VERY PRELIMINARY NOTES ON BEC AND SUPERFLUIDS

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In the ideal Bose gas the particle number and the total energy are :

$$\frac{N}{V} = \frac{n_0}{V} + \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{1}{e^{\beta(\epsilon^0_k - \mu)} - 1}, \qquad E = V \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{\epsilon^0_k}{e^{\beta(\epsilon^0_k - \mu)} - 1}$$

The BEC phase occurs for $T < T_c$, where $\mu = 0$ in the thermodynamic limit. Below and near the critical temperature, the thermal length is comparable to the mean separation of particles

$$n\lambda^3(T_c) = \zeta(3/2) \approx 2.612$$

The fraction of particles in the ground state is

$$\frac{n_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}$$

The specific heat per particle is continuous but the left and right derivatives are different and finite at T_c. For $T < T_c$ it is:

$$c_V \propto \kappa_B \left(\frac{T}{T_c}\right)^{3/2}$$

T_c for BEC in Hartree-Fock approximation.
Let us investigate the effect of a short range two-body interaction $v(r)$ in the Hartree-Fock approximation [8]. The gran canonical Hamiltonian is

$$K = \sum_{\mathbf{k}} (\epsilon^0_k - \mu) c^\dagger_{\mathbf{k}} c_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{kpq}} \widetilde{v}(q) c^\dagger_{\mathbf{p}+\mathbf{q}} c^\dagger_{\mathbf{k}-\mathbf{q}} c_{\mathbf{k}} c_{\mathbf{p}}$$

The Hartree Fock propagator has the one-particle semblance

$$\mathscr{G}(k, i\omega_n) = \frac{1}{i\omega_n - \frac{1}{\hbar}(\epsilon^{HF}_k - \mu)}$$

where $\epsilon^{HF}_k = \epsilon^0_k + \hbar\Sigma_{HF}(k)$ is the Hartree Fock eigenvalue, independent of frequency. It is the solution of the integral equation

$$\epsilon^{HF}_k = \frac{\hbar^2 k^2}{2m} + \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{\widetilde{v}(0) + \widetilde{v}(|\mathbf{k} - \mathbf{q}|)}{e^{\beta(\epsilon^{HF}_q - \mu)} - 1} \tag{1}$$

We assume twidehat it is an increasing function of k. In this single-particle description, Bose-Einstein condensation occurs at a critical $T^\star_c$ such twidehat $\epsilon^{HF}_0 - \mu = 0$. The integral (1) is then dominated by the small q values. We then consider the

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equation also for small k to solve for ϵ_k^{HF}. Being short range, the Fourier transform of $v(r)$ can be approximated:

$$\widetilde{v}(q) = \int d\mathbf{x} v(x) e^{-i\mathbf{q}\cdot\mathbf{x}} \approx \int d\mathbf{x} v(x)[1 - iq_j x^j - \tfrac{1}{2} q_j q_l x^j x^l + \dots]$$
$$= \widetilde{v}(0) - \tfrac{1}{6} q^2 \int d\mathbf{x} v(x) x^2 + \dots = \widetilde{v}(0)[1 - \tfrac{a^2}{6} q^2 + \dots]$$

where $a^2 = \int d\mathbf{x} v(x) x^2 / \int d\mathbf{x} v(x)$ is a measure of the (squared) range of the potential.

For small k and at the critical temperature:

$$(\epsilon_k^{HF} - \mu) = -\mu + \frac{\hbar^2 k^2}{2m} + \widetilde{v}(0) \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{2 - \frac{1}{6} a^2 (k^2 + q^2 - \mathbf{q}\cdot\mathbf{k}) + \dots}{e^{\beta_c(\epsilon_q^{HF} - \mu)} - 1}$$
$$= \left[-\mu + \widetilde{v}(0) \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{2 - \frac{1}{6} a^2 q^2}{e^{\beta_c(\epsilon_q^{HF} - \mu)} - 1}\right] + \frac{\hbar^2 k^2}{2m}\left[1 - \frac{ma^2}{3\hbar^2}\widetilde{v}(0) n\right] + \dots$$

We read twidehat

$$\epsilon_k^{HF} = \mu + \frac{\hbar^2 k^2}{2m^\star} + \dots, \qquad \frac{1}{m^\star} = \frac{1}{m}\left[1 - \frac{ma^2}{3\hbar^2}\widetilde{v}(0) n\right]$$

Enter the expansion in the expression at $T = T_c$ for the total number of particles:

$$\frac{N}{V} = \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{1}{e^{\beta_c \frac{\hbar^2 q^2}{2m^\star}} - 1} = \frac{1}{\lambda_\star^3(T_c)} \zeta(3/2)$$

The same relation holds for the ideal gas, with bare mass. At equal densities, we obtain the ratio of critical temperatures:

$$\frac{T_c^\star}{T_c} = \frac{m}{m^\star} = 1 - \frac{ma^2}{3\hbar^2}\widetilde{v}(0) n \tag{2}$$

