1. The BCS Hamiltonian

In electronic systems the low temperature properties are determined by the long-lived quasi-particles in an energy shell $\sim k_B T$ near the Fermi surface. Because of the Debye cutoff, the interaction mediated by phonons

$$U_{ph}(k,\omega) = g v_s^2 k^2 \omega^2 - v_s^2 k^2 \theta(\omega_D - v_s k)$$

is attractive in the energy shell $|\epsilon - \epsilon_F| < \hbar \omega_D$ and this, at low enough temperatures, leads to the formation of Cooper pairs with binding energy $\Delta \sim \hbar \omega_D \exp(-2/g \rho_F)$ characterising a superconductive phase, with critical temperature $k_B T_C \sim \Delta$.

Cooper obtained this important result (1956) by solving a 2-particle problem in presence of a filled Fermi sea [3].

The BCS theory (1957)\(^1\) is a full many-electron model, characterized by the attractive interaction that arises in the static limit, $-g \delta(x - x')$, which captures the essence [2]:

$$\hat{K} = \sum_{\mu \nu} \int d\mathbf{x} \delta_{\mu \nu} (\hat{\psi}_\nu^\dagger(k \mathbf{x}) \hat{\psi}_\nu)(\mathbf{x}) - \frac{g}{2} (\hat{\psi}_\mu^\dagger \hat{\psi}_\nu^\dagger \hat{\psi}_\nu \hat{\psi}_\mu)(\mathbf{x})$$

where $k_x = \frac{1}{m}(\mathbf{p} + \mathbf{A})^2 + U(x) - \mu$ and the Debye cut-off is understood for the interaction. By the exclusion principle, only two spin configurations are allowed, and are equivalent: $\hat{\psi}_\uparrow \hat{\psi}_\downarrow \hat{\psi}_\downarrow \hat{\psi}_\uparrow = \hat{\psi}_\downarrow \hat{\psi}_\uparrow \hat{\psi}_\uparrow \hat{\psi}_\downarrow$. The quartic interaction gets simplified by replacing pairs of operators with their mean values,

$$\hat{\psi}_\uparrow \hat{\psi}_\downarrow \hat{\psi}_\downarrow \hat{\psi}_\uparrow \approx \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle \hat{\psi}_\downarrow \hat{\psi}_\uparrow + \hat{\psi}_\downarrow \hat{\psi}_\uparrow \langle \hat{\psi}_\uparrow \hat{\psi}_\downarrow \rangle$$

This introduces a complex field $\Delta$, which behaves as an order parameter that can be related to the Ginzburg Landau field:

$$\Delta(x) = -g \langle \hat{\psi}_\uparrow(x) \hat{\psi}_\downarrow(x) \rangle$$

The thermal average is calculated with the effective Hamiltonian, that no longer conserves the number of electrons

$$\hat{K}_{eff} = \hat{K}_0 + \int d\mathbf{x} \overline{\Delta}(\mathbf{x}) \hat{\psi}_\uparrow(\mathbf{x}) \hat{\psi}_\downarrow(\mathbf{x}) + \hat{\psi}_\downarrow(\mathbf{x}) \hat{\psi}_\uparrow(\mathbf{x}) \Delta(\mathbf{x}).$$

\(^1\)At the time, Leon Cooper and Robert Schrieffer were respectively post-doc and graduate student of John Bardeen. Read the nice historical account by Hoddeson [7].
1.1. **The Hartree approximation.** To gain some understanding of the approximation, let $K = K_0 + K_1$, where $K_1$ is the quartic term, and consider the (thermal) interaction picture. A thermal average of field operators is

$$\langle T \hat{\psi}(x_1) \hat{\psi}(x_2) \ldots \rangle_K = \frac{\langle T \hat{\psi}_I(h\beta,0) \hat{\psi}(x_1) \hat{\psi}(x_2) \ldots \rangle_{K_0}}{\langle \hat{\psi}_I(h\beta,0) \rangle_{K_0}},$$

where $x = (x, \tau)$. Consider the discretization of Dyson’s T-product expansion:

$$\hat{\psi}_I(h\beta,0) = T \exp \left( -\frac{1}{\hbar} \sum_x -g \left( \hat{\psi}_1^\dagger \hat{\psi}_1 \right)(x^+) \left( \hat{\psi}_1 \hat{\psi}_1^\dagger \right)(x) \right)$$

$$= T \prod_x \exp \left[ \frac{g}{\hbar} \left( \hat{\psi}_1^\dagger \hat{\psi}_1 \right)(x^+) \left( \hat{\psi}_1 \hat{\psi}_1^\dagger \right)(x) \right]$$

where, because with time-ordering, the quadratic operators commute. The four-fermion interaction may be splitted with the introduction of an auxiliary complex field $\Delta'(x)$. At each point $x$ the following complex integral applies

$$\exp \left[ \frac{g}{\hbar} AB \right] = \int \frac{d^2 z}{\pi g} \exp \left[ -\frac{1}{\hbar g} \left( \frac{1}{2} |z|^2 + \bar{z}A + Bz \right) \right]$$

