EFFECTIVE INTERACTION AND POLARIZATION
NOTES BY L. G. MOLINARI

1. The effective interaction

In a many body system the "bare" interaction U^0 between two particles in empty space is "dressed" by polarization insertions. I give here a derivation of the expression of the polarization based on the diagrammatic expansion of the propagator.

The one-particle Green function (propagator) $iG(x,y) = \langle E_0|TS\psi(x)\psi^\dagger(y)|E_0\rangle_\star$ is the sum of all diagrams with a particle being created at y and destroyed at x that do not contain vacuum factors (spin is included in the variable. For the interaction each variable has a spin pair). The first-order rainbow diagram is

$$\frac{i}{\hbar}\int dx_1 dx_2 G^0(x,x_1)G^0(x_1,x_2^+)G^0(x_2,y)U^0(x_1,x_2)$$

If we select the diagrams of $G(x,y)$ with fixed configuration of three bare propagators, the sum of such diagrams defines the effective interaction $U(x_1,x_2)$:

$$\frac{i}{\hbar}\int dx_1 dx_2 G^0(x,x_1)G^0(x_1,x_2)G^0(x_2,y)U(x_1,x_2)$$

The zero-order term of $U(x_1,x_2)$ is $U^0(x_1,x_2)$. The next ones arise in diagrams of second and higher orders of $G(x,y)$, and necessarily contain two U^0 lines as follows:

$$\frac{i}{\hbar}\int dx_1 dx_2 dy_1 dy_2\; G^0(x,x_1)G^0(x_1,x_2)G^0(x_2,y)\left[U^0(x_1,y_1)\Pi(y_1,y_2)U^0(y_2,x_2)\right]$$

$\Pi(y_1,y_2)$ is the polarization, that sums all possible insertions among the two factors U^0. Therefore, the effective potential is:

$$U(x_1,x_2) = U^0(x_1,x_2) + \int dy_1 dy_2 U^0(x_1,y_1)\Pi(y_1,y_2)U^0(y_2,x_2) \tag{1}$$

The expression for Π is now derived by considering the diagrams associated to contractions of $G(x,y)$ that maintain the product of three bare propagators. Accordingly, we write the expansion of S from second order, where only $V(t_1)$ and $V(t_2)$ are specified in second quantization[1] and the contractions $\psi(x)$ with $\psi^\dagger(x_1)$, $\psi(x_1)$ with $\psi^\dagger(x_2)$ and $\psi(x_2)$ with $\psi^\dagger(y)$ are frozen:

$$\frac{1}{i}\sum_{N=2}^{\infty}\frac{1}{(i\hbar)^N}\int dx_1 dx_2 dy_1 dy_2 U^0(x_1,y_1)U^0(y_2,x_2)\int_{-\infty}^{+\infty}dt_2' \ldots\int_{-\infty}^{+\infty}dt_{N-1}'\times$$
$$\langle T\psi^\dagger(x_1)\psi^\dagger(y_1)\psi(y_1)\psi(x_1)V(t_2')\ldots V(t_{N-1}')\psi^\dagger(x_2)\psi^\dagger(y_2)\psi(y_2)\psi(x_2)\psi(x)\psi^\dagger(y)\rangle_{\star C}$$

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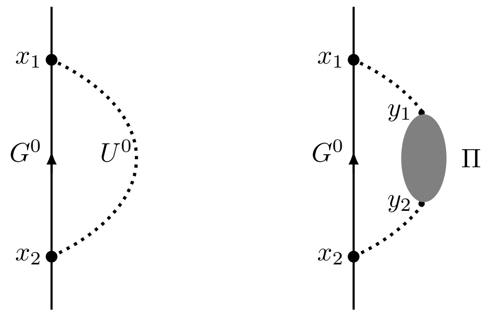
[1] Note the absence of factorials and powers of 2, as we only consider topologically different diagrams.

The star means that we avoid diagrams with vacuum factors; C means making only contractions linking x_1 to x_2.



1st diagram with bare interaction and with polarization insertions.