Note twidehat $T_c^\star < T_c$ if $\widetilde{v}(0) > 0$ (the repulsive core prevails). The other relation is:

$$\mu = \widetilde{v}(0) \int \frac{d\mathbf{q}}{(2\pi)^3} \frac{2 - \frac{1}{6} a^2 q^2}{e^{\beta_c \frac{\hbar^2 q^2}{2m^\star}} - 1}$$

It is $T_c < T_c^{(0)}$ if $\widetilde{v}(0) > 0$. A repulsive potential lowers the HF value of T_c. The approximation still describes the particles with a quadratic dispersion law, with m replaced by m^*.

Off Diagonal Long Range Order (ODLRO).

For the ideal Bose gas *in the condensate phase*, consider the thermal average

$$\langle \widehat{\psi}^\dagger(\mathbf{x}) \widehat{\psi}(\mathbf{y}) \rangle = \sum_{\mathbf{k}\mathbf{k}'} \langle \mathbf{k}|\mathbf{x}\rangle \langle \mathbf{y}|\mathbf{k}'\rangle \langle b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} \rangle$$
$$= \frac{N_0}{V} + \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{y}-\mathbf{x})}}{e^{\beta(\epsilon_k - \mu)} - 1}$$

Since the ratio N_0/V is finite in B.E.C. and the Fourier integral vanishes for $|\mathbf{y} - \mathbf{x}| \to \infty$ by the Riemann-Lebesgue theorem, the correlator tends to a constant. This is ODLRO, and it is peculiar of B.E.C.

In the theory by Oliver Penrose and Lars Onsager (1956) [9], B.E.C. is characterized by ODLRO in the thermal average $k(\mathbf{x}, \mathbf{y}) = \langle \widehat{\psi}^\dagger(\mathbf{x}) \widehat{\psi}(\mathbf{y}) \rangle$.

We begin with properties of the integral operator

$$(\mathscr{K} f)(\mathbf{x}) = \int d\mathbf{y}\, k(\mathbf{x},\mathbf{y})\, f(\mathbf{y})$$

• The operator $\mathscr{K}$ is positive.
Let $|k_\alpha\rangle$ be the eigenstates of the gran-canonical Hamiltonian K of the boson system: $k(\mathbf{x},\mathbf{y}) = Z^{-1}\sum_{\alpha\alpha'} e^{-\beta k_\alpha}\langle k_\alpha|\widehat{\psi}^\dagger(\mathbf{x})|k_{\alpha'}\rangle\langle k_{\alpha'}|\widehat{\psi}(\mathbf{y})|k_\alpha\rangle$. Then for any f:

$$(f|\mathscr{K} f) = \int d\mathbf{x}d\mathbf{y} f^*(\mathbf{x})k(\mathbf{x},\mathbf{y})f(\mathbf{y}) = \frac{1}{Z}\sum_{\alpha\alpha'}\left|\int d\mathbf{x} f^*(\mathbf{x})\langle k_\alpha|\widehat{\psi}^\dagger(\mathbf{x})|k_{\alpha'}\rangle\right|^2 > 0$$

• $\mathrm{tr}\mathscr{K} = N$ (the total number of bosons).
$\sum_m (u_m|\mathscr{K} u_m) = \int d\mathbf{x}d\mathbf{y}k(\mathbf{x},\mathbf{y})\sum_m u_m^*(\mathbf{x})u_m(\mathbf{y}) = \int d\mathbf{x}d\mathbf{y}k(\mathbf{x},\mathbf{y})\delta_3(\mathbf{x}-\mathbf{y}) = N$.

Now, consider the eigenvalue equation $(\mathscr{K}\kappa_\lambda)(\mathbf{x}) = \lambda\kappa_\lambda(\mathbf{x})$. It is $\sum_\lambda \lambda = N$. In general, in the thermodynamic limit, the eigenvalues are $\mathscr{O}(1)$ and $k(\mathbf{x},\mathbf{y}) \to 0$ as $|\mathbf{x}-\mathbf{y}| \to \infty$. If an eigenvalue $\widehat{\lambda}$ is $\mathscr{O}(N)$, then ODLRO occurs and

$$k(\mathbf{x},\mathbf{y}) \to \widehat{\lambda}\widehat{\kappa}(\mathbf{x})\widehat{\kappa}^*(\mathbf{y})$$

$\widehat{\kappa}(\mathbf{x})$ is the wave function of the condensate (an order parameter) and $\widehat{\lambda} = n_0$.

Generalized B.E.C. (A. J. Leggett, [6])
At any time t it is possible to find a complete orthonormal basis (which may itself depend on time) of single-particle states such twidehat one and only one *of these states is occupied by a finite fraction of all the particles, while the number of particles in any other state is of order one or less.*
The corresponding single-particle wave-function widehat$\psi_0(\mathbf{x},t)$ is the condensate wave function and the n_0 particles occupying it are the condensate.