With $z = \Delta'(x)$, we obtain a product of integrals which defines a Gaussian functional integral, where all pairs of operators commute because of $T$-ordering:

$$\langle \hat{\psi}_I(h\beta,0) \rangle_{K_0} = T \prod_x \int \frac{d^2 \Delta'(x)}{\pi g} \exp \left[ -\frac{1}{\hbar g} |\Delta'(x)|^2 - \frac{1}{\hbar} \left( \Delta'(x) + \hat{\psi}_1^\dagger \hat{\psi}_1 \right) \right]$$

$$S = \int dx \left( \frac{1}{\hbar} |\Delta'|^2 + \bar{\Delta} \hat{\psi}_1 \hat{\psi}_1 + \hat{\psi}_1^\dagger \hat{\psi}_1 \Delta' \right), \quad Z_\Delta = \int \mathcal{D} \Delta' \exp \left[ -\frac{1}{\hbar g} \int dx |\Delta'(x)|^2 \right]$$

The partition function is $Z = Z_0 \langle \hat{\psi}_I(h\beta,0) \rangle_{K_0}$, with $Z_0 = \text{tr} \left( e^{-\beta K_0} \right)$ and

$$\langle \hat{\psi}_I(h\beta,0) \rangle_{K_0} = \frac{1}{Z_\Delta} \int \mathcal{D} \Delta' \left( \langle \hat{\psi}_I \rangle_{K_0} \right)$$

Now comes the approximation: the main contribution to the functional integral comes from the auxiliary field $\Delta(x)$ that maximises the Boltzmann weight $\langle T e^{-S/\hbar} \rangle$. The extremum condition for a variation $\delta \Delta'$ is an equation for $\Delta(x)$. By retaining only linear terms:

$$\langle T \exp(-\frac{1}{\hbar} S[\Delta', \Delta' + \delta \Delta']) \rangle_{K_0}$$

$$= \langle T \exp(-\frac{1}{\hbar} S[\Delta', \Delta']) \rangle \left[ 1 + \int dx \delta \Delta'(x) \left[ g^{-1} \Delta'(x) + \hat{\psi}_1(x) \hat{\psi}_1^\dagger(x) + \ldots \right] \rangle_{K_0}$$

The first variation is zero for

$$\Delta(x) = -g \frac{\langle T e^{-\frac{1}{\hbar} S[\Delta, \Delta']} \hat{\psi}_1(x) \hat{\psi}_1^\dagger(x) \rangle_{K_0}}{\langle e^{-\frac{1}{\hbar} S[\Delta, \Delta']} \rangle_{K_0}}$$

This is an equation for $\Delta(x)$, which appears on both sides (time-dependence cancels because of equal times). The integral (6) simplifies:

$$\langle \hat{\psi}_I(h\beta,0) \rangle_{K_0} \approx \langle T e^{-\frac{1}{\hbar} S[\Delta, \Delta']} \rangle_{K_0} = e^{\beta K_0} e^{-\beta K_{\text{eff}}}$$. 


The Hamiltonian is now written in a matrix form introduced by Nambu [8]:

\[
\hat{K}_{\text{eff}} = \int dx \left[ -\hat{\psi}_x \hat{\psi}_x^\dagger + \hat{\psi}_x^\dagger \hat{k}_x \hat{\psi}_x^\dagger + \Delta \hat{\psi}_x^\dagger \hat{\psi}_x^\dagger \Delta \right]
\]

The Hamiltonian is now written in a matrix form introduced by Nambu [8]:

\[
\hat{K}_{\text{eff}} = \int dx \Psi^\dagger(x) \mathbb{K}_x(x) \Psi(x)
\]

1.2. Matrix formulation. In the operator \( \hat{K}_0 \) an integration by parts and an anticommutation bring \( \hat{\psi}_x^\dagger \hat{k}_x \hat{\psi}_x^\dagger \) to \( -\hat{\psi}_x \hat{k}_x \hat{\psi}_x^\dagger \) up to a constant\(^2\). Then:

\[
\hat{K}_{\text{eff}} = \int dx \left[ -\hat{\psi}_x \hat{k}_x \hat{\psi}_x^\dagger + \hat{\psi}_x^\dagger \hat{k}_x \hat{\psi}_x^\dagger + \Delta \hat{\psi}_x \hat{\psi}_x^\dagger \Delta \right]
\]

The components of \( \Psi \) and \( \Psi^\dagger \) anticommutate (note that \( (\Psi_r)^\dagger = (\Psi^\dagger)_r \)):

\[
\{\Psi_r(x), \Psi^\dagger_s(y)\} = \delta_{rs} \delta(x - y), \quad \{\Psi_r, \Psi_s\} = \{\Psi^\dagger_r, \Psi^\dagger_s\} = 0
\]

As the effective Hamiltonian is quadratic in the fields, the model can be solved like a theory of independent particles or a Hartree theory, with the self-consistency eq.(2), named \textit{gap equation}.

Two equivalent approaches are presented: one, by de Gennes, generalises the canonical transformation introduced by Bogoljubov and Valatin (1958) for homogeneous systems; the other one is based on Green functions, introduced by Gor’kov in 1958 [5] and here expressed with Nambu’s matrix formalism.