The frozen contractions $iG^0(x, x_1)$, $iG^0(x_1, x_2)$, $iG^0(x_2, y)$ are specified:

$$\frac{1}{i}\frac{i^3}{(i\hbar)^2}\int dx_1 dx_2 G^0(x, x_1)G^0(x_1, x_2)G^0(x_2, y)\int dy_1 dy_N U^0(x_1, y_1)U^0(y_2, x_2)$$
$$\times\sum_{k=0}^{\infty}\frac{1}{(i\hbar)^k}\int dt_1...dt_k\langle T\psi^\dagger(y_1)\psi(y_1)V(t_1)...V(t_k)\psi^\dagger(y_2)\psi(y_2)\rangle_{\star C}$$
$$= \frac{i}{\hbar}\int dx_1 dx_2 G^0(x, x_1)G^0(x_1, x_2)G^0(x_2, y)$$
$$\times\left[\frac{1}{i\hbar}\int dy_1 dy_2 U^0(x_1, y_1)U^0(x_2, y_2)\langle E_0|TS\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_2)\psi(y_2)|E_0\rangle_{\star C}\right]$$

We leave the interaction picture:

$$\langle E_0|TS\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_2)\psi(y_2)|E_0\rangle_{\star C} = \langle E|T\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_2)\psi(y_2)|E\rangle_C$$

The unwanted contractions are those that factor (disconnect) into a function of y_1 and a function of y_2[2].

(2) $$\langle E|T\psi^\dagger_\mu(x)\psi_{\mu'}(x)\psi^\dagger_\nu(y)\psi_{\nu'}(y)|E\rangle_C = \langle E|T\delta[\psi^\dagger_\mu(x)\psi_{\mu'}(x)]\,\delta[\psi^\dagger_\nu(y)\psi_{\nu'}(y)]|E\rangle$$

We now write the expression of the effective potential (1) in full detail:

(3) $$U_{\mu\mu',\nu\nu'}(x_1, x_2) = U^0_{\mu\mu',\nu\nu'}(x_1, x_2) + \sum_{\rho\rho'\sigma\sigma'}\int dy_1 dy_2\, U^0_{\mu\mu',\rho\rho'}(x_1, y_1) \times\Pi_{\rho\rho',\sigma\sigma'}(y_1, y_2)U^0_{\sigma\sigma',\nu\nu'}(y_2, x_2)$$

(4) $$\boxed{\Pi_{\rho\rho',\sigma\sigma'}(x, y) = \frac{1}{i\hbar}\langle E|T\psi^\dagger_\rho(x^+)\psi_{\rho'}(x)\psi^\dagger_\sigma(y^+)\psi_{\sigma'}(y)|E\rangle_C}$$

[2] A 2-point correlator can be decomposed into connected and disconnected parts:

$$\langle E|TA(x)B(y)|E\rangle = \langle E|TA(x)B(y)|E\rangle_C + \langle E|A(x)|E\rangle\langle E|B(y)|E\rangle$$

By defining $\delta A(x) \equiv A(x) - \langle E|A(x)|E\rangle$, the connected correlator is

$$\langle E|TA(x)B(y)|E\rangle_C = \langle E|T\delta A(x)\delta B(y)|E\rangle$$

The polarization $\Pi_{\rho\rho',\sigma\sigma'}(x,y)$ is the sum of all topologically distinct connected diagrams where a particle is being created with spin ρ and one destroyed with spin ρ' at the space-time point x and another pair of similar events occurs at y.

Because of time-ordering, the polarization is symmetric: $\Pi_{\rho\rho',\sigma\sigma'}(x,y) = \Pi_{\sigma\sigma'\rho\rho'}(y,x)$. The symmetry implies that *the exchange symmetry of the bare interaction U^0 is inherited by the effective potential*:

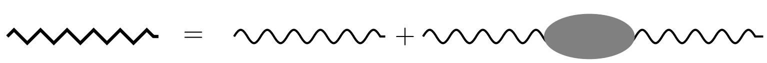
$$U_{\mu\mu',\nu\nu'}(x,y) = U_{\nu\nu',\mu\mu'}(y,x) \tag{5}$$

If the bare interaction does not modify the spin of the particles, i.e. $U^0_{\mu\mu'\nu\nu'}(x,y) = \delta_{\mu\mu'}\delta_{\nu\nu'}U^0(x,y)$, then the same property holds for the effective interaction: $U_{\mu\mu'\nu\nu'}(x,y) = \delta_{\mu\mu'}\delta_{\nu\nu'}U(x,y)$.
In this case, equation (3) is:

$$U(x_1,x_2) = U^0(x_1,x_2) + \sum_{\rho\sigma}\int dy_1 dy_2 U^0(x_1,y_1)\Pi(y_1,y_2)U^0(y_2,x_2) \tag{6}$$

$$\boxed{\Pi(x,y) = \sum_{\mu\nu}\Pi_{\mu\mu,\nu\nu}(x,y) = \frac{1}{i\hbar}\langle E|T\delta n(x)\delta n(y)|E\rangle} \tag{7}$$

$\Pi(x,y)$ is the scalar polarization.