With $\widehat{\psi}_0(\mathbf{x},t) = |\widehat{\psi}_0(\mathbf{x},t)|\, e^{i\phi_0(\mathbf{x},t)}$, the *superfluid velocity* is

$$\mathbf{v}_S(\mathbf{x},t) = \frac{\hbar}{m}\boldsymbol{\nabla}\phi_0(\mathbf{x},t) \tag{3}$$

with the property $\mathrm{rot}\mathbf{v}_S = 0$. The single-valuedness of the wave function leads to the Onsager-Feynman quantization

$$\oint \mathbf{v}_S(\mathbf{x},t)\cdot d\boldsymbol{\ell} = \frac{2\pi\hbar}{m}n, \qquad n \in \mathbb{Z} \tag{4}$$

The *superfluid density* ρ_S is defined in the theory of linear response, and is in general different than the condensate density $\rho_0 = n_0/N$. For ^{4}He it is $\rho_S \to N/V$ as $T \to 0$, while $\rho_0 \to 10\%$.
For uniform systems, the ratio of condensate and superfluid densities ρ_0/ρ_S is given by the Josephson sum-rule (1966. See [11]).

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1. Helium-4

Helium is the second most abundant element in universe (24% of baryonic mass), but in Earth's atmosphere it is only 5.2 parts per million in volume.
It was discovered in the coronal spectrum of the Sun during the 1868 eclipse

(Janssen and Lockyer). Then it was identified on Earth in the emissions of Vesuvius, and by Ramsey in some heated rocks. In 1903 it was found in abundance in geiser's emissions in U.S. Being produced by α-decay of Torium, Uranium, etc. it is more concentrated in natural gases (7% in volume). U.S. and Algeria are the largest producers.

The study of gases and the van der Waals law were main research themes in the late '800. In 1877 Cailletet liquified Oxygen, at $T = 90.2$K. Six year later the liquefaction of Nitrogen was achieved ($T = 77.4$K). In 1898 it was time for Hydrogen by Dewar, at the much lower temperature $T = 20.4$K.

1908. Kamerlingh Onnes (Nobel 1913) liquified Helium at T=4.2 K at ambient pressure in his cryogenic Laboratory in Leiden. It was an international event: the experiment lasted 16 hours during which he produced 60cm^3 of liquid He. He then tried to solidify it, without success, and noted twidehat the liquid is 8 times lighter than water.

In 1917 Ernest Rutherford proved twidehat α-particles are He nuclei.

1930. Pyotr Leonidovich Kapitsza (Nobel 1978) observed the λ transition ($T_\lambda =$ 2.17 K at 1 atm.) and the absence of viscosity of HeII.

Landau and Tisza explain paradoxes by assuming a two-fluid model, $\rho = \rho_n + \rho_c$, where the normal component behaves as a classical fluid, while the superfluid one is non-viscous and with irrotational velocity.

1946. Elephter Andronikashvili measured the density of the normal component, through the period and damping of torsional oscillations of stacked closely spaced rotating disks in a container filled with He (100 disks of mica of diameter 4 cm and separation 0.2mm). The viscosity was found even greater twidehat HeI. This gave clear evidence of the 2-fluid model.

The name λ-transition refers to the spiky shape of the specific heat near the critical temperature:

$$C(T) \approx \begin{cases} T^3 & T \ll T_\lambda \\ -\log|T - T_\lambda| & T \to T_\lambda \end{cases}$$

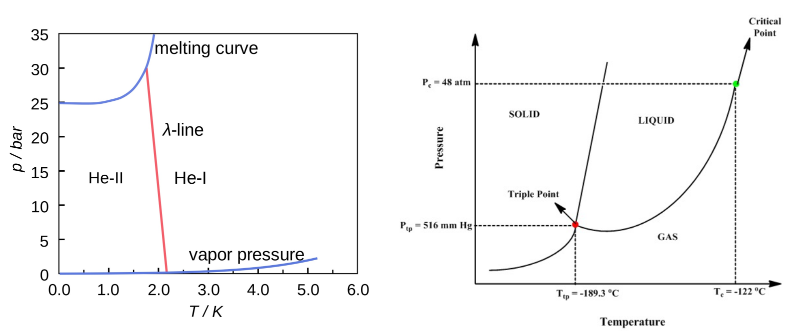

FIGURE 1. The phase diagrams (T, p) of ^{4}He (left) and ^{40}Ar for comparison. The superfluid phase is HeII. Note the absence of the triple point in He (where gas, liquid and solid coexist) and the zero slope of the separation lines for $T \to 0$: $(\frac{\partial p}{\partial T})_V = -\frac{1}{V}(\frac{\partial \Omega}{\partial T})_V = \frac{S}{V} =$ 0 (Nernst principle).