1.3. The Bogoljubov - de Gennes equations. The matrix operator \( \mathbb{K}_x \) acts on the Hilbert space \( L^2(\mathbb{R}^3) \times \mathbb{C}^2 \) and is self-adjoint. It has real eigenvalues, and the eigenvectors form an orthonormal basis. The eigenvalue equation

\[
\begin{bmatrix}
  k_x & \Delta(x) \\
  \Delta(x) & -\mathbb{K}_x
\end{bmatrix}
\begin{bmatrix}
  u_a(x) \\
  v_a(x)
\end{bmatrix} = E_a
\begin{bmatrix}
  u_a(x) \\
  v_a(x)
\end{bmatrix}
\]

gives the pair of Bogoljubov - de Gennes equations:

\[
(k u_a)(x) + \Delta(x)v_a(x) = E_a u_a(x)
\]
\[
(\mathbb{K} v_a)(x) - \Delta(x)u_a(x) = -E_a v_a(x)
\]

If \((u_a, v_a)\) solve them with eigenvalue \(E_a > 0\), then \((-\mathbb{K} a, \mathbb{K} a)\) are a solution with eigenvalue \(-E_a\). The equations (10) with eigenvalues \(\pm E_a\) may be written jointly:

\[
\mathbb{K}_x
\begin{bmatrix}
  u_a(x) \\
  v_a(x)
\end{bmatrix}
\mathbb{K}_x
\begin{bmatrix}
  -\mathbb{K} a(x) \\
  \mathbb{K} a(x)
\end{bmatrix} =
\begin{bmatrix}
  u_a(x) \\
  v_a(x)
\end{bmatrix}
\begin{bmatrix}
  -\mathbb{K} a(x) \\
  \mathbb{K} a(x)
\end{bmatrix}
\begin{bmatrix}
  E_a & 0 \\
  0 & -E_a
\end{bmatrix}
\]

\(^2\)The operators \(k_x\) and \(\mathbb{K}_x\) differ by the sign of the term linear in \(p\), if any.
The ortho-normalization and completeness of the doublets in Hilbert space may be expressed in matrix form:

\[
\int dx \left[ \begin{array}{cc} \pi_b(x) & \pi_b(x) \\ -\nu_b(x) & u_b(x) \end{array} \right] \left[ \begin{array}{cc} u_a(x) & -\pi_a(x) \\ v_a(x) & \pi_a(x) \end{array} \right] = \delta_{ab} I_2
\]

\[
\sum_a \left[ \begin{array}{cc} u_a(x) & -\pi_a(x) \\ v_a(x) & \pi_a(x) \end{array} \right] \left[ \begin{array}{cc} \pi_a(y) & \pi_a(y) \\ -v_a(y) & u_a(y) \end{array} \right] = \delta(x-y) I_2
\]

1.4. **Diagonalization of the many-body Hamiltonian.** The matrix relation (11) suggests that the many body Hamiltonian is diagonalized by the following transformation to new operators:

\[
\left[ \begin{array}{c} \hat{\psi}_0(x) \\ \hat{\psi}_1(x) \end{array} \right] = \sum_a \left[ \begin{array}{cc} u_a(x) & -\pi_a(x) \\ v_a(x) & \pi_a(x) \end{array} \right] \left[ \begin{array}{c} \hat{\alpha}_a \\ \hat{\beta}_a \end{array} \right]
\]

This and the adjoint are, in detail:

\[
\hat{\psi}_0(x) = \sum_a u_a(x) \hat{\alpha}_a - \pi_a(x) \hat{\beta}_a, \quad \hat{\psi}_1(x) = \sum_a \pi_a(x) \hat{\alpha}_a - v_a(x) \hat{\beta}_a
\]

\[
\hat{\psi}_0(x) = \sum_a \pi_a(x) \hat{\alpha}_a + u_a(x) \hat{\beta}_a, \quad \hat{\psi}_1(x) = \sum_a v_a(x) \hat{\alpha}_a + \pi_a(x) \hat{\beta}_a
\]

Inversion is done with the aid of (12):

\[
\left[ \begin{array}{c} \hat{\alpha}_a \\ \hat{\beta}_a \end{array} \right] = \int dx \left[ \begin{array}{cc} \pi_a(x) & \pi_a(x) \\ -v_a(x) & u_a(x) \end{array} \right] \left[ \begin{array}{c} \hat{\psi}_0(x) \\ \hat{\psi}_1(x) \end{array} \right]
\]

The adjoint operators are also obtained. The transformation is canonical i.e. the new operators have canonical anticommutation relations:

\[
\{ \hat{\alpha}_a, \hat{\alpha}_b^\dagger \} = \delta_{ab}, \quad \{ \hat{\beta}_a, \hat{\beta}_b^\dagger \} = \delta_{ab}
\]

(all other anticommutators vanish). By eq.(11)

\[
(\mathbb{K}_x \Psi)(x) = \sum_a \left[ \begin{array}{cc} u_a(x) & -\pi_a(x) \\ v_a(x) & \pi_a(x) \end{array} \right] \left[ \begin{array}{cc} E_a & 0 \\ 0 & -E_a \end{array} \right] \left[ \begin{array}{c} \hat{\alpha}_a \\ \hat{\beta}_a^\dagger \end{array} \right]
\]

Evaluation of \( \hat{K}_{eff} = \int dx \Psi^\dagger \mathbb{K}_x \Psi \) and (12) give a diagonal operator for quasiparticles (bogolons):

\[
\hat{K}_{eff} = U_0 + \sum_a E_a [\hat{\alpha}_a \hat{\alpha}_a + \hat{\beta}_a^\dagger \hat{\beta}_a^\dagger]
\]

where \( U_0 = -\sum_a E_a \). The ground state is defined by \( \hat{\alpha}_a |BCS\rangle = 0 \) and \( \hat{\beta}_a |BCS\rangle = 0 \) for all \( a \).

