Devil's staircase phase diagram of the FQHE in the thin torus limit

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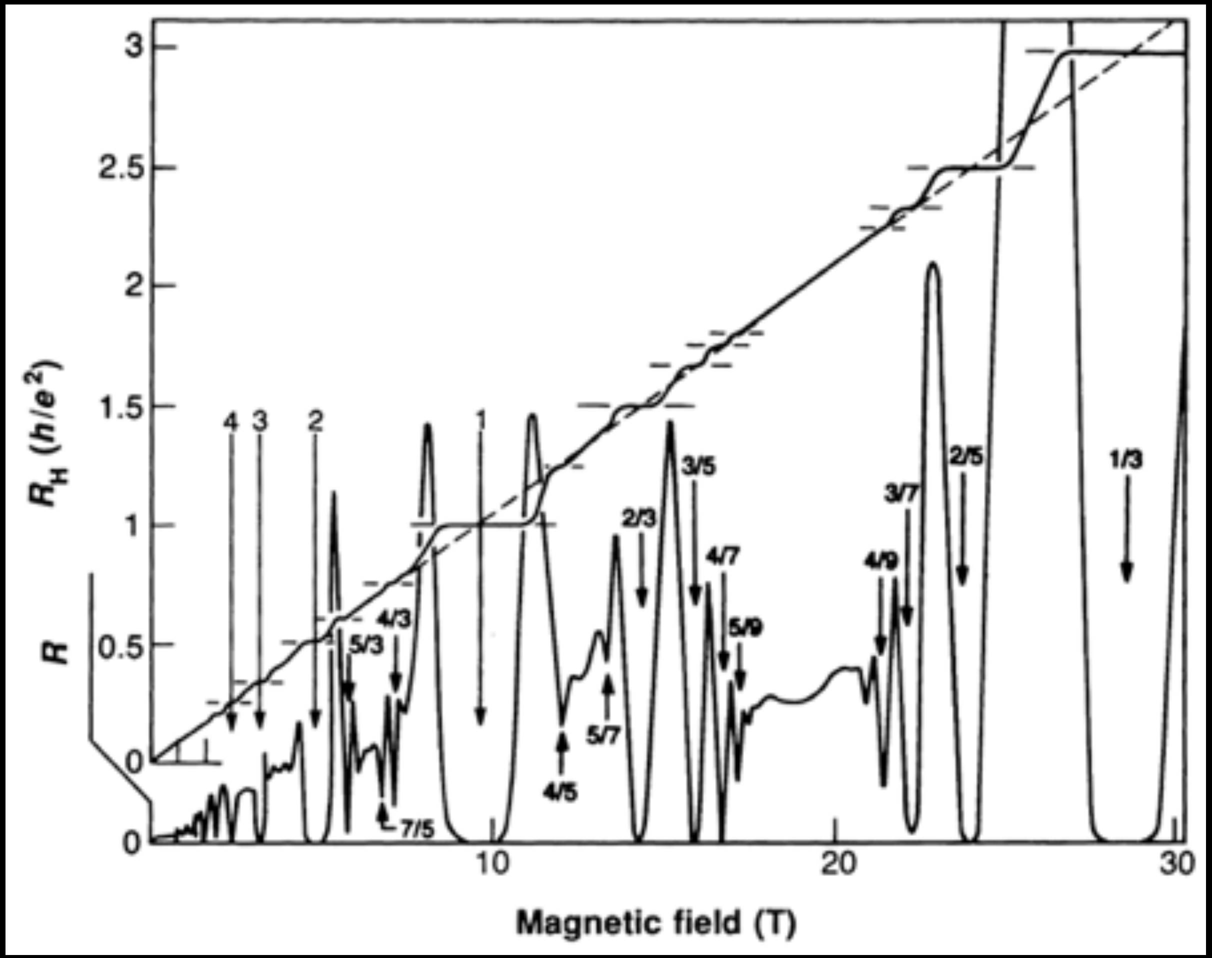
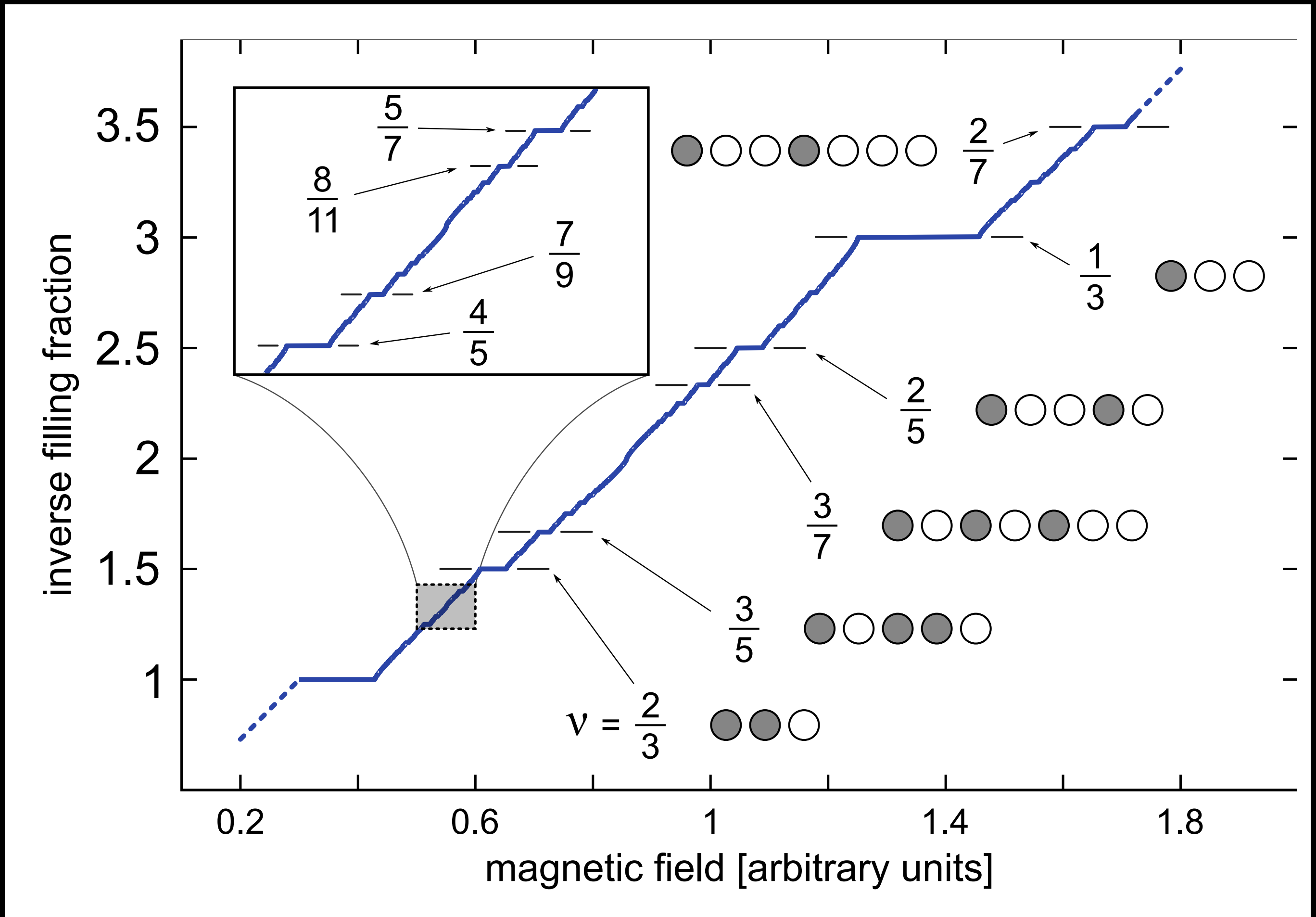
SUMMARY

