The Ginzburg-Landau theory of superconductivity (1950)

Notes by Luca G. Molinari

Before any clue on a microscopic mechanism, and just by assuming that superconductivity can be described by a complex order parameter $\psi(\mathbf{x})$, Vitaly Ginzburg (Nobel 2003) and Lev Landau (Nobel 1962) developed a theory[1] valid near T_c. It was inspired by the recent theory of second-order phase transitions developed by Landau. The free energy is a functional of the order parameter but, since the latter is small near the transition, the free energy is written as an expansion. Being a phenomenological theory based on thermodynamics, it is quite universal.

For a uniform infinite superconductor in absence of magnetic field, the free energy density is a function of the order parameter ψ, independent of position, and is expanded as follows:

$$f_s[\psi,\bar\psi] = f_n + a|\psi|^2 + \frac{b}{2}|\psi|^4$$

f_n is the free energy of the normal phase, a and b are parameters that describe the material and depend on temperature. For stability, it is necessary that $b>0$.
Minimization in the order parameter gives $\psi(a+b|\psi|^2)=0$. Them either $\psi=0$ or

$$\psi_\infty^2 = -\frac{a}{b}$$

meaning that $a(T)<0$ for $T<T_c$ (the free energy is a double well in $|\psi|^2$). The difference in free energies is the condensation energy density

$$f_s(T) - f_n(T) = -\frac{a^2}{2b} = -\frac{H_c(T)^2}{8\pi}$$

Therefore:

$$a^2 = \frac{b}{4\pi}H_c(0)^2\left[1-\frac{T^2}{T_c^2}\right]^2$$

Ginzburg and Landau put $a(T)=\alpha(T-T_c)$ and b constant near T_c. This gives $\psi\propto\sqrt{T_c-T}$.
The jump of specific heat per unit volume at the transition is:

$$c_s - c_n = T_c\frac{\alpha^2}{b}$$

In presence of an external magnetic field $\mathbf{H}$, the order parameter ψ and the induction field $\mathbf{B}=\nabla\times\mathbf{A}$ are not uniform. While $\mathbf{B}$ approaches the external field $\mathbf{H}$ near the surface, the order parameter approaches in modulus the uniform value ψ_∞ deep in the superconductor.

The free energy is postulated as

$$F_s[\psi,\overline{\psi},\mathbf{A}] = F_n[0] + \int d^3x\, \frac{1}{2m^\star}\left|\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi(\mathbf{x})\right|^2 + a|\psi(\mathbf{x})|^2 + \frac{b}{2}|\psi(\mathbf{x})|^4 + \frac{1}{8\pi}(\nabla\times\mathbf{A})^2 \qquad (1)$$

F_n is the free energy of the normal phase in absence of the field H; a and b are the parameters of the superconductor, $-e^\star$ and $m^\star$ are the charge and mass of the particles that compose the superfluid with density $|\psi(\mathbf{x})|^2$.
The integral extends to the whole space.

A. The first GL equation

Minimization of F_s with respect to $\overline{\psi}$ or ψ gives the first G.L. equation or its complex conjugate:

$$\boxed{\frac{1}{2m^\star}\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)^2\psi + a\psi + b|\psi|^2\psi = 0} \qquad (2)$$

In deriving the equation, integration by parts produces a boundary term that vanishes if

$$\mathbf{n}\cdot(\mathbf{p}+\frac{e^\star}{c}\mathbf{A})\psi = 0$$

The left-multiplication of the GL equation by $\overline{\psi}$ is an equation whose imaginary part can be written in the form $\mathrm{div}\mathbf{J_s}=0$, where

$$\mathbf{J}_s = -\frac{e^\star}{2m^\star}\bar\psi\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi + c.c. \qquad (3)$$

will be soon understood as the supercurrent density. It is analogue of the probability current density of the Schrödinger equation. It reinforces the picture that $|\psi|^2$ is the density of some conserved fluid of particles with charge $-e^\star$ and mass $m^\star$. The net flux of $\mathbf{J}_S$ out of a closed surface is zero.
The afore b.c. is sufficient for ensuring the physical condition that the supercurrent flows parallel to the boundary surface: $\mathbf{n}\cdot\mathbf{J}_S=0$. Pierre Gilles de Gennes stated a more general condition[5]:

$$\mathbf{n}\cdot\left(\mathbf{p}+\frac{e^\star}{c}\mathbf{A}\right)\psi = i\lambda\psi \qquad (4)$$

where $\lambda=0$ for contact with an insulator and λ is real non-zero for contact with a metal. In the second case the condition allows for the "proximity effect", whence a layer near the superconductor gains superconducting properties.