1.5. **The gap equation.** The change of basis (15) simplifies the gap equation:

\[
\Delta(x) = -g \sum_{ab} u_a(x) \pi_b(x) \langle \hat{\alpha}_a \hat{\alpha}_b \rangle - \pi_a(x) u_b(x) \langle \hat{\beta}_a \hat{\beta}_b \rangle = g \sum_a u_a(x) \pi_a(x) [1 - 2n(E_a)] \]

where \( n(E_a) \) is the Fermi-Dirac occupation number of the state with energy \( E_a \). Then:

\[
\Delta(x) = g \sum_a u_a(x) \pi_a(x) \tanh \left( \frac{\beta}{2} E_a \right)
\]

The equation must be solved self-consistently with the Bogolubov - de Gennes equations for \( u_a \) and \( v_a \).
Remark 1.2. As the gap function depends on temperature, the amplitudes $u_a$, $v_a$ as well as the energies $E_a$ and |BCS| depend on $T$.

Exercise 1.3. Show that $\Omega = -\frac{2}{\tau} \sum_a \log \left(2 \cosh \frac{1}{2} \beta E_a\right)$.

Exercise 1.4. Show that the average density of electrons is:

$$n(x) = \sum_a |u_a(x)|^2 n_a + |v_a(x)|^2 (1 - n_a), \quad n_a = \frac{1}{e^{\beta E_a} + 1}$$

1.6. Nambu - Gorkov theory. There are advantages in studying the BCS model with the Green function formalism. The imaginary time evolution of operators is $O(\tau) = e^{\tau \bar{K}/\hbar} O e^{-\tau \bar{K}/\hbar}$, where $\bar{K}$ is the effective hamiltonian (7). The equation of motion of $\Psi(x, \tau)$ is:

$$-\hbar \frac{\partial}{\partial \tau} \Psi_r(x, \tau) = e^{\frac{i}{2} \tau \bar{K}} [\Psi_r(x), \bar{K}] e^{-\frac{i}{2} \tau \bar{K}}$$

$$= e^{\frac{i}{2} \tau \bar{K}} \int dx' [\Psi_r(x), \Psi_s(\bar{x}')] (\bar{K}_{s', s} \Psi_s)(\bar{x}') e^{-\frac{i}{2} \tau \bar{K}}$$

$$= e^{\frac{i}{2} \tau \bar{K}} \int dx' \{\Psi_r(x), \Psi_s(\bar{x}')\} (\bar{K}_{s', s} \Psi_s)(\bar{x}') e^{-\frac{i}{2} \tau \bar{K}}$$

$$= (\bar{K}_{s', s} \Psi_s)(x, \tau)$$

Let us introduce the thermal Nambu propagator

$$G(x, x') = \langle T \psi(x) \psi^\dagger(x') \rangle$$

It is a matrix with components

$$G(x, x') = \begin{bmatrix}
\langle \psi(x) \psi^\dagger(x') \rangle & \langle \psi(x) \psi^\dagger(x') \rangle \\
\langle \psi^\dagger(x) \psi^\dagger(x') \rangle & \langle \psi^\dagger(x) \psi^\dagger(x') \rangle
\end{bmatrix} = \begin{bmatrix}
G(x, x') & F(x, x') \\
F^\dagger(x, x') & -G(x', x)
\end{bmatrix}$$

Note the sign and the exchange of $x$ and $x'$ in one component. The correlators $F$ and $F^\dagger$ are named **anomalous** and vanish in the normal phase. In particular:

$$\Delta(x) = -\beta F(x, x^+)$$

The equation of motion of the Nambu propagator,

$$\left[\hbar \frac{\partial}{\partial \tau} + \bar{K}_s\right] G(x, x') = -\hbar \delta_2(x - x')$$

simplifies in Matsubara (odd) frequency space:

$$\left[\begin{array}{cc}
-i \hbar \omega_n + k_x & \Delta(x) \\
\Delta(x) & -i \hbar \omega_n - k_x
\end{array}\right] G(x, x'; i \omega_n) = -\hbar \delta_2(x - x')$$

$$G(x, x', i \omega_n) = \begin{bmatrix}
\mathcal{G}(x, x', i \omega_n) & \mathcal{F}(x, x', i \omega_n) \\
\mathcal{F}^\dagger(x, x', i \omega_n) & -\mathcal{G}(x', x, -i \omega_n)
\end{bmatrix}$$