- We map the Quantum Hall Hamiltonian restricted to the subspace of the lowest Landau level (in the thin torus limit: Lx << magn. length << Ly) to a 1D long-range lattice model (exactly solved by Hubbard)
- Going back, we qualitatively reproduce the experimental diagram of transverse resistance vs B of FQHE

FQHE - EXPERIMENT

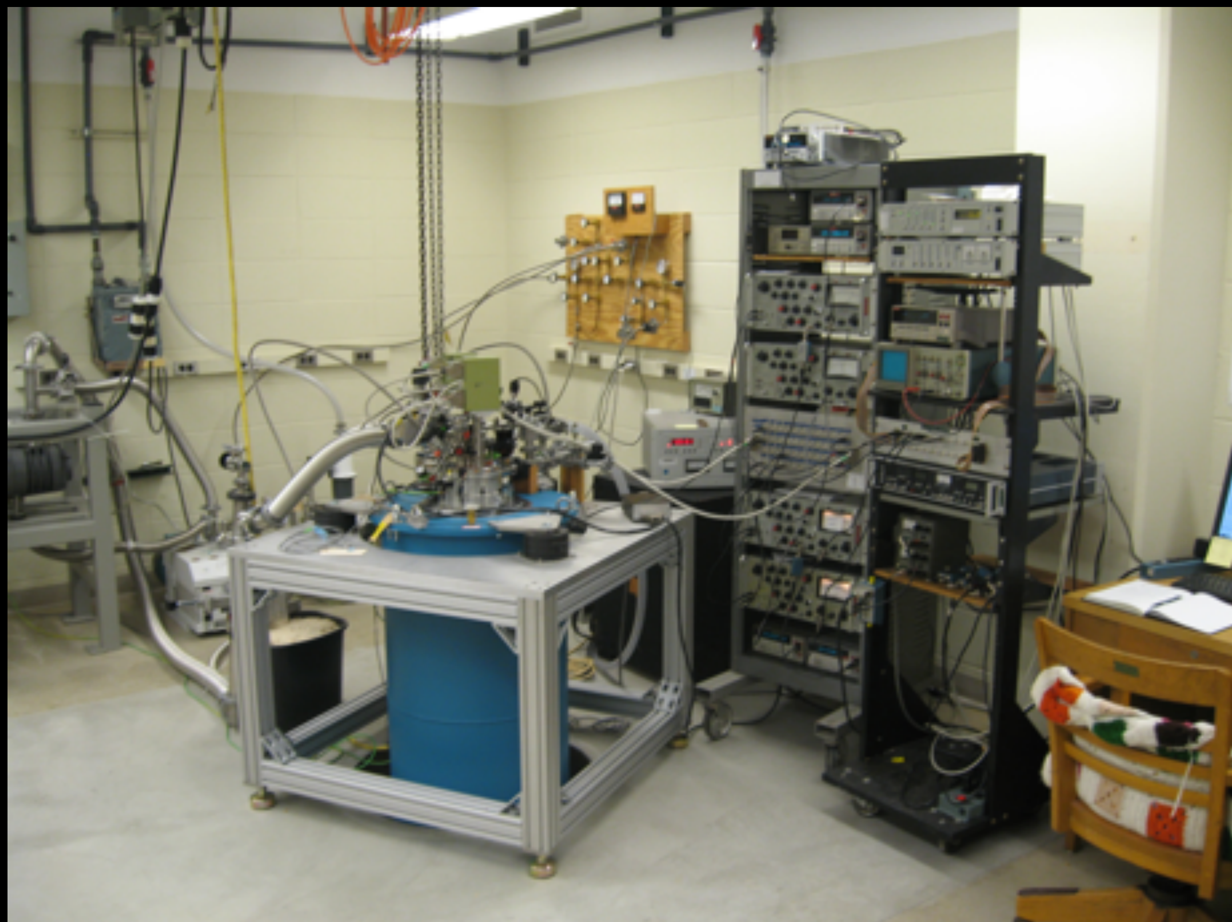


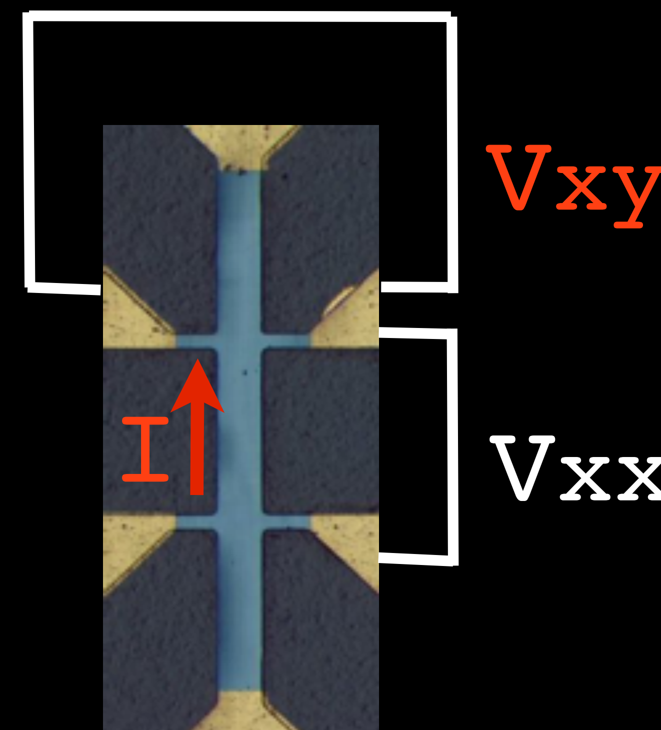
LATTICE CRYSTAL



Hall effect

$$R_{xy} = V_{xy}/I$$
$$= B/ne$$

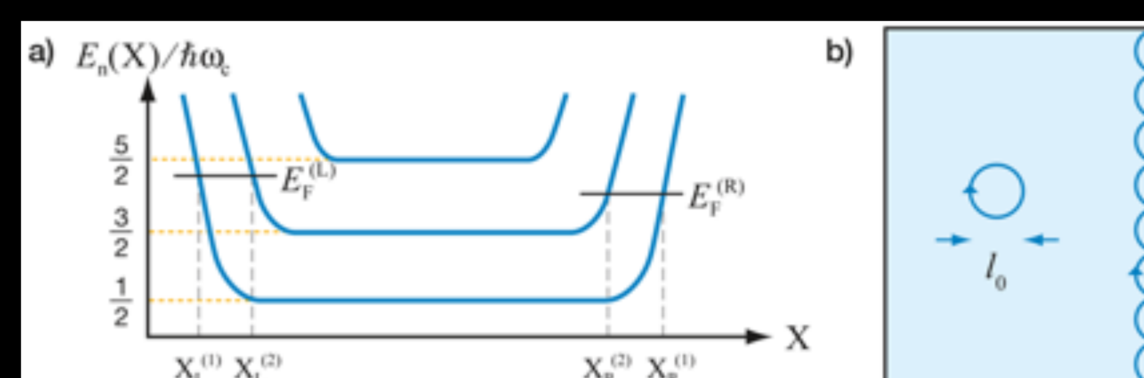




2D electron gas
T ~ 100 mK or less
B ~ tens of Tesla

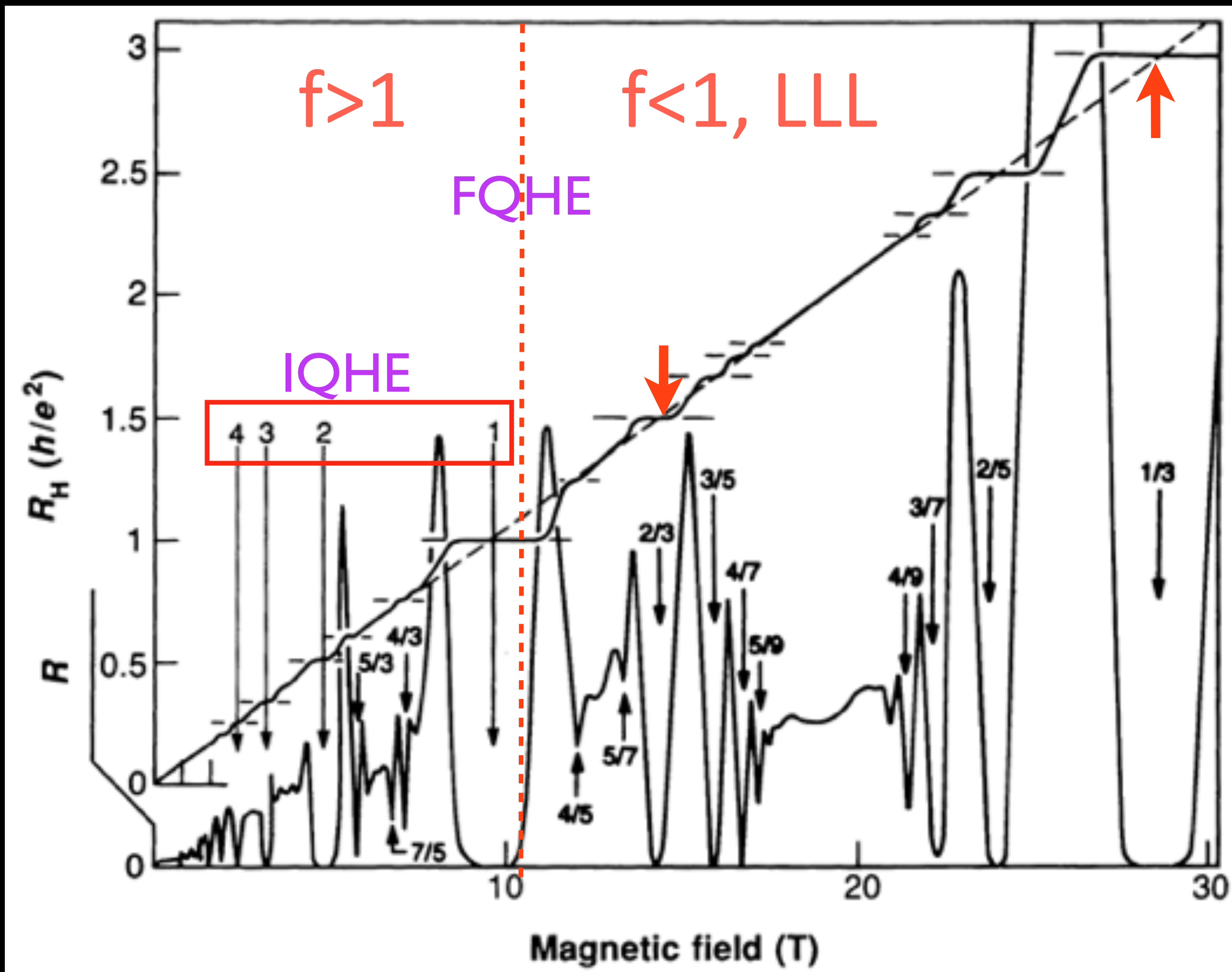
$$R_{xy} = [h/e^2]\,1/f$$

f = filling fraction = N/deg of LL



IQHE (1980)

