IDENTITIES AND EXPONENTIAL INEQUALITIES FOR TRANSFER MATRICES

(Properties of a single transfer matrix)

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- Duality (an identity for determinants of band matrices)
- Theorem by Demko Moss Smith on inverse of band matrices
- The singular values of a transfer matrix are large/small for large n (with exp bounds)

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Difference equation & transfer matrix

$$C_k u_{k-1} + A_k u_k + B_k u_{k+1} = E u_k, \quad u_k \in \mathbb{C}^m, \quad k = 1 \dots n.$$

$$\begin{bmatrix} u_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} B_k^{-1}(E - A_k) & -B_k^{-1} C_k \\ \mathbb{I}_m & 0 \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \end{bmatrix}$$

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix}$$

* Boundary conditions

* Large n asymptotics of u_n

The Lyapunov spectrum

Singular values $\sigma_1 \geq \ldots \geq \sigma_{2m}$ (eigenvalues of $[T(E)^\dagger T(E)]^{1/2}$):

$$\sigma_1 \cdots \sigma_p = \|\Lambda^p T(E)\| = \sup_{v_1 \ldots v_p} \frac{\text{Volume P}\{Tv_1, \ldots, Tv_p\}}{\text{Volume P}\{v_1, \ldots, v_p\}}$$

where $\mathrm{P}\{v_1, \ldots, v_p\}$ = parallelogram with sides $v_i \in \mathbb{R}^{2m}$ (linear algebra).

If $T(E)$ is the product of $n >> 1$ random matrices, $\sigma_k \approx e^{\pm n\lambda_k}$ (**Oseledec's theorem**) with Lyapunov exponents λ_k *independent of the realization*. Then:

$$\lambda_1 + \ldots + \lambda_p = \lim_{n\to\infty} \frac{1}{n} \ln \|\Lambda^p T\|$$

Kunz and Souillard:

$$\frac{\lambda_1 + \ldots + \lambda_m}{m} = \int dE' \rho(E') \ln |E - E'| + \text{const.}$$

b.c. and spectrum of a single $T(E)$

$$\begin{bmatrix} u_{n+1} \\ u_n \end{bmatrix} = T(E) \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} = z \begin{bmatrix} u_1 \\ u_0 \end{bmatrix} \qquad H(z) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z} C_1 \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ z B_n & & C_n & A_n \end{bmatrix}$$

z is eigenvalue of $T(E)$ [with eigenvector $(u_1, u_n/z)^t$]

$\Leftrightarrow$

E is eigenvalue of $H(z)$ [with eigenvector $(u_1, \ldots, u_n)^t$]

Theorem (**Duality** Molinari 1997)

$$\boxed{\det[z\mathbb{I}_{2m} - T(E)] = (-z)^m \frac{\det[E\mathbb{I}_{nm} - H(z)]}{\det(B_1 \cdots B_n)}}$$

similarity: b.c. $\Leftrightarrow$ ring geometry

$$H(z^n) = \begin{bmatrix} A_1 & B_1 & & \frac{1}{z^n}C_1 \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ z^n B_n & & C_n & A_n \end{bmatrix} \sim \begin{bmatrix} A_1 & zB_1 & & \frac{1}{z}C_1 \\ \frac{1}{z}C_2 & \ddots & \ddots & \\ & \ddots & \ddots & zB_{n-1} \\ zB_n & & \frac{1}{z}C_n & A_n \end{bmatrix}$$