Exercise 1.5. The propagators can be represented as expansions in the Bogolubov - de Gennes eigenstates. Show that:

$$\mathcal{G}(x, x', i \omega_n) = \sum_a \frac{u_a(x) \overline{v}_a(x')}{{\omega_n - E_a}/\hbar} + \frac{\overline{u}_a(x) v_a(x')}{{\omega_n + E_a}/\hbar}$$

$$\mathcal{F}(x, x', i \omega_n) = \sum_a \frac{-u_a(x) \overline{v}_a(x')}{{\omega_n - E_a}/\hbar} + \frac{\overline{u}_a(x) u_a(x')}{{\omega_n + E_a}/\hbar}$$
and recover the gap equation (20) by evaluating the Matsubara sum
\[ \Delta(x) = -g \frac{1}{\hbar \beta} \sum_n \mathcal{F}(x, x, i\omega_n) e^{i\omega_n \eta} \]

1.7. Perturbative expansion. When \( \Delta = 0 \), eq.(25) is solved by the normal Nambu propagator \( \mathbb{G}_n(x, x') \), which can be used to transform (25) into a Dyson equation (integration and summation of repeated variables is implicit):
\[ \mathbb{G}(x, y) = \mathbb{G}_n(x, y) + \frac{1}{\hbar} \mathbb{G}_n(x, x') \mathbb{D}(x') \mathbb{G}(x', y), \quad \mathbb{D}(x) = \begin{bmatrix} 0 & \Delta \\ \Delta & 0 \end{bmatrix} \]

In BCS model the self-energy \( \mathbb{D} \) is local and time-independent. When this description is inadequate, one has to consider a microscopic model with the actual phonon-electron interaction. The Dyson’s equation becomes
\[ \mathbb{G}(x, y) = \mathbb{G}_n(x, y) + \mathbb{G}_n(x, x') \mathbb{S}(x', x'') \mathbb{G}(x'', y) \]
where \( \mathbb{S} \) is a non-local self-energy matrix. In a 1-phonon exchange approximation, \( \mathbb{S}(x, y) = -\frac{1}{\hbar} \mathbb{G}(x, y) U_{ph}^0(x - y) \), the coupled equations for \( \mathbb{G} \) and \( \mathcal{F} \) are:
\[ \mathcal{G}(x, y) = \mathcal{G}_n(x, y) + \mathcal{G}_n(x, x') \mathcal{S}_{11}(x', x'') \mathcal{F}(x'', y) + \mathcal{G}_n(x, x') \mathcal{S}_{12}(x', x'') \mathcal{F}(x'', y) \]
\[ \mathcal{F}(x, y) = -\mathcal{G}_n(x, x') \mathcal{S}_{12}(x', x'') \mathcal{G}(y, x'') + \mathcal{G}_n(x, x') \mathcal{S}_{11}(x', x'') \mathcal{F}(x'', y) \]
with the addition of the gap equation.

2. Homogeneous systems

In homogeneous problems there is no external field and \( \Delta \) is constant. An analytic solution is found in momentum space.

2.1. The Bogoljubov - Valatin canonical transformation. We seek for a solution of the Bogoljubov - de Gennes equations of the form
\[ \begin{bmatrix} u_k(x) \\ v_k(x) \end{bmatrix} = \frac{e^{i k \cdot x}}{\sqrt{V}} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \]
Then
\[ \begin{bmatrix} \xi_k & \Delta \\ -\Delta & -\xi_k \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} = E_k \begin{bmatrix} u_k \\ v_k \end{bmatrix} \]
where \( \xi_k = \epsilon_k - \mu \) are the single-particle energies (normal phase) measured with respect to the chemical potential. The homogeneous system admits a nontrivial solution if
\[ E_k = \sqrt{\xi_k^2 + |\Delta|^2} \]
(the positive root is selected for stability). The energy gap \( |\Delta| \) separating the Fermi surface \( \xi = 0 \) from the lowest excitation, profoundly modifies the properties of the electron gas at low temperatures.

The amplitudes solve the normalization condition \( |u_k|^2 + |v_k|^2 = 1 \) and the condition \( \xi_k u_k + \Delta v_k = E_k u_k \). The latter gives \( |\Delta| |v_k| = (E_k - \xi_k) |u_k| \), with solutions
\[ |u_k|^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{E_k} \right), \quad |v_k|^2 = \frac{1}{2} \left( 1 - \frac{\xi_k}{E_k} \right) \]
simplified the theory by their canonical transformation [1, 9].