Klaus von Klitzing
Max Planck
Nobel 1985



FQHE (1981)

Horst Stormer
Columbia

Nobel 1988
For their discovery of a new form of quantum fluid with fractionally charged excitations

Daniel Tsui
Princeton

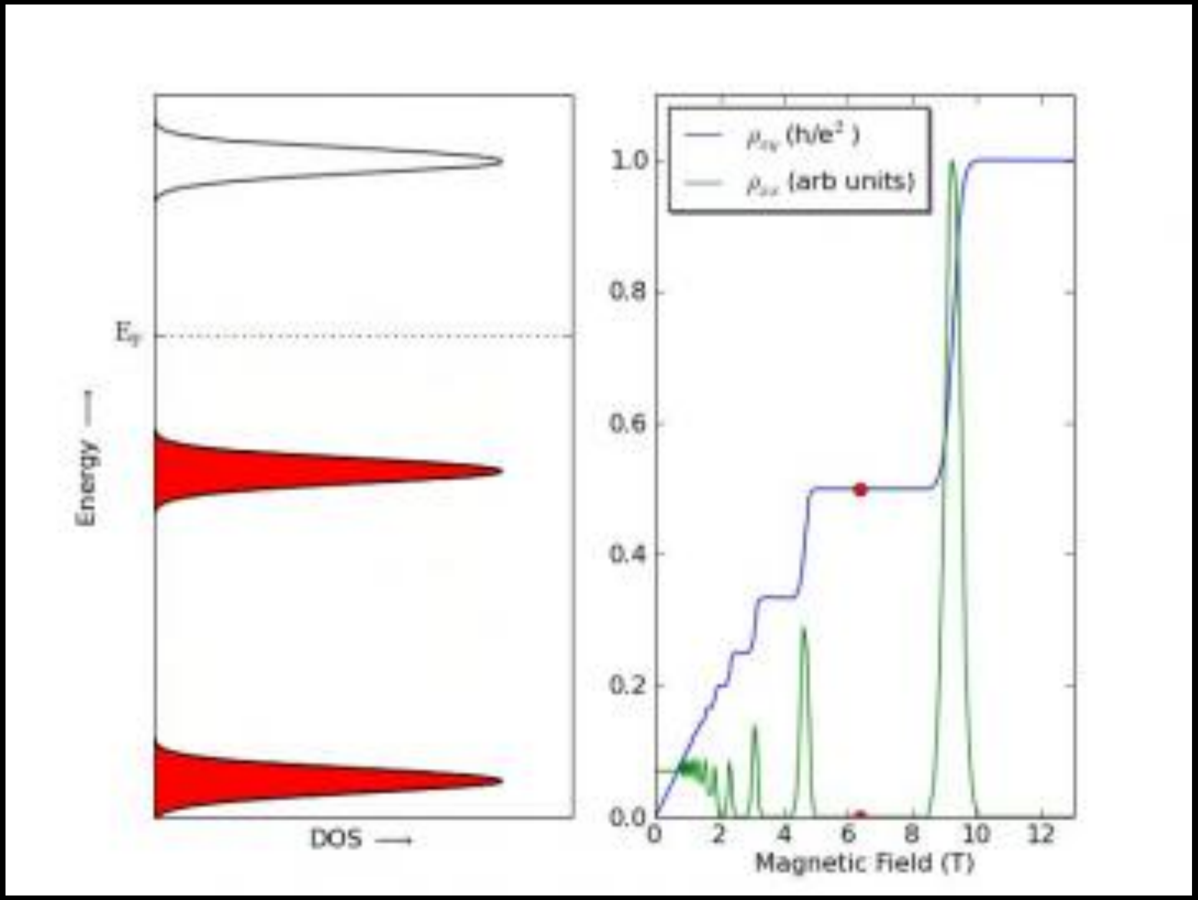
Robert Laughlin
Stanford

- more than 60 plateaux in LLL
- particle-hole asymmetry of plateaux (see 2/3 and 1/3)
- absence of even denominators (few exceptions as 5/2)
- average linearity
- R_{xx} small where R_{xy} is flat (energy gap)

IQHE (edge or bulk currents?)

Disorder broadens LL into bands of localized states and a core of conducting states

Increase B
= increase degeneracy of LL
= Fermi Energy moves down (fixed density) and crosses localized states (plateaux) and LL cores (jump of R)



FQHE Coulomb interaction and formation of quasiparticles that undergo IQHE
(Haldane, Jain, Moore, Halperin, Laughlin, Wen, ...)

