|^2 \tag{8.4}$$

In the position representation of ψ Wigner's function takes the familiar expression

$$\rho_\psi(p,q) = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} dx e^{2ipx/\hbar}\psi(x+q)^*\psi(q-x) \tag{8.5}$$

For a coherent state $|z\rangle$ centered in (p_0,q_0), the expression is particularly simple:

$$\rho_z(p,q) = \frac{1}{\pi\hbar}e^{-\frac{i}{\hbar}[(q-q_0)^2+(p-p_0)^2]} \tag{8.6}$$

When $\hbar$ goes to zero, the function ρ_z converges to $\delta(q-q_0)\delta(p-p_0)$. The average value of an observable taken on a coherent state

$$\langle\alpha|\hat{F}|\alpha\rangle = \frac{1}{\pi\hbar}\int dpdq f(p,q)e^{-\frac{i}{\hbar}[(q-q_0)^2+(p-p_0)^2]}$$

has a nice interpretation as a diffusion process over a time $t = \hbar/4$.

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