Deep inside a superconductor it is $\mathbf{A}=0$ and the order parameter $|\psi|$ equals ψ_∞. It is convenient to put $\psi(\mathbf{x})=\psi_\infty f(\mathbf{x})$, where f is complex and $|f|\le 1$ (the bulk value).

The GL equation for f, after division by $|a|$, is greatly simplified in appearance:

$$\boxed{\xi^2\left(-i\boldsymbol{\nabla}+\frac{e^\star}{\hbar c}\mathbf{A}\right)^2 f - f + |f|^2 f = 0} \tag{5}$$

where ξ is the coherence length:

$$\xi(T) = \sqrt{\frac{\hbar^2}{2m^\star|a(T)|}} \tag{6}$$

It diverges at the critical temperature. It is the scale of decay of the order parameter in a s-to-n transition.

Consider the GL equation in 1D, in absence of field H with b.c. $f = 1$ and $f' = 0$ for $x \ll 0$:

$$-\xi^2 f'' - f + f^3 = 0$$

Multiply by $-4f'$: $\frac{d}{dx}(2\xi^2 f'^2 + 2f^2 - f^4) = 0$. Then $2\xi^2 f'^2 + 2f^2 - f^4 = C$. The b.c. give $C = 1$. Noting that $f' < 0$ in the transition from s to n, the square root is: $\xi\sqrt{2}f' = -(1-f^2)$. The integral is:

$$f(z) = \tanh\left(\frac{x_0 - x}{\xi\sqrt{2}}\right), \qquad x < x_0 \tag{7}$$

B. The second G-L equation

It is a general statement that, at equilibrium and away from the macrosopic currents that generate $\mathbf{H}$, it is[14]:

$$\frac{\delta F}{\delta \mathbf{A}(\mathbf{x})} = 0.$$

This condition gives the second G.L. equation. Amazingly, it is a Maxwell equation with the supercurrent density:

$$\boxed{\nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}_s} \tag{8}$$

The boundary condition is continuity of the tangent component of $\mathbf{B}$ and $\mathbf{H}$.

With $\psi = |\psi|e^{i\theta}$ the supercurrent becomes:

$$\mathbf{J}_S = -\frac{e^{\star 2}}{m^\star c}|\psi|^2\left(\frac{\hbar c}{e^\star}\boldsymbol{\nabla}\theta + \mathbf{A}\right) \tag{9}$$

A first interesting result is now obtained: divide by $|\psi|^2$ and integrate on a closed circuit. Since the order parameter is single-valued, the phase θ can only change by an integer multiple of 2π:

$$-\frac{m^\star c}{e^{\star 2}}\oint_C \frac{\mathbf{J}_S\cdot d\boldsymbol{\ell}}{|\psi|^2} - \frac{hc}{e^\star}n = \int_\sigma \mathbf{B}\cdot\mathbf{n}\,da \tag{10}$$

If the path C is in a region where the first integral is negligible, it turns out that the flux of B through the surface σ is quantized in units of the fundamental flux

$$\phi_0 = \frac{hc}{e^\star} \tag{11}$$

The identity $\mathrm{rot\,rot}\mathbf{B} = \mathrm{grad\,div}\mathbf{B} - \nabla^2\mathbf{B}$ and the Maxwell equation div $\mathbf{B}$=0 give $-\nabla^2\mathbf{B} = \frac{4\pi}{c}\mathrm{rot}\mathbf{J}_S$.
Now use the property $\mathrm{rot}(\lambda\mathbf{v}) = \lambda\mathrm{rot}\mathbf{v} - \mathbf{v}\times\boldsymbol{\nabla}\lambda$:

$$\nabla^2\mathbf{B} = \frac{4\pi e^{\star 2}}{m^\star c^2}\psi_\infty^2|f|^2\mathbf{B} - \frac{8\pi}{c}\mathbf{J}_S\cdot\frac{\boldsymbol{\nabla}|f|}{|f|}$$