Electrons in the lowest Landau level

$$\psi_s(\mathbf{r}) = \frac{1}{\sqrt{L_y}} \frac{1}{\pi^{\frac{1}{4}}\sqrt{\ell}} \exp\left[-\frac{1}{2}\left(\frac{x}{\ell} - \frac{2\pi\ell}{L_y}s\right)^2 - i\frac{2\pi}{L_y}sy\right], \quad 0 \le s \le N_s - 1$$

(Jacobi theta functions in torus geometry)

$$H = E_{LL}(B)N + V(ee,B)$$

$$\langle s_1, s_2|v|s_3, s_4\rangle = \frac{e^2}{L_y}\delta_{s_1+s_2,s_3+s_4} e^{-\frac{2\pi^2\ell^2}{L_y^2}(s_3-s_1)^2} \int_{-\infty}^{\infty} dq \frac{e^{-\frac{\ell^2}{2}q^2 + iq\frac{2\pi\ell^2}{L_y}(s_3-s_2)}}{\sqrt{q^2 + \frac{4\pi^2}{L_y^2}(s_3-s_1)^2}}$$

(Tao and Thouless, torus geometry)

It is function of s(3) - s(1) and s(3) - s(2) mod Ns>>1

V can be diagonalized if s(3)-s(2)=0

(exact in the thin torus limit, Bergholtz-Karlhede, 2008)

THIN TORUS LIMIT Lx << ell << Ly

$$V(s_{13}) = \frac{e^2}{L_y} e^{-\frac{\pi^2 \ell^2}{L_y^2}(s_3 - s_1)^2} K_0\left(\frac{\pi^2 \ell^2}{L_y^2}(s_3 - s_1)^2\right)$$

Change from periodic lattice s= 1...Ns to dual latt. via DFT

$$\ell^2 [H_{LLL} - \mu' N] = -\left(\frac{\mu'}{eB} - \frac{1}{m}\right) N + L_x \sum_{k,k'} \frac{e^2}{|k-k'|} n_k n_{k'}$$

The right-hand-side is a lattice Hamiltonian with long-range interaction, chemical potential mu(B)
k=1, ... , Ns >> 1 (degeneracy of LLL)
The exact g.s. was obtained by Hubbard.

PHYSICAL REVIEW B VOLUME 17, NUMBER 2 15 JANUARY 1978

Generalized Wigner lattices in one dimension and some applications to tetracyanoquinodimethane (TCNQ) salts

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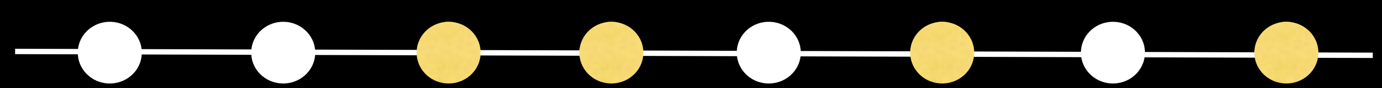
(Received 7 September 1977)

Estimates show that both the on-site and the near-neighbor electrostatic interactions in tetracyanoquinodimethane chains may be much greater than the bandwidth. A method of determining the exact ground state when the interactions are dominant is described; the electrons are found to have a periodic arrangement which may be regarded as a generalization of the classical Wigner lattice. It is shown how the optical spectra may be interpreted in terms of such a configuration; also that such arrangements may give rise to lattice distortions manifested as satellites in the x-ray diffraction pattern.



$$\boxed{E[n_1, n_2 \ldots] = \textstyle\sum_{a,b} V(|b-a|) n_a n_b - \mu \sum_a n_a}$$

Find the GS occupation numbers n = 0,1
(minimum of E with given density)
A universal answer if
1) V(m) > 0 and decreases to zero
2) V(m+1) + V(m-1) > 2 V(m)

Ground state of period q with p particles

(filling ratio p/q)

- particles wish to stay as far as possible
  but the lattice constrains their positions;
- q-fold degeneracy of gr. state for 1-site shift;
- particle-hole symmetry (exchange n with 1-n).

q=5

p=1 10000 10000 ....

p=2 10100 10100 ....

p=3 11010 11010 ....

q=7

p=1 1000000 1000000 ....

p=2 1001000 1001000 ....

p=3 1010100 1010100 ....

Spacings are n, n-1, n+1, with n = [q/p].
The sequence is obtained from the continued fraction expansion of q/p (Hubbard)

A hierarchy of ground states
by continued fraction expansion of p/q

TABLE II. Generalized-Wigner-lattice configurations.

	Density	Period		Configuration
(a)	$\frac{1}{3}$	3	3	100100100. . .
(b)	$\frac{2}{5}$	5	32	100101001010010. . .
(c)	$\frac{3}{7}$	7	$2^2 3$	10101001010100. . .
(d)	$\frac{3}{8}$	8	$3^2 2$	1001001010010010. . .
(e)	$\frac{10}{23}$	23	$(2^2 3)^2 2^3 3$	10101001010100101010100. . .
(f)	$\frac{3}{5}$	5	12^2	1101011010. . .
(g)	$\frac{3}{4}$	4	$1^2 2$	111011101110. . .
(h)	$\frac{4}{7}$	7	12^3	11010101101010. . .
(i)	$\frac{1}{2}$	2	2	1010101010. . .
(j)	$\frac{1}{2}$	4	13	110011001100. . .

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One-Dimensional Ising Model and the Complete Devil's Staircase

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(Received 18 March 1982)

It is shown rigorously that the one-dimensional Ising model with long-range antiferromagnetic interactions exhibits a complete devil's staircase.