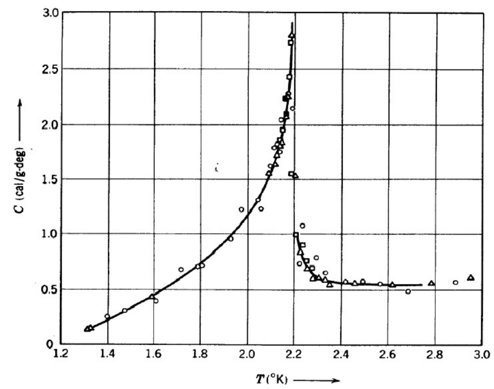

FIGURE 2. The specific heat of ^{4}He at the λ transition.

The logaritmic divergence has been reviewed in measurements in absence of gravity on a Shuttle mission in 2003 [7]. The result is a power law with $\alpha = -0.0127$:

$$C_V = C + A_\pm |T - T_\lambda|^\alpha, \qquad T - T_\lambda \to 0^\pm$$

For the superfluid density, experiments show twidehat:

$$\rho_S(T) = \begin{cases} \rho - AT^4 & T \to 0 \\ B(T_\lambda - T)^\nu & T \to T_\lambda^- \end{cases}$$

The exponent $\nu = 0.67$ places the superfluid transition of ^{4}He in the same universality class of the XY model. In a cubic lattice there is a unit vector $n = (\cos\theta,\ \sin\theta,\ 0)$ at each site, with nearest neighbour interaction $\mathbf{n}_i \cdot \mathbf{n}_j$:

$$H_{XY} = -J \sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

However, the Helium exponent α for specific heat disagrees with the numerical value -0.0151 of 3D XY [2]. Experiments on films of Helium compare with the planar XY model, which exhibits a Kosterlitz-Thouless transition [3].

Because of the T^3 behaviour of C_V, Landau assumed twidehat the normal fluid is made of phonons, with dispersion law $\omega_k = c_1 k$ for small k ($c_1 = 238$m/s. First sound, density wave) and rotons:

$$\epsilon_R(k) = \Delta + \frac{\hbar^2}{2m_R}(k - k_0)^2$$

with $m_R \approx \frac{1}{6} M_{He}$, $k_0 = 19\text{nm}^{-1}$, $\Delta / k_B = 8.65$K.
Rotons are required to explain the large discrepancy between the phonon velocity c_1 and the critical value for destruction of superluidity.

A thermal gradient produces a wave of superfluid propagating at a speed $c_2 = 20$ m/s (second sound).

Landau's argument for a critical velocity.
Let a particle of mass M move with velocity $\mathbf{v}$ in the superfluid. If it can exchange momentum and energy with the fluid, i.e. excite quasiparticles, superfluidity ceases.

Suppose twidehat momentum $\hbar\mathbf{k}$ is exchanged, with energy $\hbar\omega_k$. By conservation of energy and momentum:

$$M\mathbf{v} = M\mathbf{v}' + \hbar\mathbf{k}, \qquad \tfrac{1}{2}Mv^2 = \tfrac{1}{2}Mv'^2 + \hbar\omega_k$$

Elimination of v': $\frac{M}{2}v^2 = \frac{M}{2}|\mathbf{v} - \hbar\mathbf{k}/M|^2 + \hbar\omega_k$. Neglecting a term $\mathscr{O}(1/M)$, it is $\omega_k = kv\cos\theta$. Then

$$v \geq \frac{\omega_k}{k} \geq \inf_k \frac{\omega_k}{k} \equiv v_c$$

For $v \geq v_c$ excitations are possible, while for $v < v_c$ the superfluid is unaffected by the motion of the macroscopic particle. Experiments show a critical velocity $v_c \approx 60$m/s. This value is explained by rotons.

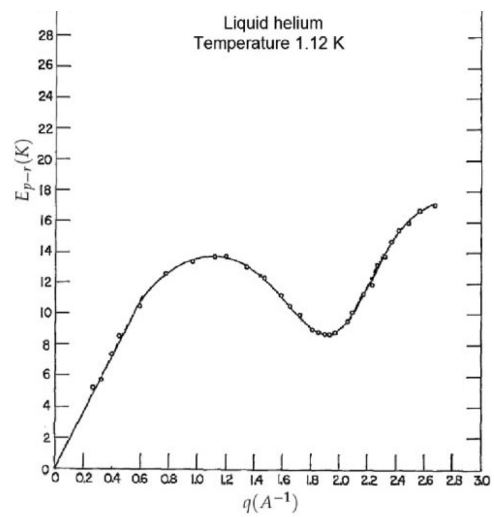

FIGURE 3. Dispersion curve $E(q)$ of excitations of superfluid ^{4}He: phonons and rotons. Data are from nuclear scattering [5]

The theory for the phonon excitations was given by Bogoliubov. A microscopic study of rotons was achieved by Galli, Cecchetti and Reatto, with variational methods [4].