If the density f slowly changes on the scale of change of the magnetic field, the rotor of the 2nd GL equation is:

$$\nabla^2\mathbf{B} = \frac{1}{\delta^2}|f|^2\mathbf{B} \tag{12}$$

where $\delta(T)$ is the penetration depth:

$$\delta(T) = \sqrt{\frac{m^\star c^2 b}{4\pi e^{\star 2}|a(T)|}} \tag{13}$$

It diverges near the transition.
Consider a superconductor in half-space $x \le 0$ in presence of a uniform field H along the z-axis in $x > 0$. If $\xi \ll \delta$ we approximate $|f| = 1$. The field B solves $B''(x) = B(x)/\delta$ with b.c. $B(0) = H$. The solution is $B(x) = He^{x/\delta}$. i.e. the field penetrates a length δ in the superconductor.

The ratio δ/ξ is independent of temperature, and defines the important Ginzburg-Landau parameter

$$\boxed{\kappa = \frac{\delta(T)}{\xi(T)} = \frac{m^\star c}{\hbar e^\star}\sqrt{\frac{b}{2\pi}}} \tag{14}$$

	T_c (∗)	δ (*nm*)	ξ (*nm*)	H_c(mT)(∗)
Cd	0.517	110	760	2.805
Al	1.175	16	1600	10.49
Sn	3.722	34	230	30.55
Pb	7.196	37	83	80.34
Nb	9.25	39	38	206

(*) Data from Springer handbook of condensed matter and materials (2005).
Nb: H_{c2}=0.42 T, H_{c1}=0.17 T (slides of A. Gurevich, Nat. High Magn. Field Lab. Florida).

C. The Gibbs potential

Consider an extended homogeneous material, in presence of a field H. Suppose that the order parameter as well as $\mathbf{B}$ may be not uniform. The Gibbs potential, after integration by parts, is

$$G_s = F_n + \int d\mathbf{x}\Big[\bar{\psi}\frac{1}{2m^\star}(\mathbf{p}+\frac{e^\star}{c}\mathbf{A})^2\psi + a|\psi|^2 + \frac{b}{2}|\psi|^4 + \frac{B^2}{8\pi} - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi}\Big]$$

Simplify with the 1st G.L. equation:

$$G_s = F_n + \int d\mathbf{x}\Big[\frac{B^2}{8\pi} - \frac{\mathbf{B}\cdot\mathbf{H}}{4\pi} - \frac{b}{2}|\psi|^4\Big]$$

In the normal phase $G_n = F_n + \int d\mathbf{x}H^2/8\pi$. Insert $\psi = \psi_\infty f = (|a|/b)f$ and use the relation $a^2/2b = H_c^2(T)/8\pi$ (condensation energy density):

$$G_s - G_n = \frac{H_c^2}{8\pi}\int d\mathbf{x}\Big[\Big(\frac{\mathbf{H}-\mathbf{B}}{H_c}\Big)^2 - |f|^4\Big] \tag{15}$$

At $T < T_c$ and $H = H_c(T)$, two cases are interesting:
• Uniform case. $G_s^u = G_n^u$ with the equivalent situations at the transition line: $|f| = 1$ and $B = 0$ or $|f| = 0$ and $B = H_c$.
• Mixed case, coexistence of s and n regions. The integral is non-zero in the transition region because both B and $|f|$ vary, and on different scales: f grows towards bulk s-region, while B decays from the value H_c in n-phase. The mixed phase occurs when $G_s^{mix} < G_s^u$, i.e. when the integral is negative.

In $d = 1$ the difference per unit area is

$$\frac{\Delta G}{A} = \frac{H_c^2}{8\pi}\int_{-\infty}^{+\infty} dx\Big[\Big(1-\frac{B(x)}{H_c}\Big)^2 - f^4(x)\Big] \tag{16}$$

The integral defines the length L of the transition region, that depends on how B and f decay in opposite directions. If it is negative a s-n interface is favoured within the material.

