EFFECTIVE INTERACTION AND POLARIZATION

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1. The effective interaction

In a many body system the "bare" interaction of two particles in empty space U_0 is "dressed" by polarization insertions.

The one-particle Green function $iG(x,y) = \langle E_0|TS\psi(x)\psi^\dagger(y)|E_0\rangle_\star$ is the sum of all diagrams with a particle being created at y and destroyed at x that do not contain vacuum factors (spin is included in the variable). Its first-order rainbow diagram is

$$\frac{i}{\hbar}\int dx_1 dx_2 G^0(x,x_1)G^0(x_1,x_2)G^0(x_2,y)U^0(x_1,x_2)$$

Among the diagrams at higher order we select and sum only the ones that dress the interaction, keeping fixed the configuration of three bare propagators. The sum of such diagrams defines the effective interaction $U(x,y)$:

$$\frac{i}{\hbar}\int dx_1 dx_2 G^0(x,x_1)G^0(x_1,x_2)G^0(x_2,y)U(x_1,x_2)$$

The first term of $U(x,y)$ is $U^0(x,y)$. The next ones arise as all possible contractions in the $T-$expansion of S in the propagator. Here, only $V(t_1)$ and $V(t_N)$ are written in second quantization)[1]:

$$\frac{1}{i}\sum_{N=2}^{\infty}\frac{1}{(i\hbar)^N}\int dx_1 dy_1 dx_N dy_N U^0(x_1,y_1)U^0(x_N,y_N)\int_{-\infty}^{+\infty}dt_2 ... \int_{-\infty}^{+\infty}dt_{N-1}$$
$$\times \langle T\psi^\dagger(x_1)\psi^\dagger(y_1)\psi(y_1)\psi(x_1)V(2)...V(N-1)\psi^\dagger(x_N)\psi^\dagger(y_N)\psi(y_N)\psi(x_N)\psi(x)\psi^\dagger(y)\rangle_{\star C}$$

with the following *fixed contractions*: $x-x_1$, x_1-x_N and x_N-y.
The star means that we avoid diagrams with vacuum factors, the C means that we avoid contractions that do not link 1 to N (we want an effective potential among points $1,N$).

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[1]Note the absence of factorials and powers of 2, as we only consider topologically different diagrams.

The fixed contractions are done with proper signs and factors:

$$\frac{1}{i}\frac{i^3}{(i\hbar)^2}\int dx_1 dx_N G^0(x,x_1)G^0(x_1,x_N)G^0(x_N,y)\int dy_1 dy_N U^0(x_1,y_1)U^0(x_N,y_N)$$
$$\times\sum_{k=0}^{\infty}\frac{1}{(i\hbar)^k}\int dt_1...dt_k\langle T\psi^\dagger(y_1)\psi(y_1)V(t_1)...V(t_k)\psi^\dagger(y_N)\psi(y_N)\rangle_{\star C}$$
$$=\frac{i}{\hbar}\int dx_1 dx_N G^0(x,x_1)G^0(x_1,x_N)G^0(x_N,y)\frac{1}{i\hbar}\int dy_1 dy_N U^0(x_1,y_1)U^0(x_N,y_N)$$
$$\times\langle E_0|TS\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_N)\psi(y_N)|E_0\rangle_{\star C}$$

We leave the interaction picture:

$$\langle E_0|TS\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_N)\psi(y_N)|E_0\rangle_{\star C}=\langle E|T\psi^\dagger(y_1)\psi(y_1)\psi^\dagger(y_N)\psi(y_N)|E\rangle_C$$

The unwanted contractions are those that factor (disconnect) into a function of y_1 and a function of y_N. They are dealt with as follows.
A 2-point correlator can be decomposed into connected and disconnected parts:

$$\langle E|TA(x)B(y)|E\rangle=\langle E|TA(x)B(y)|E\rangle_C+\langle E|A(x)|E\rangle\langle E|B(y)|E\rangle$$