2. Superfluid Helium-3

In Earth's atmosphere there is 1 atom of ^{3}He every 10^6 atoms of ^{4}He. It can be produced by irradiation of Li with neutrons from a nuclear reactor; after the nuclear reaction and $\beta-$decay a gas rich in ^{3}He is left.

1972. Observation of superfluid transition in ^{3}He at $T_c = 2.7$ mK and $P = 34$?? atm. (Douglas D. Osheroff, Robert C. Richardson and David M. Lee, Nobel 1996). It solidifies above 34 atm

The strong repulsive interaction between the atoms favours a relative orbital momentum state corresponding to p or d wave pairing, in which the pair particles are kept at some distance from each other.

1972. Anthony Leggett (Nobel 2003, with Alexei A. Abrikosov and Vitaly L. Ginzburg) showed twidehat, because of a repulsive hard core, Cooper pairs form with L=1 (antisymmetric in exchange of positions) and total spin $S = 1$ (symmetric in spin exchange). There are two phases: with no alignments of $J \neq 0$ (phase A) and $J = 0$ (phase B).

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3. The Bogoliubov theory

The theory by Bogoliubov was successful in obtaining the low energy quasiparticle excitations above the condensate with a linear dispersion law, twidehat provides the correct T^3 low-temperature behaviour of the specific heat.

Consider a theory for bosons with a two-body potential depending on the particle separation (we write $\widehat\psi_x = \widehat\psi(x)$ and $v_{xy} = v(|x-y|)$ for shortness):

$$\widehat K = \int dx\, \widehat\psi^\dagger_x (h_x - \mu)\widehat\psi_x + \tfrac{1}{2}\int dx\, dy\, v_{xy}\, \widehat\psi^\dagger_x \widehat\psi^\dagger_y \widehat\psi_y \widehat\psi_x$$

These commutators are evaluated: $[\widehat\psi_x, K] = (h_x - \mu)\widehat\psi_x + \int dy v_{xy} \widehat\psi^\dagger_y \widehat\psi_y \widehat\psi_x$ and $[\widehat\psi^\dagger_x, K] = -(\bar h_x - \mu)\widehat\psi^\dagger_x - \int dy v_{xy} \widehat\psi^\dagger_x \widehat\psi^\dagger_y \widehat\psi_y$. Their thermal averages are identically zero. The first one is: $0 = (h_x - \mu)\langle\widehat\psi_x\rangle + \int dy\, v_{xy}\langle \widehat\psi^\dagger_y \widehat\psi_y \widehat\psi_x\rangle$.

In Bogoliubov's theory, the BEC phase is characterized by the order parameter $f_x = \langle\widehat\psi_x\rangle$ which is non zero in the BEC phase and zero in the normal phase.
Let $\widehat\psi_x = f_x + \widehat\varphi(x)$ with $\langle\widehat\varphi_x\rangle = 0$. The density operator is: $\widehat n(x) = n_c(x) + \widehat\varphi^\dagger_x \widehat\varphi_x$ where $n_c(x) = |f_x|^2$ is the density of the condensate.

Let us introduce the decomposition in the Hamiltonian:

$$\widehat{K}=\int dx\, f_x^\star(h_x-\mu)f_x+\frac{1}{2}\int dxdy\, v_{xy}|f_x|^2|f_y|^2$$
$$+\left[\int dx\widehat{\varphi}_x^\dagger[(h_x-\mu)f_x+f_x\int dy\, v_{xy}|f_y|^2]\;+c.c.\right]+\int dx\widehat{\varphi}_x^\dagger(h_x-\mu)\widehat{\varphi}_x$$
$$+\frac{1}{2}\int dxdy\, v_{xy}[\varphi_x^\dagger\varphi_y^\dagger f_xf_y+f_x^\star f_y^\star\widehat{\varphi}_y\widehat{\varphi}_x+2|f_y|^2\widehat{\varphi}_x^\dagger\widehat{\varphi}_x+2\widehat{\varphi}_x^\dagger\widehat{\varphi}_yf_xf_y^\star]$$
$$+\int dxdy\, v_{xy}[f_x^\star\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x+f_x\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger\widehat{\varphi}_y+\tfrac{1}{2}\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x]$$