A 1-dimensional problem[1]

Even in $d = 1$ the coupled GL equations cannot be exactly solved. Consider an infinite superconducting material that is normal (n) for $x \gg 0$ with a magnetic field H_c in direction z, and superconducting (s) for $x \ll 0$.

$$f(x) \to \begin{cases} 1 & x\to-\infty \\ 0 & x\to+\infty\end{cases} \qquad B(x) \to \begin{cases} 0 & x\to-\infty \\ H_c & x\to+\infty\end{cases}$$

The induction field $B(x)$, with orientation z, is obtained from a vector potential $\mathbf{A}(x) = (0, A(x), 0)$. The equation $\mathbf{B} = \mathrm{rot}\mathbf{A}$ is $A'(x) = B(x)$.
With $f = |f(x)|e^{i\theta(x)}$ the supercurrent is

$$\mathbf{J}_S = -\frac{e^{\star 2}}{m^\star c}\frac{|a|}{b}|f(x)|^2(\frac{\hbar c}{e^\star}\theta'(x), A(x), 0)$$

The x component of the 2nd GL equation gives: $\theta'(x) = 0$. We set $\theta = 0$. The y component and $A' = B$ give:

$$A''(x) = \frac{1}{\delta^2}f^2 A \tag{17}$$

The z component is zero. The first GL equation is

$$-\xi^2 f''(x) + \frac{e^{\star 2}A^2(x)}{2m^\star c^2|a|}f(x) - f(x) + f^3(x) = 0$$

Multiply by $2f'$ and show a total derivative:

$$\frac{d}{dx}\left[-\xi^2 f'^2 + \frac{e^{\star 2}A^2}{2m^\star c^2|a|}f^2 - f^2 + \frac{f^4}{2}\right] = f^2\frac{e^{\star 2}AA'}{m^\star c^2|a|}$$

Note that $f^2AA' = \delta^2 A'A''$ is a total derivative. Then:

$$-\xi^2 f'^2 + \frac{e^{\star 2}A^2}{2m^\star c^2|a|}f^2 - f^2 + \frac{f^4}{2} - \frac{e^{\star 2}}{m^\star c^2|a|}\delta^2\frac{A'^2}{2} = C$$

The constant $C = -1/2$ is found with the b.c. at $-\infty$. This is the final integral:

$$f'^2 - \frac{(1-f^2)^2}{2\xi^2} = \Big(\frac{e^\star}{\hbar c}\Big)^2(A^2f^2 - \delta^2 A'^2) \tag{18}$$

Limit cases.
• $\kappa \ll 1$. Let $B(x) = H_c\theta(x)$ ($\delta = 0$). The function f is the solution (7) with $x_0 = 0$ because for $x > 0$ B is const. ($f = 0$ in (17) gives $B' = 0$). $L = \int_{-\infty}^0 dx(1-f^4) > 0$ (the mixed phase does not occur). The integral can be done: $L = \int_{-\infty}^0 dx(1-f^2)(1+f^2) = -\xi\sqrt{2}\int_{-\infty}^0 dx f'(x)(1+f^2) = \xi\sqrt{2}\int_0^1 df(1+f^2) = \frac{4}{3}\xi\sqrt{2}$.
• $\kappa = 1/\sqrt{2}$. The condition $f^2(x) = (1 - B(x)/H_c)$ in (16) makes $L = 0$. With the equations of the 1D problem, it is found to occur at $\kappa = 1/\sqrt{2}$.
• $\kappa \gg 1$. Now $f(x) = \theta(-x)$. The field $B(x)$ decays from H_c in $x < 0$: $B(x) = H_c e^{x/\delta}$. Then: $L = \int_{-\infty}^0 dx[(1-e^{x/\delta})^2 - 1] = -1.5\delta$ (there is the mixed phase).
(texts report the result $L = -\frac{8}{3}(\sqrt{2}-1)\delta \approx -1.1\delta$)

D. Mixed superconductors (1957).

Alexei Abrikosov (Nobel 2003) discovered that the linearized G.L. first equation admits a mixed solution[2], where the external field penetrates the bulk of the superconductor in quantized fluxes. *I made my derivation of the vortex lattice in 1953 but publication was postponed since Landau at first disagreed with the whole idea. Only after R. Feynman published his paper on vortices in superfluid Helium (1955) and Landau accepted the idea of vortices, did he agree with my derivation, and I published my paper*[3].