PACS numbers: 05.50.+q, 75.10.Hk

$$\Delta\mu = 2q\sum_{k=1}^{\infty} k[V(kq+1) + V(kq-1) - 2V(kq)]$$

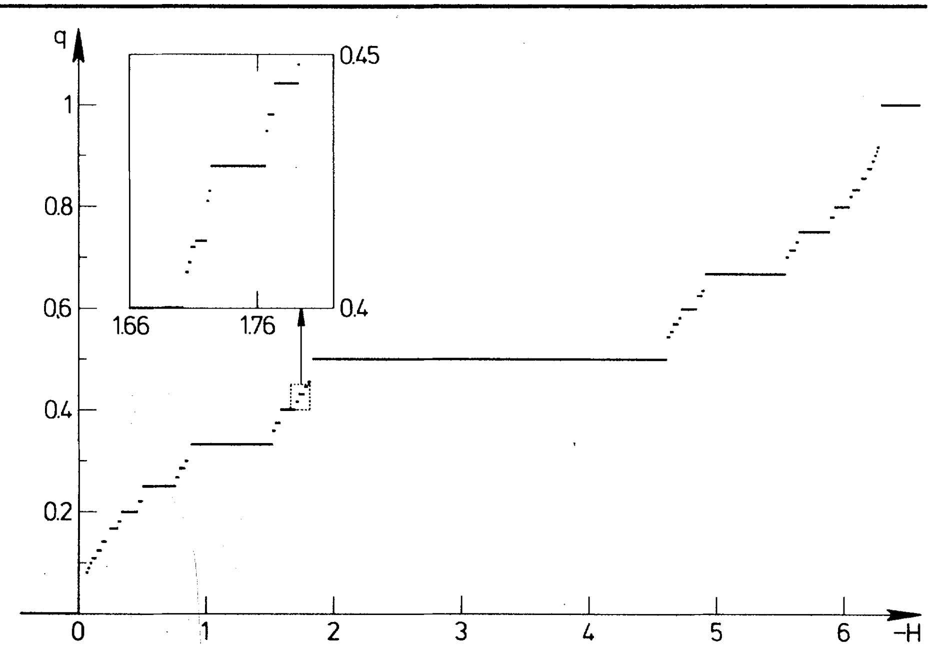
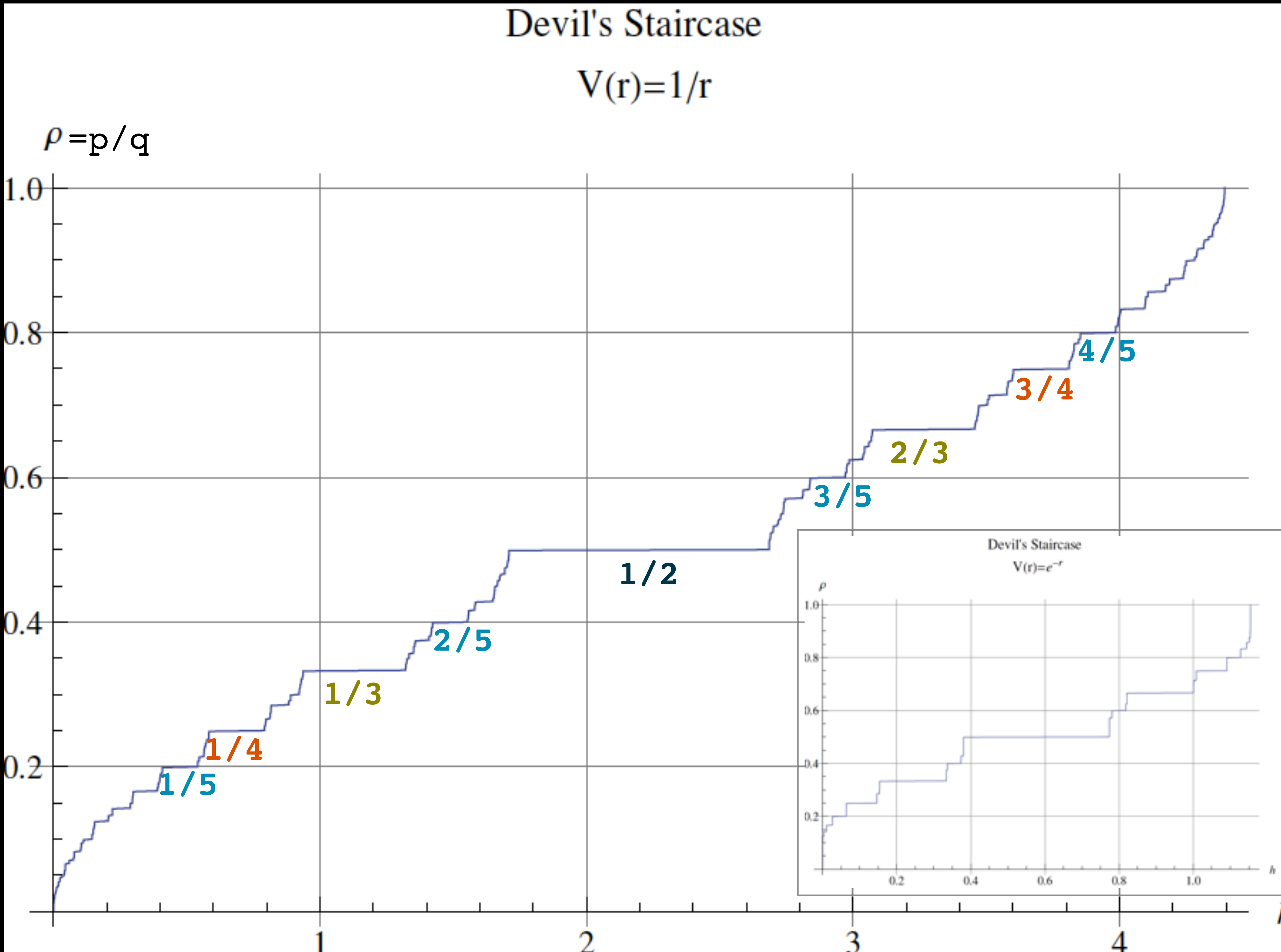


FIG. 2. The devil's staircase. The ratio of up spins over down spins q is plotted vs the applied field H for an interaction $J(i) = i^{-2}$. Inset: The area in the square magnified 10 times.