The equation \( \xi_k u_k + \Delta v_k = E_k u_k \) gives \( \Delta |v_k|^2 = (E_k - \xi_k)u_k v_k \) i.e. the useful relation:

\[
(34) \quad u_k v_k = \frac{\Delta}{2E_k}
\]

The expansion of the field operators in the two canonical basis,

\[
\begin{bmatrix}
\hat{\psi}_1(x) \\
\hat{\psi}_2(x)
\end{bmatrix} = \sum_k \frac{e^{ik \cdot x}}{\sqrt{V}} \begin{bmatrix}
\hat{a}_{k, \downarrow} \\
\hat{a}_{-k, \uparrow}
\end{bmatrix} = \sum_k \frac{e^{ik \cdot x}}{\sqrt{V}} \begin{bmatrix}
u_k & u_k \\
\bar{v}_k & \bar{u}_k
\end{bmatrix} \begin{bmatrix}
\hat{a}_k \\
\hat{\beta}_{-k, \uparrow}
\end{bmatrix}
\]

implies the Bogoljubov - Valatin transformation:

\[
(35) \quad \begin{bmatrix}
\hat{a}_{k, \downarrow} \\
\hat{a}_{-k, \uparrow}
\end{bmatrix} = \begin{bmatrix}
u_k & u_k \\
\bar{v}_k & \bar{u}_k
\end{bmatrix} \begin{bmatrix}
\hat{a}_k \\
\hat{\beta}_{-k, \uparrow}
\end{bmatrix}
\]

and the Hermitian conjugate.

\[
\hat{\alpha}_k = \bar{u}_k \hat{a}_{k, \downarrow} + v_k \hat{a}_{-k, \uparrow}, \quad \hat{\alpha}_k^\dagger = u_k \hat{a}_{k, \downarrow}^\dagger + v_k \hat{a}_{-k, \uparrow}^\dagger
\]

\[
\hat{\beta}_k = -v_k \hat{a}_{-k, \downarrow}^\dagger + \bar{u}_k \hat{a}_{k, \uparrow}, \quad \hat{\beta}_k^\dagger = -v_k \hat{a}_{-k, \downarrow}^\dagger + u_k \hat{a}_{k, \uparrow}^\dagger
\]

The operators \( \hat{\alpha}_k \) and \( \hat{\beta}_k \) annihilate, for all vectors \( \mathbf{k} \), the state

\[
(38) \quad |BCS\rangle = \prod_k (\bar{u}_k + v_k \hat{a}_{k, \uparrow}^\dagger \hat{a}_{-k, \downarrow}^\dagger) |0\rangle
\]

which reads as a sea of Cooper pairs\(^3\). In the normal phase \( (\Delta = 0) \) it coincides with the filled Fermi sphere.

\[
\hat{\alpha}_k |BCS\rangle = \prod_{q \neq k} (\bar{u}_q + v_q \hat{a}_{q, \uparrow}^\dagger \hat{a}_{-q, \downarrow}^\dagger) \hat{\alpha}_k (\bar{u}_k + v_k \hat{a}_{-k, \uparrow}^\dagger \hat{a}_{k, \downarrow}^\dagger) |0\rangle
\]

\[
= \prod_{q \neq k} (\bar{u}_q + v_q \hat{a}_{q, \uparrow}^\dagger \hat{a}_{-q, \downarrow}^\dagger) (\bar{u}_k \bar{v}_k \hat{a}_{k, \downarrow} \hat{a}_{-k, \uparrow} + \bar{v}_k \bar{u}_k \hat{a}_{-k, \downarrow} \hat{a}_{k, \uparrow}) |0\rangle = 0
\]

\(^3\)In [2] Bardeen, Cooper and Schrieffer (1957) introduced the state with variational parameters \( u_k \) and \( v_k \) with \( |u_k|^2 + |v_k|^2 = 1 \) for normalization. Minimization of \( \langle BCS | \hat{H}_{\text{eff}} | BCS \rangle \) with respect to the parameters yields the same results presented here. Bogoljubov and Valatin independently simplified the theory by their canonical transformation [1, 9].
\[ \hat{\beta}_{-k}^{\uparrow} |BCS \rangle = \prod_{q \neq k} (\bar{u}_q + \bar{v}_q \hat{a}^{\dagger}_{-q} \hat{a}^{\dagger}_{q}) \hat{\beta}_{-k}(\bar{u}_k + \bar{v}_k \hat{a}^{\dagger}_{-k} \hat{a}^{\dagger}_{k}) |0\rangle \]

\[ = \prod_{q \neq k} (\bar{u}_q + \bar{v}_q \hat{a}^{\dagger}_{-q} \hat{a}^{\dagger}_{q}) (-\bar{v}_k \bar{u}_k \hat{a}^{\dagger}_{k} \hat{a}^{\dagger}_{-k} + \bar{u}_k \bar{v}_k \hat{a}^{\dagger}_{-k} \hat{a}^{\dagger}_{k}) |0\rangle = 0 \]

Creation operators create excited states (bogolons) consisting of Cooper pairs and unpaired electrons. For example:

\[ \hat{a}^{\dagger}_{k} |BCS \rangle = \prod_{q \neq k} (\bar{u}_q + \bar{v}_q \hat{a}^{\dagger}_{-q} \hat{a}^{\dagger}_{q}) \hat{a}^{\dagger}_{k} |0\rangle \]

\[ \hat{\beta}_{-k}^{\dagger} |BCS \rangle = \prod_{q \neq k} (\bar{u}_q + \bar{v}_q \hat{a}^{\dagger}_{-q} \hat{a}^{\dagger}_{q}) \hat{a}^{\dagger}_{k} |0\rangle \]

\[ \hat{\alpha}_{k}^{\dagger} \hat{\beta}_{-k}^{\dagger} |BCS \rangle = \bar{u}_k \prod_{q \neq k} (\bar{u}_q + \bar{v}_q \hat{a}^{\dagger}_{-q} \hat{a}^{\dagger}_{q}) \hat{a}^{\dagger}_{k} \hat{a}^{\dagger}_{k} |0\rangle \]