By defining $\delta A(x)\equiv A(x)-\langle E|A(x)|E\rangle$, the connected correlator is

$$\langle E|TA(x)B(y)|E\rangle_C=\langle E|T\delta A(x)\delta B(y)|E\rangle_C$$

In our specific case it is:

$$\langle E|T\psi^\dagger_\mu(x)\psi_{\mu'}(x)\psi^\dagger_\nu(y)\psi_{\nu'}(y)|E\rangle_C=\langle E|T\delta[\psi^\dagger_\mu(x)\psi_{\mu'}(x)]\,\delta[\psi^\dagger_\nu(y)\psi_{\nu'}(y)]|E\rangle \tag{1}$$

We now write the expression of the effective potential in full detail:

$$U_{\mu\mu',\nu\nu'}(x_1,x_2)=U^0_{\mu\mu',\nu\nu'}(x_1,x_2)+\frac{1}{i\hbar}\sum_{spin}\int dx_3dx_4U^0_{\mu\mu',\rho\rho'}(x_1,x_3)$$
$$\times\langle E|T\psi^\dagger_\rho(x_3^+)\psi_{\rho'}(x_3)\psi^\dagger_\sigma(x_4^+)\psi_{\sigma'}(x_4)|E\rangle_C U^0_{\sigma\sigma',\nu\nu'}(x_4,x_2) \tag{2}$$

The connected correlator defines the Polarization:

$$\boxed{U(1,2)=U^0(1,2)+\int d3d4\,U^0(1,3)\,\Pi(3,4)\,U^0(4,2)} \tag{3}$$

$$\Pi_{\rho\rho',\sigma\sigma'}(x,y)=\frac{1}{i\hbar}\langle E|T\psi^\dagger_\rho(x^+)\psi_{\rho'}(x)\psi^\dagger_\sigma(y^+)\psi_{\sigma'}(y)|E\rangle_C \tag{4}$$

The polarization $\Pi_{\rho\rho',\sigma\sigma'}(x,y)$ is the sum of all topologically distinct connected diagrams where a particle is being created with spin ρ and one destroyed with spin ρ' at the space-time point x and another pair of similar events occurs at y.

The polarization is symmetric: $\Pi_{\rho\rho',\sigma\sigma'}(x,y)=\Pi_{\sigma\sigma'\rho\rho'}(y,x)$. The symmetry implies that *the exchange symmetry of the bare interaction U^0 is inherited by the effective potential*:

$$U_{\mu\mu',\nu\nu'}(x,y)=U_{\nu\nu',\mu\mu'}(y,x) \tag{5}$$

If the bare interaction does not modify the spin of the particles, i.e. $U^0_{\mu\mu'\nu\nu'}(x,y)=\delta_{\mu\mu'}\delta_{\nu\nu'}U^0(x,y)$, then the same form holds for the effective interaction: $U_{\mu\mu'\nu\nu'}(x,y)=\delta_{\mu\mu'}\delta_{\nu\nu'}U(x,y)$.

Equation (2) now is:

$$U(x_1,x_2) = U^0(x_1,x_2) + \sum_{\rho\sigma}\int dx_3dx_4U^0(x_1,x_3)\Pi(x_3,x_4)U^0(x_4,x_2) \tag{6}$$

$$\boxed{\Pi(x,y) = \sum_{\mu\nu}\Pi_{\mu\mu,\nu\nu}(x,y) = \frac{1}{i\hbar}\langle E|T\delta n(x)\delta n(y)|E\rangle} \tag{7}$$

$\Pi(x,y)$ is the scalar polarization.