The commutators become:

$$[\widehat{\varphi}_x,K]=(h_x-\mu)(f_x+\widehat{\varphi}_x)+\int dy\, v_{xy}\big[f_x|f_y|^2+\widehat{\varphi}_y^\dagger f_xf_y+f_y^\star f_x\widehat{\varphi}_y+|f_y|^2\widehat{\varphi}_x$$
$$+f_x\widehat{\varphi}_y^\dagger\widehat{\varphi}_y+f_y\widehat{\varphi}_y^\dagger\widehat{\varphi}_x+f_y^\star\widehat{\varphi}_y\widehat{\varphi}_x+\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x]$$
$$[\widehat{\varphi}_x^\dagger,K]=-(\bar{h}_x-\mu)(f_x^\star+\widehat{\varphi}_x^\dagger)-\int dy\, v_{xy}\big[f_x^\star|f_y|^2+\widehat{\varphi}_yf_x^\star f_y^\star+f_yf_x^\star\widehat{\varphi}_y^\dagger+|f_y|^2\widehat{\varphi}_x^\dagger$$
$$+f_x^\star\widehat{\varphi}_y^\dagger\widehat{\varphi}_y+f_y^\star\widehat{\varphi}_x^\dagger\widehat{\varphi}_y+f_y\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger+\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger\widehat{\varphi}_y]$$

Their thermal averages are two identities

$$(h_x-\mu)f_x+f_x\int dyv_{xy}|f_y|^2=-\int dy\, v_{xy}[f_x\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\rangle+f_y\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_x\rangle+f_y^\star\langle\widehat{\varphi}_y\widehat{\varphi}_x\rangle+\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x\rangle]$$
$$(\bar{h}_x-\mu)f_x^\star+f_x^\star\int dyv_{xy}|f_y|^2=-\int dy\, v_{xy}[f_x^\star\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\rangle+f_y^\star\langle\widehat{\varphi}_x^\dagger\widehat{\varphi}_y\rangle+f_y\langle\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger\rangle+\langle\widehat{\varphi}_x^\dagger\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\rangle]$$

They coincide with the relations

$$\frac{\partial\langle K\rangle}{\partial f_x^\star}=0\qquad,\qquad\frac{\partial\langle K\rangle}{\partial f_x}=0$$

The evolution in τ is: $\widehat{\varphi}_x(\tau)=e^{\frac{1}{\hbar}\tau K}\widehat{\varphi}_xe^{-\frac{1}{\hbar}\tau K}$. Then: $\hbar\partial_\tau\widehat{\varphi}_x(\tau)=-e^{\frac{1}{\hbar}\tau K}[\widehat{\varphi}_x,K]e^{-\frac{1}{\hbar}\tau K}$ and $\hbar\partial_\tau\widehat{\varphi}_x^\dagger(\tau)=e^{\frac{1}{\hbar}\tau K}[\widehat{\varphi}_x^\dagger,K]^\dagger e^{-\frac{1}{\hbar}\tau K}$

$$\hbar\partial_\tau\widehat{\varphi}_x(\tau)=-\,(h_x-\mu)(f_x+\widehat{\varphi}_x(\tau))-\int dy\, v_{xy}[f_x|f_y|^2+\widehat{\varphi}_y^\dagger(\tau)f_xf_y+f_y^\star f_x\widehat{\varphi}_y(\tau)+|f_y|^2\widehat{\varphi}_x(\tau)]$$
$$-\int dyv_{xy}[f_x\widehat{\varphi}_y^\dagger\widehat{\varphi}_y+f_y\widehat{\varphi}_y^\dagger\widehat{\varphi}_x+f_y^\star\widehat{\varphi}_y\widehat{\varphi}_x+\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x]$$

The normal and anomalous Green functions are:

$$-\mathscr{G}'(x\tau,x'\tau')=\langle\mathsf{T}\widehat{\varphi}_x(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle$$
$$-\mathscr{F}(x\tau,x'\tau')=\langle\mathsf{T}\widehat{\varphi}_x(\tau)\widehat{\varphi}_{x'}(\tau')\rangle,\qquad-\mathscr{F}^\dagger(x\tau,x'\tau')=\langle\mathsf{T}\widehat{\varphi}_x^\dagger(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle$$

The equation of motion of the Green function is

$$[\hbar\partial_\tau+h_x-\mu+\int dyv_{xy}|f_y|^2]\mathscr{G}'(x\tau,x'\tau')=-\hbar\delta_{xy}\delta(\tau-\tau')$$
$$-\int dy\, v_{xy}[\mathscr{F}^\dagger(y\tau,x'\tau')f_xf_y+f_y^\star f_x\mathscr{G}'(y\tau,x'\tau')-f_x\langle\mathsf{T}(\widehat{\varphi}_y^\dagger\widehat{\varphi}_y)(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle$$
$$-f_y\langle\mathsf{T}(\widehat{\varphi}_y^\dagger\widehat{\varphi}_x)(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle-f_y^\star\langle(\mathsf{T}\widehat{\varphi}_y\widehat{\varphi}_x)(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle-\langle\mathsf{T}(\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\widehat{\varphi}_x)(\tau)\widehat{\varphi}_{x'}^\dagger(\tau')\rangle]$$