N. Hatano and D. R. Nelson, Localization transitions in non-Hermitian quantum mechanics, PRL **77** (1996)

$$e^{\xi} u_{k+1} + \epsilon_k u_k + e^{-\xi} u_{k-1} = E u_k,$$

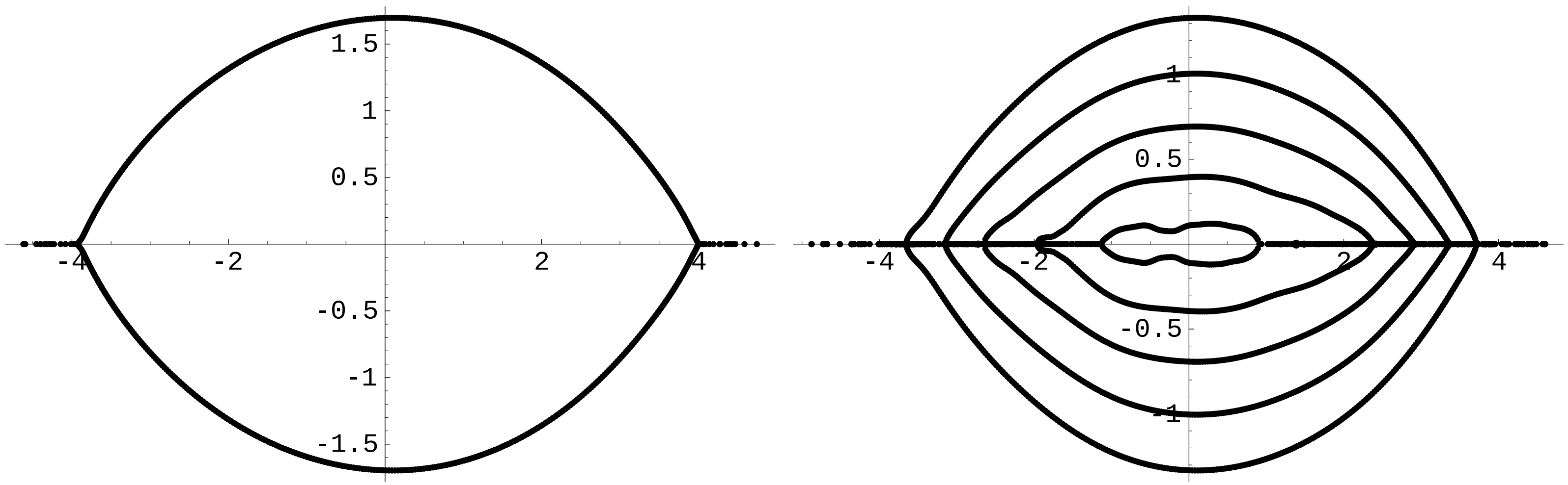

Figure 1: Left: the complex eigenvalues of a Hatano Nelson tridiagonal matrix ($n = 600$, $\xi = 1$) with diagonal elements in $[-3.5, +3.5]$. Real eigenvalues correspond to states with localization length less than $1/\xi$. Right: same system with ξ from 0 to 1, to show the expanding curve.

L. G. Molinari and G. Lacagnina, Disk-annulus transition and localization in random non-Hermitian tridiagonal matrices, JPA **42** (2009)

$$e^{\xi} c_k u_{k+1} + a_k u_k + e^{-\xi} b_k u_{k-1} = E u_k, \quad a_k, b_k, c_k \in [-1, 1]$$