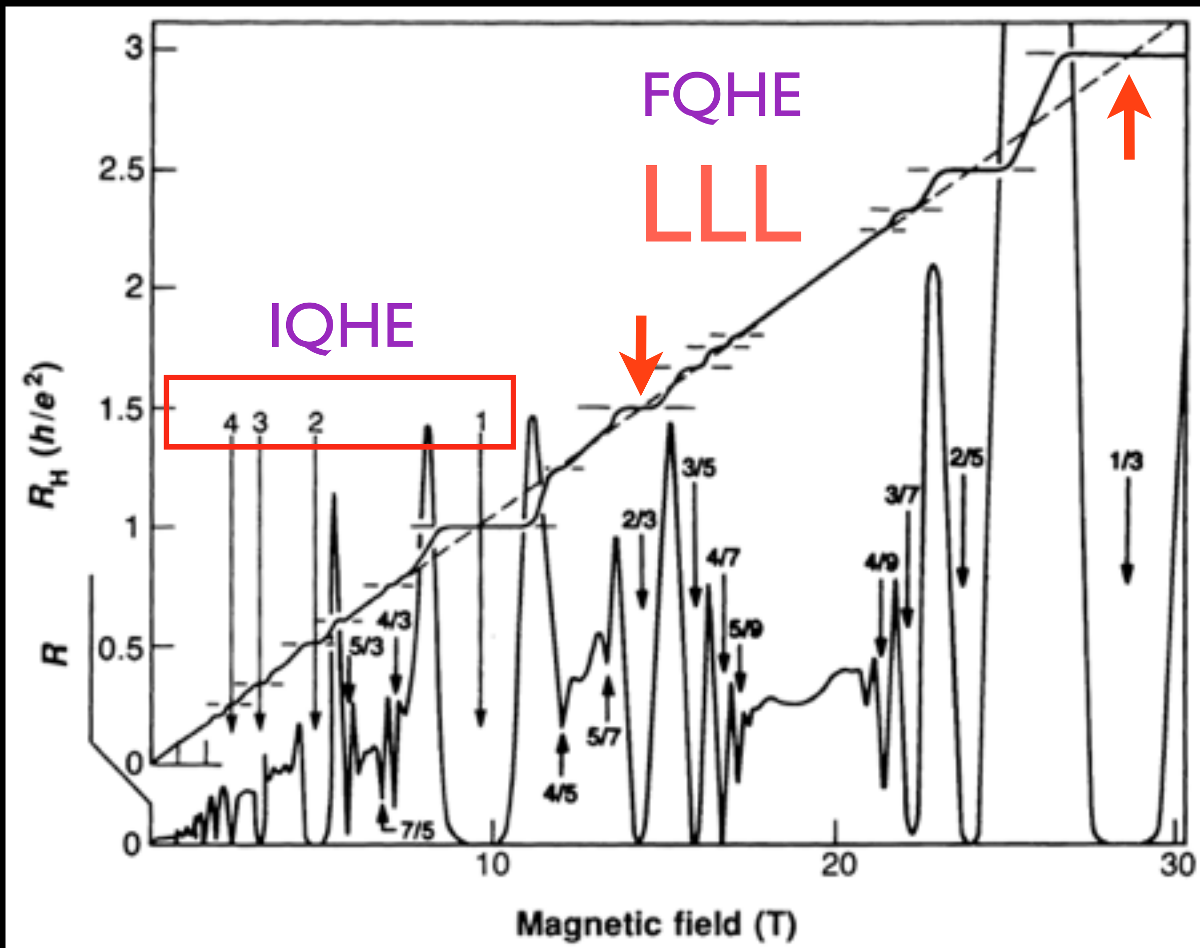


Devil's Staircase
V(r)=1/r
ρ=p/q
1.0
0.8
0.6
0.4
0.2
1
2
3
4
h
1/5
1/4
1/3
2/5
1/2
3/5
2/3
3/4
4/5
Devil's Staircase
V(r)=e^{-r}
ρ
1.0
0.8
0.6
0.4
0.2
0.2
0.4
0.6
0.8
1.0
h

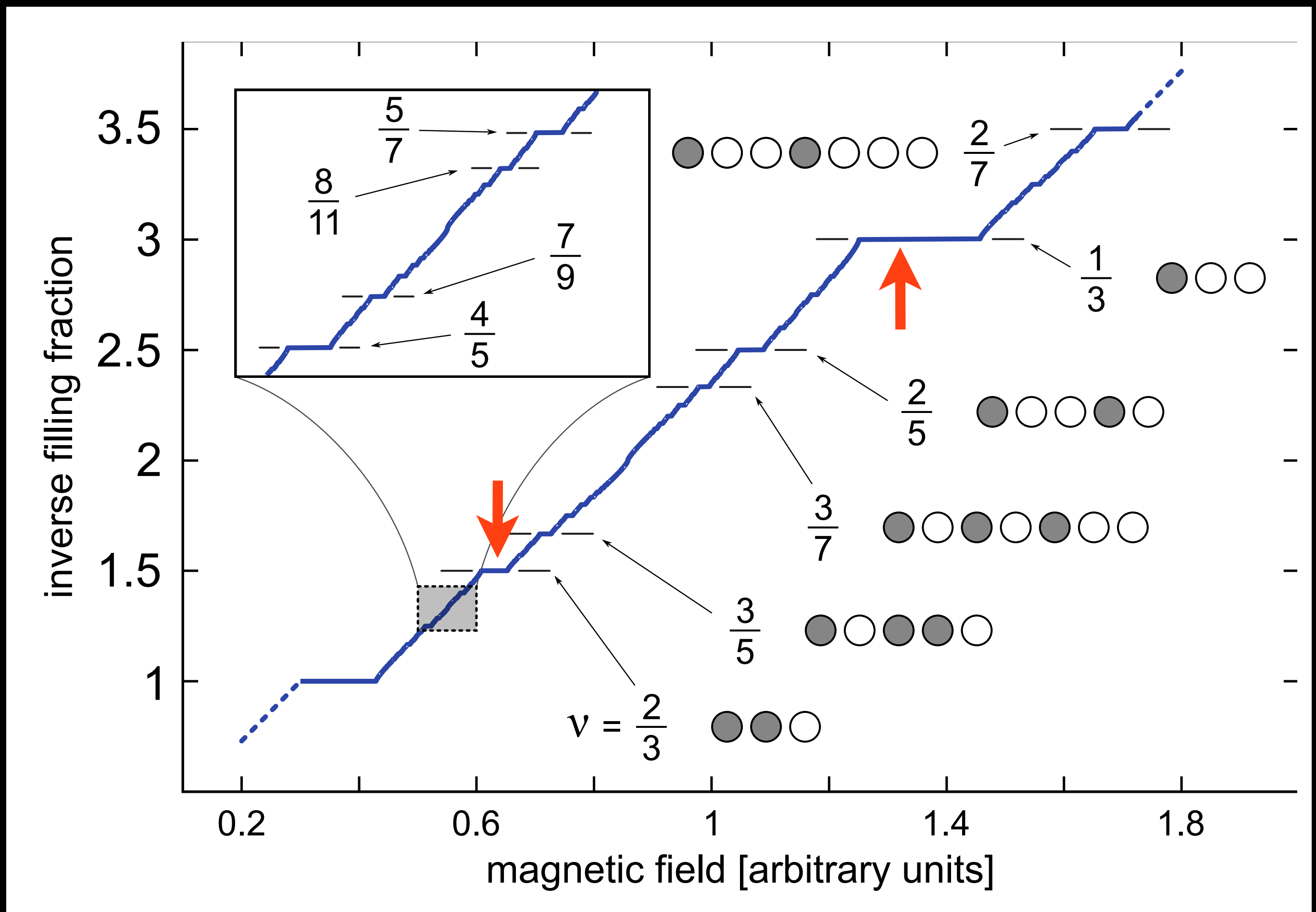
The g. s. configuration `[n(1),n(2),...]`
for p/q is independent of the potential.
The widths of the plateaux depend on the potential.
Same period q gives same widths.