For $T < T_C$ consider the problem of lowering the intensity of a uniform magnetic field H to a value where the first solution of the GL solution appears. This highest

possible value corresponds to a small field ψ, and justifies the omission of the cubic term in the GL equation.
In the Landau gauge, for H aligned in the z direction, a choice is $\mathbf{A} = (0, Hx, 0)$. The linearized equation is that of a harmonic oscillator

$$\xi^2\left[-\frac{\partial^2}{\partial x^2} + (-i\frac{\partial}{\partial y} + \frac{e^\star H}{\hbar c}x)^2 - \frac{\partial^2}{\partial z^2}\right] f - f = 0$$

Let us introduce the magnetic length $\ell^2 = \hbar c/e^\star H$ and measure lengths with this unit. The solutions have the form $f(x, y, z) = e^{iky+iqz}u(s)$ where u solves

$$-\frac{d^2u}{ds^2} + (k\ell + s)^2 u + q^2\ell^2 u = \frac{\ell^2}{\xi^2}u$$

The equation admits integrable solutions only for the discrete values of the harmonic oscillator:

$$\ell^2/\xi^2 = q^2\ell^2 + 2n + 1, \quad n = 0, 1, 2, ...$$

The largest possible H (smallest ℓ) occurs for $q = 0$ and $n = 0$. In this situation $\xi^2 = \ell^2$ i.e. there is a unit flux of the critical field in an area $2\pi\xi^2$

$$2\pi\xi^2 H_{c2} = \frac{hc}{e^\star}$$

Remembering that $H_c^2(T) = 4\pi|a|/b$, the equation is

$$\frac{H_{c2}(T)}{H_c(T)} = \kappa\sqrt{2} \tag{19}$$

The solution is translation invariant along the z-axis, and is a superposition of shifted Gaussians:

$$f(x, y) = \sum_k c_k \exp\left[iky - \frac{1}{2\ell^2}(x + k\ell^2)^2\right]$$

It is periodic in y if $k = 2\pi p/L$:

$$f(x, y) = \sum_p c_p \exp\left[i\frac{2\pi}{L}py - \frac{1}{2\ell^2}(x + p\frac{2\pi}{L}\ell^2)^2\right]$$

It is also periodic in x, up to a phase, if c_p are all equal.

$$f(x, y + L) = f(x, y), \quad f(x + \tfrac{2\pi}{L}\ell^2, y) = f(x, y)e^{-i\frac{2\pi}{L}y}$$

The unit cell of the lattice has area $2\pi\xi^2$. It is a square if $L = \xi\sqrt{2\pi}$. Each cell is crossed by a unit flux of the critical field.
The periodic solution (up to a phase) is now rewritten:

$$\begin{aligned} f(x, y) =& e^{-\frac{x^2}{2\xi^2}}\left[1 + 2\sum_{p=1}^{\infty} e^{-\pi p^2}\cos\left(p\sqrt{2\pi}\frac{y + ix}{\xi}\right)\right] \\ =& e^{-\frac{x^2}{2\xi^2}}\theta_3\left(\sqrt{\frac{\pi}{2}}\frac{y + ix}{\xi}; e^{-\pi}\right) \end{aligned} \tag{20}$$

where $\theta_3(z, q) = 1 + \sum_{n=1}^{\infty} q^{n^2}\cos(2nz)$ is a Jacobi Theta function. The periods are $\xi\sqrt{2\pi}$ and $i\xi\sqrt{2\pi}$.

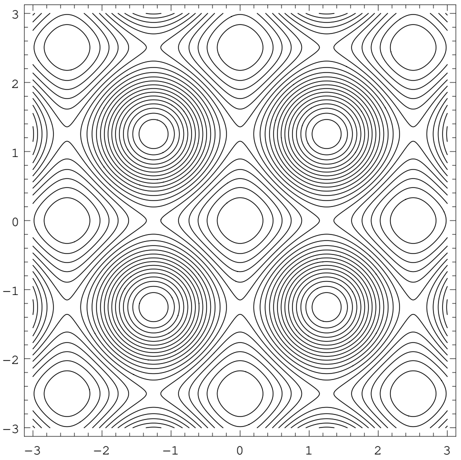

FIG. 1: Level curves of $|f(x, y)|^2$ for the square Abrikosov lattice, eq.(20) with $\xi = 1$. The central contour is a peak, while the concentric circles point to a zero, where the critical field is stronger. The phase of f changes by 2π along a circuit around a single zero. The lattice period is $\sqrt{2\pi} \approx 2.50$