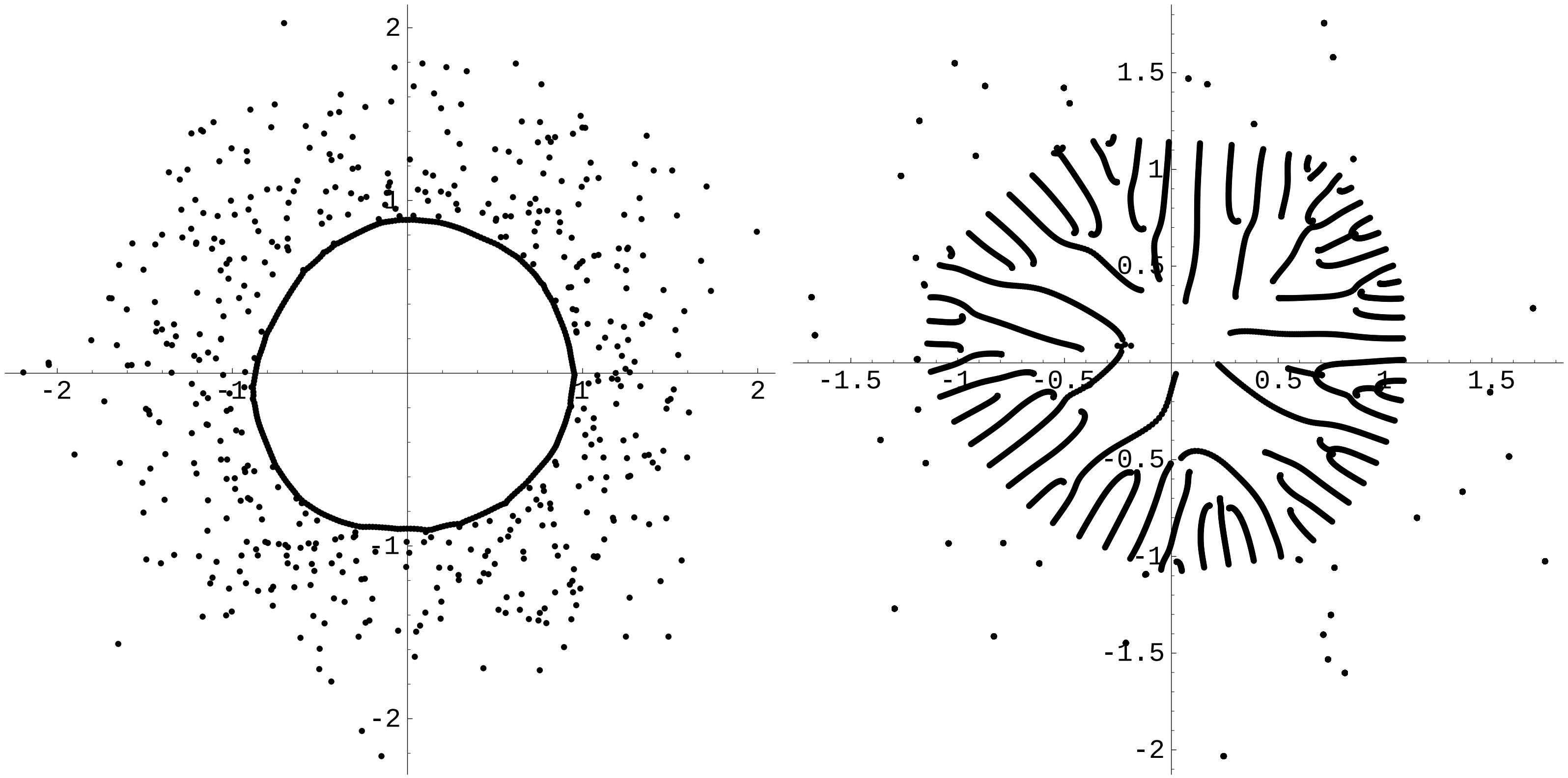

Figure 2: Left: the eigenvalues of a tridiagonal matrix ($n = 800$, $\xi = 0.5$). Right: $n = 100$; motion of eigenvalues for ξ from 0.3 to 0.6. Outer eigenvalues numerically unaffected before being reached.

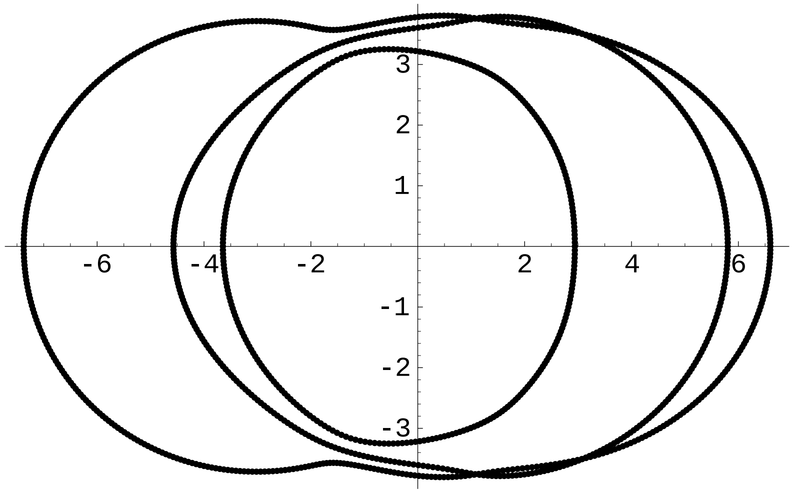

Figure 3: 2D Anderson model on lattice $(n = 8) \times (m = 3)$, disorder $w = 7$, $|z| = \exp 1.5$. As arg z changes, the 24 eigenvalues trace $m = 3$ closed loops of equations $|z_k(E)| = |z|$.

Transfer matrix and resolvent

$$h = \begin{bmatrix} A_1 & B_1 & & \\ C_2 & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ & & C_n & A_n \end{bmatrix}, \quad [h-E]^{-1} =: \begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \vdots \\ g_{n,1} & \cdots & g_{n,n} \end{bmatrix}$$

$$\boxed{T(E) = \begin{bmatrix} -B_n^{-1}(g_{1,n})^{-1} & -B_n^{-1}(g_{1,n})^{-1}g_{1,1}C_1 \\ g_{n,n}(g_{1,n})^{-1} & g_{n,n}(g_{1,n})^{-1}g_{1,1}C_1 - g_{n,1}C_1 \end{bmatrix}}$$

S. Demko, W. F. Moss and P. W. Smith, Decay rates for inverses of band matrices, *Math. Comp.* (1984) **43** 491-499.

1) Error of best approximation of $1/x$ on $[a, b]$ by a monic polynomial p_k (Chebyshev):

$$\inf_{p_k \in P_k} \left\{ \sup_{x \in [a,b]} \left| \frac{1}{x} - p_k(x) \right| \right\} = C\, q^{k+1},$$

$$C = \frac{(\sqrt{b} + \sqrt{a})^2}{2ab}, \qquad q = \frac{\sqrt{b} - \sqrt{a}}{\sqrt{b} + \sqrt{a}}$$

2) In block tridiagonal matrix A_{ij}, blocks $[p_k(A)]_{i,j} = 0$ for $|i - j| > k$.