2. Proper polarization

The polarization diagrams may be reordered as

$$\Pi = \Pi^\star + \Pi^1 + \Pi^2 \dots$$

where $\Pi^\star$ is the sum of *proper* or *irreducible* polarization diagrams, i.e. diagrams that cannot be disconnected into two polarisation diagrams by removal of a single U^0 line. Π^1 is the sum of polarization diagrams that may be disconnected (into polarization diagrams) in a unique way, i.e. there is just one line U^0 whose removal disconnects the diagram into two proper ones ($\Pi^1 = \Pi^\star U^0 \Pi^\star$), and so on. Therefore:

$$\begin{aligned}\Pi =& \Pi^\star + \Pi^\star U^0\Pi^\star + \Pi^\star U^0\Pi^\star U^0\Pi^\star + \dots\\ =& \Pi^\star + \Pi^\star U^0(\Pi^\star + \Pi^\star U^0\Pi^\star + \dots)\\ =& \Pi^\star + \Pi^\star U^0\Pi.\end{aligned}$$

In the same way one obtains $\Pi = \Pi^\star + \Pi U^0\Pi^\star$. These are the Dyson's equations for the polarization Π, in terms of the proper polarization. One of them is:

$$\boxed{\Pi(x_1,x_2) = \Pi^\star(x_1,x_2) + \int dx_3x_4\; \Pi^\star(x_1,x_3)U^0(x_3,x_4)\Pi(x_4,x_2)} \tag{8}$$

As a consequence, one obtains a Dyson equation for the effective interaction in terms of the proper polarization:

$$\boxed{U(x_1,x_2) = U^0(x_1,x_2) + \int dx_3dx_4\; U^0(x_1,x_3)\Pi^\star(x_3,x_4)U(x_4,x_2)} \tag{9}$$

Exercise 2.1. *Show that* $U(1,2) = U^0(1,2) + \int d3\, d4\, U(1,3)\Pi^\star(3,4)U^0(4,2)$ *and* $\int d3\, U^0(1,3)\Pi(3,2) = \int d3\, U(1,3)\Pi^\star(3,2)$.

Exercise 2.2. *Show that* $\int d1\, \Pi(1,2) = 0$. *Does this imply* $\int d1\, \Pi^*(1,2) = 0$*?*

3. Space-time translation invariance

If both U^0 and Π are space-time translation-invariant (i.e. $f(x+y, x'+y) = f(x,x')$ for all y), then also U and $\Pi^\star$ are invariant.
It is convenient to expand the functions in $k = (\mathbf{k}, \omega)$ space:

$$f(x,x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} f(k), \qquad kx = \mathbf{k}\cdot\mathbf{x} - \omega t$$

The Dyson equations become algebraic:

$$U(k) = U^0(k) + U^0(k)\Pi^\star(k)U(k)$$
$$\Pi(k) = \Pi^\star(k) + \Pi^\star(k)U^0(k)\Pi(k)$$

They are matrix equations in spin variables. If they are scalar equations, the solutions are:

$$\boxed{U(k) = \frac{U^0(k)}{1 - U^0(k)\Pi^\star(k)}\,, \qquad \Pi(k) = \frac{\Pi^\star(k)}{1 - U^0(k)\Pi^\star(k)}} \tag{10}$$

For a static two-particle potential $U^0(x,x') = v(\mathbf{x}-\mathbf{x}')\delta(t-t')$, it is $U^0(k) = v(\mathbf{k})$. Then:

$$U(\mathbf{k},\omega) = \frac{v(\mathbf{k})}{\epsilon(\mathbf{k},\omega)}, \qquad \Pi(\mathbf{k},\omega) = \frac{\Pi^*(\mathbf{k},\omega)}{\epsilon(\mathbf{k},\omega)} \tag{11}$$

$$\boxed{\epsilon(\mathbf{k},\omega) = 1 - v(\mathbf{k})\Pi^\star(\mathbf{k},\omega)} \tag{12}$$

$\epsilon(k)$ is the (time ordered) *generalised dielectric function*.
Despite the bare interaction being static, the effective interaction is time-dependent through the dielectric function, which describes the response of the medium. For the Coulomb interaction,

$$U(\mathbf{k},\omega) = \frac{4\pi e^2}{|\mathbf{k}|^2 - 4\pi e^2 \Pi^\star(\mathbf{k},\omega)}$$

The long-range Coulomb interaction is modified by the screening produced by the polarized medium.