Surprisingly, nobody ever translated the plot density - mu of the lattice gas to a plot inverse density - B for the thin torus FQHE
While rescaling mu ~ 1/B, plateaux with same q become narrower for higher p as experimentally observed.

QHE - experiment



Lattice crystal



- Rescaling in B gives particle-hole asymmetry of plateaux (see 2/3 and 1/3)
- absence of even denominators (few exceptions, as 5/2)
- average linearity

Absence of even denominators

symmetries & Fermi statistics

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S-Duality Constraints on 1D Patterns Associated with Fractional Quantum Hall States

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Using the modular invariance of the torus, constraints on the 1D patterns are derived that are associated with various fractional quantum Hall ground states, e.g., through the thin torus limit. In the simplest case, these constraints enforce the well-known odd-denominator rule, which is seen to be a necessary property of all 1D patterns associated to quantum Hall states with minimum torus degeneracy. However, the same constraints also have implications for the non-Abelian states possible within this framework. In simple cases, including the $\nu = 1$ Moore-Read state and the $\nu = 3/2$ level 3 Read-Rezayi state, the filling factor and the torus degeneracy uniquely specify the possible patterns, and thus all physical properties that are encoded in them. It is also shown that some states, such as the "strong *p*-wave pairing state," cannot in principle be described through 1D patterns.

Symmetries of Hamiltonian on the torus

Let $a_k^\dagger$, a_k, $k = 1 \dots L$ be canonical Fermi or Boson operators that create/destroy a particle at a site k of a ring. The dual canonical basis of operators is

$$c_r^\dagger = \frac{1}{\sqrt{L}} \sum_{k=1}^{L} e^{i\frac{2\pi k}{L} r} a_k^\dagger \qquad (1)$$

Discrete Fourier transform (L=Ns)

it corresponds to a unitary operator

$$\boxed{c_k^\dagger = \mathbf{U}^\dagger a_k^\dagger \mathbf{U} = F_{kl} a_l^\dagger}$$

$$\mathbf{U}^\dagger \mathbf{H}(r)\, \mathbf{U} = \mathbf{H}\left(\frac{1}{r}\right), \qquad r = \frac{L_x}{L_y}$$

$$\mathbf{T} c_k^\dagger \mathbf{T}^\dagger = c_{k+1}^\dagger, \quad \mathbf{S}^\dagger a_k^\dagger \mathbf{S} = a_{k+1}^\dagger$$
$$\mathbf{T} a_k^\dagger \mathbf{T}^\dagger = e^{i\frac{2\pi}{N_s}k} a_k^\dagger, \quad \mathbf{S}^\dagger c_k^\dagger \mathbf{S} = e^{-i\frac{2\pi}{N_s}k} c_k^\dagger$$

Magnetic Translations commute with U and H(r)

$$\boxed{\mathbf{TS} = \mathbf{ST}\, e^{-i2\pi\nu}, \qquad \nu = \frac{N}{N_s} \equiv \frac{p}{q} \text{ (coprime numbers)}}$$

Let $\mathbf{H}$ be a Hamiltonian that commutes with $\mathbf{T}$, $\mathbf{S}$, $\mathbf{N}$, and $\mathcal{E}$ be the eigenspace of ground states of $\mathbf{H}$ with N particles, with projector $\mathbf{P}$. If $\nu = p/q$ and $\mathcal{E}$ has exactly dimension q, then:
1) $\mathbf{PT}^q\mathbf{P} = \mathbf{P}\lambda_T$, $\mathbf{PS}^q\mathbf{P} = \mathbf{P}\lambda_S$, $|\lambda_T| = |\lambda_S| = 1$;
2) If $\mathbf{PUP} \neq 0$ then $\boxed{\lambda_T = \lambda_S}$.
3) If $[\mathbf{H}, \mathbf{U}^2] = 0$ then $\lambda_T = \pm 1$ and $\lambda_S = \pm 1$

$$\mathbf{PT}^q\mathbf{P}|\boldsymbol{n}\rangle = \exp\left[iq\frac{2\pi}{N_s}\sum_{j=0}^{M-1}\sum_{k=1}^{q}(jq+k)n_k\right]|\boldsymbol{n}\rangle = (-1)^{qp(M-1)}|\boldsymbol{n}\rangle$$

$$\mathbf{PS}^q\mathbf{P}|\boldsymbol{n}\rangle = a_{q+1}^{\dagger n_1}\dots a_{2q}^{\dagger n_q}\dots a_1^{\dagger n_1}\dots a_q^{\dagger n_q}|0\rangle = (-1)^{p^2(M-1)}|\boldsymbol{n}\rangle \quad \text{(fermions)}$$

$$(-1)^{qp(M-1)} = (-1)^{p^2(M-1)}$$

true for all M if q=odd (p and q are coprime)

What is the analogous of the Thin Torus limit in the symmetric gauge ?
QHE as a 2D Wigner crystal?

$$\Psi(z_1 \dots z_n) = \prod_{i<j} (z_i - z_j)^m \exp(-\tfrac{1}{4\ell^2} \sum_k |z_k|^2)$$

Laughlin's ansatz for GS for filling 1/m is surprisingly good. What is its expansion on a basis of Slater states?