2.2. The gap equation. By means of (34) the gap equation (20) becomes:

\[ \Delta = g \frac{1}{V} \sum_{k} \frac{\Delta}{2E_k} \text{tanh} \left( \frac{\beta E_k}{2} \right) \theta(\hbar \omega_D - |\xi_k|) \]

By introducing the density of states per unit volume and single spin component of the normal phase,

\[ \rho_n(\xi) = \frac{1}{V} \sum_{k} \delta(\xi - \xi_k) \]

the sum in \( \mathbf{k} \)-space is changed into an integral in energy,

\[ 1 = \frac{g}{2} \int d\xi \rho_n(\xi) \frac{\text{tanh} \left( \frac{\beta}{2} \sqrt{\xi^2 + |\Delta|^2} \right)}{\sqrt{\xi^2 + |\Delta|^2}} \theta(\hbar \omega_D - |\xi|) \]

With the assumption that the density is almost constant in the energy shell \( |\xi| < \hbar \omega_D \) it simplifies to:

\[
\frac{1}{g \rho_n(0)} = \int_0^{\hbar \omega_D} d\xi \frac{\text{tanh} \left( \frac{\beta}{2} \sqrt{\xi^2 + |\Delta|^2} \right)}{\sqrt{\xi^2 + |\Delta|^2}} \]

where \( \rho(0) \) is the density of states at the Fermi energy, \( g \) is the squared coupling constant of the phonon to the electron. According to the microscopic theory:

\[ \sqrt{g} = \frac{z_c}{v_s} \sqrt{\frac{n_i}{M_i}} \pi^2 e^2 \frac{a_0}{k_F} \]

\( z_c \) is the number of conducting electrons per ion, \( n_i \) is the number of ions per unit volume, \( M_i \) is the ionic mass, \( v_s \) is the speed of sound.

Exercise 2.1. Show that the density of states per unit volume and spin component of the free electron gas at the Fermi energy is \( \rho_n(0) = \frac{\pi}{3} n / E_F \), where \( n \) is the density of electrons and \( E_F \) is the Fermi energy. Then show that

\[ g \rho_n(0) = \frac{z_c}{6} \frac{m}{M_i} \left( \frac{v_F}{v_s} \right)^2 \]

where \( m \) is the electron’s mass and \( v_F \) is the Fermi velocity.
2.3. The Green functions. In $k$-space the equation of motion (25) for the Nambu propagator is algebraic

$$\left[ \begin{array}{cc} i\hbar \omega_n - \xi_k & -\Delta \\ -\Delta & i\hbar \omega_n + \xi_k \end{array} \right] G(k; \omega_n) = \hbar I_2$$

$$G(k; \omega_n) = \frac{-\hbar}{\hbar^2 \omega_n^2 + \xi_k^2 + |\Delta|^2} \left[ \begin{array}{cc} i\hbar \omega_n + \xi_k & \Delta \\ \Delta & i\hbar \omega_n - \xi_k \end{array} \right]$$

The normal and anomalous propagators are obtained, with $E_k = \sqrt{\xi_k^2 + |\Delta|^2}$:

$$\mathcal{G}(k, \omega_n) = -\hbar i\hbar \omega_n + \xi_k \frac{|u_k|^2}{\omega_n - (E_k/\hbar) + i\omega_n + (E_k/\hbar)} + |v_k|^2$$

$$\mathcal{F}(k, \omega_n) = -\hbar \frac{\Delta}{\hbar^2 \omega_n^2 + E_k^2} = \frac{u_k\overline{v}_k}{\omega_n - (E_k/\hbar)} - \frac{u_k\overline{v}_k}{\omega_n + (E_k/\hbar)}$$

The Matsubara sum in the gap equation

$$\Delta = -\frac{g}{\hbar^2} \sum_{\eta} \int \frac{d^3k}{(2\pi)^3} \mathcal{F}(k, \omega_n) e^{i\omega_n \eta}$$

yields the expression (39).

**Example 2.2.** Show that the average number of electrons in a state $(k, \sigma)$ is

$$n_k = \frac{1}{\beta} \sum_{\eta} \mathcal{G}(k, \omega_n) = \frac{1}{2} - \frac{1}{2} \frac{\xi_k}{E_k} \tanh \frac{\beta E_k}{2}$$

**Exercise 2.3** (spectral density). Evaluate the spectral density of the superconducting phase

$$\rho_s(E) = \sum_k |u_k|^2 \delta(E - E_k) + |v_k|^2 \delta(E + E_k)$$

(use the approximation $|\Delta| \ll \mu$). Note the presence of an energy gap of width $2\Delta$ centred at $E = 0$ (chemical potential).

$$\rho_s(E) = \frac{\rho_s(0)}{\rho_n(0)} = \begin{cases} \frac{E + \sqrt{E^2 - \Delta^2}}{\sqrt{1 + \frac{E}{\mu}} + \sqrt{1 - \frac{E}{\mu}}} & -\mu < E < -\Delta \\ 0 & |E| < \Delta \\ \frac{E - \sqrt{E^2 - \Delta^2}}{\sqrt{1 - \frac{E}{\mu}} + \sqrt{1 + \frac{E}{\mu}}} & \Delta < E < \mu \\ \frac{2}{\sqrt{1 + \frac{E}{\mu}}} & \mu < E \end{cases}$$

$$\rho_n(E) = \frac{1}{4\pi^2} \left( \frac{2m}{\hbar^2} \right)^{\frac{3}{2}} \sqrt{E + \mu}$$

Near the gap $\rho_s(E) \approx 2\rho_n(0)|E|/\sqrt{E^2 - \Delta^2}$.