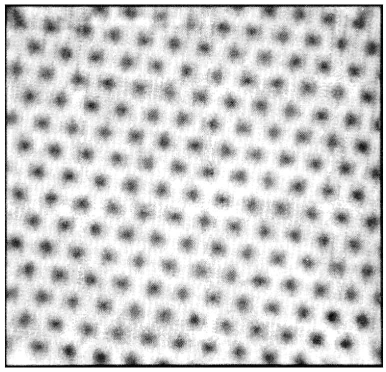

FIG. 2: STM picture of the triangular Abrikosov lattice in $NbSe_2$ ($T_c = 7.2$, $H_{c2} = 3.2$T). The width of the scan is 6000 Å. Previous observations used fine magnetic particles to mark the flux lines. The STM allows observation of the lattice in the full range of magnetic fields, and to observe the variation of the density of the states in and around a single flux line. (Hess et al.[8]).

It was later discovered, by considering the quartic term, that the triangular lattice has a lower free energy[10].
The highest field allowed in a superconductor is

$$H_{c2} = \kappa\sqrt{2}H_c(T) \tag{21}$$

The relation requires $\kappa > 1/\sqrt{2}$, which defines type II superconductors.
A higher threshold $H_{c3} \approx 1.7H_{c2}$ describes the onset of superconductivity in a surface layer of width $\xi(T)$.
The lower critical field for the mixed phase is

$$H_{c1} = H_c\frac{\log\kappa}{\kappa\sqrt{2}} \tag{22}$$

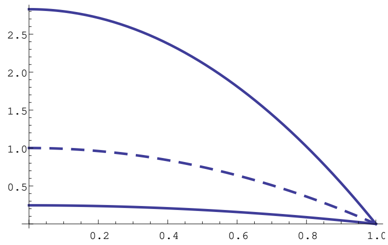

FIG. 3: The critical fields (from above) H_{c2}/H_c and H_{c1}/H_c (continuous lines), as functions of T/T_c. Below H_{c1} there is no mixed phase ($\kappa = 2$).

It is the value at which a single vortex grows in the type II superconductor when the H field is increased above the pure diamagnetic (Meissner) phase. ($G_s - G_n = 0$).

For $H_{c1} < H < H_{c2}$ the field penetrates in tubes where the phase is normal and the flux is quantized. The flux tubes form a triangular array (Abrikosov lattice). Each tube is surrounded by the superconducting phase and a thin layer of supercurrents that shield it. For $H > H_{c2}$ the normal phase occurs. Nothing occurs at the value $H_c(T)$. Type II superconductors resist fields of some tesla up a record field of 45.5T.

All superconducting chemical elements are Type I, with the exception of Vanadium (V, $T_c = 5.46$), Niobium (Nb, $T_c = 9.25$) and Technetium (Tc, $T_c = 7.77$) that are type II.

The Jacobi function $\theta_3(z,q)$

$$\theta_3(z,q) = \sum_{n\in\mathbb{Z}} e^{i(2nz+n^2\pi\tau)} \qquad \mathrm{Im}\tau > 0 \tag{23}$$

$$= 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos(2nz) \tag{24}$$

with $q = \exp(i\pi\tau)$. The function is analytic in $\mathbb{C}$ and doubly-periodic up to a factor, with periods π and $i\pi\tau$:

$$\theta_3(z+\pi,q) = \theta_3(z,q), \qquad \theta_3(z+\pi\tau,q) = \frac{1}{q}e^{-2iz}\theta_3(z,q)$$

A representation as infinite product:

$$\theta_3(z,q) = G\prod_{n=1}^{\infty}(1 + 2q^{2n-1}\cos(2z) + q^{4n-2})$$

$$G = \prod_{n=1}^{\infty}(1 - q^{2n})$$

By putting to zero the factor $n = 1$ we find a zero of the function at $z_0 = \frac{\pi}{2}(1+i\tau)$, that replicates with the periodicity of the lattice.

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