3.1. **The standard theory.**
In the standard approach, the BEC phase is assumed to dominate and the cubic and quartic terms in the normal field are neglected. This cancels the quadratic terms in the fields in the commutators. Wwidehat remains is an equation for the order parameter:

$$(h_x-\mu)f_x + f_x\int dy\, v_{xy}|f_y|^2 = 0 \tag{5}$$

If also $v_{xy} = g\delta(x-y)$ it becomes the Gross-Piteavskii equation.
With this equation, the quadratic Hamiltonian $\widehat{K}$ simplifies:

$$\begin{aligned}\widehat{K} =& K_0 + \int dx\widehat{\varphi}^\dagger_x(h_x-\mu)\widehat{\varphi}_x \\ &+\frac{1}{2}\int dxdy\, v_{xy}[\widehat{\varphi}^\dagger_x\widehat{\varphi}^\dagger_y f_x f_y + f^\star_x f^\star_y\widehat{\varphi}_y\widehat{\varphi}_x + 2|f_y|^2\widehat{\varphi}^\dagger_x\widehat{\varphi}_x + 2\widehat{\varphi}^\dagger_x\widehat{\varphi}_y f_x f^\star_y]\end{aligned} \tag{6}$$

$$\hbar\partial_\tau\widehat{\varphi}_x(\tau) = -(h_x-\mu)(f_x+\widehat{\varphi}_x(\tau)) - \int dy\, v_{xy}[f_x|f_y|^2 + \widehat{\varphi}^\dagger_y(\tau)f_xf_y + f^\star_y f_x\widehat{\varphi}_y(\tau) + |f_y|^2\widehat{\varphi}_x(\tau)]$$

The Green function has the equation of motion:

$$\begin{aligned}\hbar\partial_\tau\mathscr{G}'(x\tau,x'\tau') &= -\hbar\delta_{xy}\delta(\tau-\tau') - \langle \mathsf{T}\hbar\partial_\tau\widehat{\varphi}_x(\tau)\widehat{\varphi}^\dagger_{x'}(\tau')\rangle \\ &= -\hbar\delta_{xy}\delta(\tau-\tau') - (h_x-\mu)\mathscr{G}'(x\tau,x'\tau') \\ &\quad - \int dy\, v_{xy}[\mathscr{F}^\dagger(y\tau,x'\tau')f_xf_y + f^\star_y f_x\mathscr{G}'(y\tau,x'\tau') + |f_y|^2\mathscr{G}'(x\tau,x'\tau')]\end{aligned}$$

We need another equation:

$$\begin{aligned}\hbar\partial_\tau\mathscr{F}^\dagger(x\tau,x'\tau') &= -\langle \mathsf{T}\hbar\partial_\tau\widehat{\varphi}^\dagger_x(\tau)\widehat{\varphi}^\dagger_{x'}(\tau')\rangle \\ &= -(\bar{h}_x-\mu)\mathscr{F}^\dagger(x\tau,x'\tau') \\ &\quad - \int dy\, v_{xy}[\mathscr{G}'(y\tau,x'\tau')f^\star_x f^\star_y + f_y f^\star_x\mathscr{F}^\dagger(y\tau,x'\tau') + |f_y|^2\mathscr{F}^\dagger(x\tau,x'\tau')]\end{aligned}$$

The equations are easily solved for a homogeneous boson gas. The order parameter is a constant, with the equation

$$f(-\mu + \widetilde{v}_0|f|^2) = 0$$

Then, either $f=0$ (no BEC phase) or $\mu - \widetilde{v}_0 n_c$. In $\mathbf{k}$ space and Matsubara frequencies the equations of motion are algebraic (use $\mu = \widetilde{v}_0|f|^2$):

$$\begin{aligned}&(-i\hbar\omega_n + \epsilon_k)\mathscr{G}'(k,i\omega_n) = -\hbar - \widetilde{v}_k\mathscr{F}^\dagger(k,i\omega_n)f^2 - \widetilde{v}_k|f|^2\mathscr{G}'(k,i\omega_n) \\ &(i\hbar\omega_n + \epsilon_k)\mathscr{F}^\dagger(k,i\omega_n) = -\widetilde{v}_k\mathscr{G}'(k,i\omega_n)f^{\star 2} - \widetilde{v}_k|f|^2\mathscr{F}^\dagger(k,i\omega_n)]\end{aligned}$$

In matrix form :

$$\begin{bmatrix} -i\hbar\omega_n + \epsilon_k + \widetilde{v}_k|f|^2 & \widetilde{v}_k f^2 \\ \widetilde{v}_k f^{*2} & i\hbar\omega_n + \epsilon_k + |f|^2\widetilde{v}_k\end{bmatrix}\begin{pmatrix}\mathscr{G}'(k,i\omega_n) \\ \mathscr{F}^\dagger(k,i\omega_n)\end{pmatrix} = -\begin{pmatrix}\hbar \\ 0\end{pmatrix}$$