2.4. Discussion of the gap equation.

$T = T_c$. At the critical temperature the order parameter $\Delta$ is zero, and the gap equation is an equation for $T_c$:

$$\frac{1}{\rho(0)} = \int_0^{\hbar \omega_D} \frac{d\xi}{\xi} \tanh \left( \frac{1}{2} \beta \xi \right) = \int_0^{\infty} \frac{dx}{x} \tanh (x) \log(x) - \int_0^{x_c} \frac{dx}{x} \frac{\log(x)}{\cosh^2 x}$$

$$\approx \log x_c - \int_0^{\infty} \frac{dx}{x} \frac{\log(x)}{\cosh^2 x} = \log x_c + \log(4eC/\pi), \quad x_c = \frac{\hbar \omega_D}{2k_B T_c} = \frac{T_D}{2T_c}$$
Figure 2. The spectral density ($\mu = 20, \Delta = 0.6$). The thin line is the square root $\sqrt{E}$ of the normal phase. The gap is centred on the chemical potential.

The approximations are justified by $T_D/T_c \gg 1$. With $C \approx 0.5772..$, the result is

$$k_B T_c = 1.134 \hbar \omega_D \exp \left( -\frac{1}{g \rho(0)} \right)$$

$T = 0$. The gap equation becomes:

$$\frac{1}{g \rho(0)} = \int_0^{\hbar \omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta_0^2}}$$

with solution

$$\Delta_0 = \frac{\hbar \omega_D}{\sinh \frac{1}{g \rho(0)}} \approx 2\hbar \omega_D \exp \left( -\frac{1}{g \rho(0)} \right)$$

The following universal ratio is then obtained:

$$\frac{\Delta_0}{k_B T_c} = \pi e^{-C} \approx 1.76$$

3. THE GINZBURG - LANDAU LIMIT OF BCS

The Ginzburg-Landau theory can be derived from the microscopic BCS model. Near the transition line $H = H_c(T)$, the function $\Delta$ is small, and the Dyson equation (29) for $G(x, y; i\omega_n)$ can be solved by iteration:

$$G = G_n + \frac{1}{\hbar} G_n D G_n + \frac{1}{\hbar^2} G_n D G_n D G_n + \frac{1}{\hbar^3} G_n D G_n D G_n D G_n$$

The truncation to third order in $\Delta$ evaluates the anomalous correlator $\mathcal{F}(x, y, i\omega_n)$ and the Green function $\mathcal{G}(x, y, i\omega_n)$ in terms of the normal Green function and the
gap function:
\[
\mathcal{F}(1, 2, i\omega_n) = -\frac{1}{\hbar} \mathcal{G}_n(1, 3, i\omega_n) \Delta(3) \mathcal{G}_n(2, 3, -i\omega_n) \\
+ \frac{1}{\hbar^3} \mathcal{G}_n(1, 3, i\omega_n) \Delta(3) \mathcal{G}_n(4, 3, -i\omega_n) \Delta(4) \mathcal{G}_n(4, 5, i\omega_n) \Delta(5) \mathcal{G}_n(2, 5, -i\omega_n);
\]
\[
\mathcal{G}(1, 2, i\omega_n) = \mathcal{G}_n(1, 2, i\omega_n)
\]
Eq.(46) with \(2 = 1^+\) and summation of Matsubara frequencies, is a cubic expansion of the gap equation for \(\Delta\), and provides the first G.L. equation with order parameter \(\psi \propto \Delta\):
\[
\frac{1}{g} \Delta(1) = Q(1, 2) \Delta(2) + R(1, 2, 3, 4) \Delta(3) \Delta(4)
\]
with weight functions
\[
Q(1, 2) = \frac{1}{\hbar^2} \sum_n \mathcal{G}_n(1, 2, i\omega_n) \mathcal{G}_n(1, 2, -i\omega_n)
\]
\[
R(1, 2, 3, 4) = -\frac{1}{\hbar^4} \sum_n \mathcal{G}_n(1, 2, i\omega_n) \mathcal{G}_n(3, 2, -i\omega_n) \mathcal{G}_n(3, 4, i\omega_n) \mathcal{G}_n(1, 4, -i\omega_n)
\]
Eq.(47) is an expansion for the Green function, that is used to estimate the super-current, and yields the second G.L. equation.
The derivation of G.L. equations relies crucially on the large difference among the length scales involved. We need some preliminaries...

References