2. Proper polarization

The polarization diagrams may be reordered as

$$\Pi = \Pi^\star + \Pi^1 + \Pi^2 \ldots$$

where $\Pi^\star$ is the sum of *proper* or *irreducible* polarization diagrams, i.e. diagrams that cannot be disconnected into two polarisation diagrams by removal of a single U^0 line. Π^1 is the sum of polarization diagrams that may be disconnected (into polarization diagrams) in a unique way, i.e. there is just one line U^0 whose removal disconnects the diagram into two proper ones ($\Pi^1 = \Pi^\star U^0\Pi^\star$), and so on. Therefore:

$$\begin{aligned}\Pi =&\Pi^\star + \Pi^\star U^0\Pi^\star + \Pi^\star U^0\Pi^\star U^0\Pi^\star + \ldots\\ =&\Pi^\star + \Pi^\star U^0(\Pi^\star + \Pi^\star U^0\Pi^\star + \ldots)\\ =&\Pi^\star + \Pi^\star U^0\Pi.\end{aligned}$$

In the same way one obtains $\Pi = \Pi^\star + \Pi U^0\Pi^\star$. These are the Dyson's equations for the polarization Π, in terms of the proper polarization. One of them is:

$$\boxed{\Pi(x_1,x_2) = \Pi^\star(x_1,x_2) + \int dx_3x_4\, \Pi^\star(x_1,x_3)U^0(x_3,x_4)\Pi(x_4,x_2)} \tag{8}$$

As a consequence, one obtains a Dyson equation for the effective interaction in terms of the proper polarization:

$$\boxed{U(x_1,x_2) = U^0(x_1,x_2) + \int dx_3dx_4\, U^0(x_1,x_3)\Pi^\star(x_3,x_4)U(x_4,x_2)} \tag{9}$$

Exercise 2.1. *Show that* $U(1,2) = U^0(1,2) + U(1,3)\Pi^\star(3,4)U^0(4,2)$ *and* $\int d3\, U^0(1,3)\Pi(3,2) = \int d3\, U(1,3)\Pi^\star(3,2)$.

Exercise 2.2. *Show that* $\int d1\Pi(1,2) = 0$. *Does this imply that* $\int d1\Pi^*(1,2) = 0$*?*

3. Space-time translation invariance

If both U^0 and Π are space-time translation-invariant functions (i.e. $f(x+y,x'+y) = f(x,x')$ for all y), then also U and $\Pi^\star$ are invariant. It is convenient to expand the functions in $k = (\mathbf{k},\omega)$ space: $f(x,x') = \int \frac{d^4k}{(2\pi)^4}e^{ik(x-x')}f(k)$, with $kx = \mathbf{k}\cdot\mathbf{x} - \omega t$. The Dyson equations become algebraic in $k-$space:

$$U(k) = U^0(k) + U^0(k)\Pi^\star(k)U(k)$$
$$\Pi(k) = \Pi^\star(k) + \Pi^\star(k)U^0(k)\Pi(k)$$

They are matrix equations in spin variables. If they are scalar equations, the solutions are:

$$\boxed{U(k) = \frac{U^0(k)}{1 - U^0(k)\Pi^\star(k)}\ , \qquad \Pi(k) = \frac{\Pi^\star(k)}{1 - U^0(k)\Pi^\star(k)}} \tag{10}$$

For a static two-particle potential $U^0(x, x') = v(\mathbf{x} - \mathbf{x}')\delta(t - t')$, it is $U^0(k) = v(\mathbf{k})$. Then:

$$U(\mathbf{k}, \omega) = \frac{v(\mathbf{k})}{\epsilon(\mathbf{k}, \omega)}, \qquad \Pi(\mathbf{k}, \omega) = \frac{\Pi^*(\mathbf{k}, \omega)}{\epsilon(\mathbf{k}, \omega)} \tag{11}$$

$$\boxed{\epsilon(\mathbf{k}, \omega) = 1 - v(\mathbf{k})\Pi^\star(\mathbf{k}, \omega)} \tag{12}$$

$\epsilon(k)$ is the (time ordered) *generalised dielectric function.*
Despite the bare interaction being static, the effective interaction is time-dependent through the dielectric function, which describes the response of the medium. For the Coulomb interaction,

$$U(\mathbf{k}, \omega) = \frac{4\pi e^2}{|\mathbf{k}|^2 - 4\pi e^2 \Pi^\star(\mathbf{k}, \omega)}$$

The long-range Coulomb interaction is modified by the screening produced by the polarized medium.