Matrix inversion gives the solution:

$$\mathscr{G}'(k,i\omega_n) = \frac{u_k^2}{i\omega_n - \frac{E_k}{\hbar}} - \frac{v_k^2}{i\omega_n + \frac{E_k}{\hbar}}, \quad \mathscr{F}^\dagger(k,i\omega_n) = -\frac{u_kv_k}{i\omega_n - \frac{E_k}{\hbar}} + \frac{u_kv_k}{i\omega_n + \frac{E_k}{\hbar}} \tag{7}$$

$$u_k^2 = \frac{1}{2}\left[\frac{\epsilon_k + n_c\widetilde{v}_k}{E_k} + 1\right], \quad v_k^2 = \frac{1}{2}\left[\frac{\epsilon_k + n_c\widetilde{v}_k}{E_k} - 1\right], \quad E_k = \sqrt{2n_c\widetilde{v}_k\epsilon_k + \epsilon_k^2} \tag{8}$$

The Green function describes quasiparticles twidehat, for small k, have a linear dispersion provided twidehat $\widetilde{v}_0 > 0$:

$$\omega_k = ck, \qquad c = \sqrt{\frac{n_c}{m}\widetilde{v}_0}$$

The uniform density of the normal phase is

$$n' = -\mathscr{G}'(x\tau, x\tau^+) = -\int \frac{dk}{(2\pi)^2}\frac{1}{\hbar\beta}\sum_{i\omega}\mathscr{G}'(k, i\omega_n)e^{i\omega_n}$$
$$= \int \frac{dk}{(2\pi)^3}\left[\frac{u_k^2}{e^{\beta E_k}-1} - \frac{v_k^2}{e^{-\beta E_k}-1}\right]$$

It is a function of the temperature.

$$n'(T) - n'(0) = \int \frac{dk}{(2\pi)^3}\frac{u_k^2 + v_k^{02} + (v_k^2 - v_k^{02})e^{\beta E_k}}{e^{\beta E_k}-1}$$

By assuming twidehat for low T the coefficients u_k and v_k are close to their values at $T = 0$:

$$n'(T) - n'(0) = \int \frac{dk}{(2\pi)^3}\frac{u_k^{02} + v_k^{02}}{e^{\beta E_k}-1}$$

Being E_k linear for small k, the main contribution in the integral comes from small k. It is $u_k^{02} + v_k^{02} = \frac{1}{E_k}(\epsilon_k + n_c\widetilde{v}_k) \approx \frac{n_c\widetilde{v}_0}{\hbar ck}$. Then

$$n'(T) = n'(0) + \frac{4\pi}{(2\pi)^3}\frac{n_c\widetilde{v}_0}{(\hbar c)^3}(k_B T)^2\int_0^\infty dx\frac{x}{e^x - 1}$$

The integral is $\Gamma(2)\zeta(2) = \pi^2/3$. It turns out twidehat the condensate density decreases as T^2 for small T, differently from the superfluid density twidehat changes as T^4.

4. Homogeneous gas in Hartree Fock

In the Hartree-Fock approximation we neglect the cubic terms and factor the quartic one:

$$[\hbar\partial_\tau + h_x - \mu + \int dy\, v_{xy}|f_y|^2]\mathscr{G}'(x\tau, x'\tau') = -\hbar\delta_{xy}\delta(\tau-\tau')$$
$$-\int dy\, v_{xy}[\mathscr{F}^\dagger(y\tau, x'\tau')f_x f_y + f_y^\star f_x\mathscr{G}'(y\tau, x'\tau')]$$
$$+\int dy v_{xy}\,[\mathscr{G}'(x\tau, y\tau^+)\mathscr{G}'(y\tau, x'\tau') + \mathscr{F}^\dagger(y\tau, x'\tau')\mathscr{F}(y\tau^+, x\tau)]$$

For a homogeneous gas f is constant. The HF equation for the order parameter is:

$$-\mu f + f|f|^2\widetilde{v}_0 = -\int dy\, v_{xy}[f\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_y\rangle + f\langle\widehat{\varphi}_y^\dagger\widehat{\varphi}_x\rangle + f^\star\langle\widehat{\varphi}_y\widehat{\varphi}_x\rangle] \tag{9}$$

Then either $f = 0$ or f solves an equation where also the normal degrees of freedom enter.

The HF equation of motion is

$$[-i\hbar\omega_n + \epsilon_k - \mu + f^2\widetilde{v}_0]\mathscr{G}'(k, i\omega_n) = -\hbar - f^2\widetilde{v}_k[\mathscr{F}^\dagger(k, i\omega_n) + \mathscr{G}'(k, i\omega_n)]$$
$$+\int dy v_{xy}\,[\mathscr{G}'(x\tau, y\tau^+)\mathscr{G}'(y, x', i\omega_n) + \mathscr{F}^\dagger(y, x', i\omega_n)\mathscr{F}(x\tau, y\tau^+)]$$