Theorem 0.1. *$A > 0$ is block tridiagonal matrix with blocks $m \times m$, $sp(A) \subseteq [a, b]$. If $A^{-1}[i, j]$ is a matrix element in block $(A^{-1})_{ij}$ then:*

$$\boxed{\left|A^{-1}[i, j]\right| \leq C\, q^{|i-j|}}$$

Proof:

$$
\begin{aligned}
|A^{-1}[i,j]| &= \left|A^{-1}[i,j] - p_k(A)[i,j]\right| \le \|A^{-1} - p_k(A)\| \\
&= \sup_{\lambda \in sp(A)} \left|\frac{1}{\lambda} - p_k(\lambda)\right| \le \sup_{\lambda \in [a,b]} \left|\frac{1}{\lambda} - p_k(\lambda)\right|
\end{aligned}
$$

inf over monic p_k is taken. Minimum exists, error gives main inequality. If $i = j$: $|A^{-1}[i,i]| \le \|A^{-1}\| = 1/a$.

If A block tridiagonal but non positive: $A^{-1} = A^\dagger (AA^\dagger)^{-1}$.

Theorem 0.2. *A is an invertible block tridiagonal matrix, $sp(AA^\dagger) \subseteq [a,b]$, let $A^{-1}[i,j]$ be* any *matrix element in the block $(A^{-1})_{ij}$. Then:*

$$|A^{-1}[i,j]| \le C' \, q^{\frac{1}{2}|i-j|}$$

$$|g[1,n]| \;\leq\; C''\, q^{\frac{1}{2}(n-3)}, \quad |g[n,1]| \;\leq\; C''\, q^{\frac{1}{2}(n-3)}$$

Lemma 0.3. *If $\theta_1 \ldots \theta_m$ singular values of T_{11}, then $\theta_k > Kq^{-n/2}$*
$\sum_{k=1}^{m} \theta_k^{-2} = \mathrm{tr}[(T_{11}^\dagger T_{11})^{-1}] = \mathrm{tr}[B_n B_n^\dagger g_{1n}^\dagger g_{1n}] \leq \mathrm{const}\, q^n \Rightarrow \theta_k > Kq^{-n/2}$.

Main Theorem 0.1. *If $q < 1$ and $n \gg m$, $T(E)$ has singular values $\sigma_1 \geq \ldots \geq \sigma_m > Kq^{-n/2}$ and singular values $\sigma_{m+1} \geq \ldots \geq \sigma_{2m} < \frac{1}{K} q^{n/2}$.*
Interlacing property:

$$\sigma_k \geq \theta_k \geq \sigma_{m+k}, \quad k = 1, \ldots, m$$

Therefore, at least m singular values of $T(E)$ are larger than $q^{-n/2}$.
Same holds true for $T(E)^{-1}$ (similar to a transfer matrix) $\Rightarrow$ precisely m singular values of $T(E)$ are larger than $q^{-n/2}$ and m are smaller than $q^{n/2}$.

Jensen's formula and exponents

$\det[z\mathbb{I}_{2m} - T(E)]$ has zeros $z_1, \ldots, z_{2m}$ (eigenvalues of $T(E)$), and zeros $E_1, \ldots, E_{nm}$ (eigenvalues of $H(z)$). *Exponents* of the transfer matrix:

$$\xi_k =: \frac{1}{n} \ln |z_k|$$

Jensen: f holomorphic with zeros z_k in disk of radius r, $f(0) \neq 0$, then

$$\int_0^{2\pi} \frac{d\theta}{2\pi} \ln |f(re^{i\theta})| = \ln |f(0)| - \sum_k \ln(|z_k|/r)$$

Proposition:

$$\frac{1}{m} \sum_{\xi_k < \xi} (\xi - \xi_k) \; - \xi =$$

$$= \frac{1}{mn} \int_0^{2\pi} \frac{d\varphi}{2\pi} \ln \left|\det[H(e^{n\xi + i\varphi}) - E]\right| - \frac{1}{mn} \sum_{j=1}^{n} \ln |\det C_j|$$

Sum of positive exponents

One matrix (exact):

$$\frac{\xi_1 + \ldots + \xi_k}{m} = \frac{1}{nm} \int_0^{2\pi} \frac{d\theta}{2\pi} \ln \left| \det[H(e^{i\theta}) - E] \right| - \frac{1}{n} \ln |\det [B_1 \cdots B_n]|$$

Anderson model (Kunz Souillard):

$$\frac{\lambda_1 + \ldots + \lambda_m}{m} = \int dE' \rho(E') \ln |E' - E| + \text{const.}$$