Chapter 24

LINEAR OPERATORS IN HILBERT SPACES

24.1 Linear Functionals

Definition 24.1.1. The space of continuous linear functionals $\mathscr{L}(\mathscr{H}, \mathbb{C})$ is $\mathscr{H}^*$, the dual space of $\mathscr{H}$.

Theorem 24.1.2 (Riesz's lemma). *For each $F \in \mathscr{H}^*$ there is a unique $x_F \in \mathscr{H}$ such that $Fx = (x_F|x)$ for all $x \in \mathscr{H}$. In addition $\|F\| = \|x_F\|$.*

Proof. If Ker$F$=$\mathscr{H}$ then $x_F = 0$. If Ker$F$ is a proper subspace, then there is a vector y orthogonal to it. Any vector x can be decomposed as $x = (x - y\frac{Fx}{Fy}) + y\frac{Fx}{Fy}$, where the first term belongs to Ker$F$ and then is orthogonal to y. The evaluation $(y|x) = Fx\frac{(y|y)}{Fy}$ shows that $x_F = y\frac{(Fy)^*}{\|y\|^2}$. The vector is unique, for suppose that $Fx = (x_F|x) = (x'_F|x)$ for all x, then $0 = (x_F - x'_F|x)$ i.e. $x_F = x'_F$.
Because of Schwarz's inequality: $\frac{|Fx|}{\|x\|} = \frac{|(x_F|x)|}{\|x\|} \le \|x_F\|$. Note that equality is attained at $x = x_F$, then: $\|F\| = \sup_x \frac{|Fx|}{\|x\|} = \|x_F\|$. □

24.2 Bounded linear Operators

The linear bounded operators with domain $\mathscr{H}$ to $\mathscr{H}$ form the Banach space $\mathscr{L}(\mathscr{H})$. The operator set is closed under the involutive action of adjunction:

Theorem 24.2.1. *For any linear operator $A \in \mathscr{L}(\mathscr{H})$ there is an operator $A^\dagger \in \mathscr{L}(\mathscr{H})$ (the adjoint of A) such that*

$$\boxed{(x|Ay) = (A^\dagger x|y) \qquad \forall x,\, y} \tag{24.0}$$

and $\|A^\dagger\| = \|A\|$.

Proof. Fix $x \in \mathscr{H}$, the map $y \to (x|Ay)$ is a linear bounded functional. Then, by Riesz' theorem, there is a vector v such that $(x|Ay) = (v|y)$ for all y. The vector v depends on x linearly, i.e. $v = A^\dagger x$.
Equality of norms follows from Schwarz's inequality and boundedness of A: $|(A^\dagger x|y)| = |(x|Ay)| \le \|x\|\,\|A\|\|y\|$, if $y = A^\dagger x$ then $\|A^\dagger x\| \le \|x\|\|A\|$ i.e. $A^\dagger$ is bounded and $\|A^\dagger\| \le \|A\|$. The proof of equality is left to the reader. □

E) Show that: $(A+B)^\dagger = A^\dagger + B^\dagger$, $(\lambda A)^\dagger = \overline{\lambda} A^\dagger$, $(AB)^\dagger = B^\dagger A^\dagger$, $(A^\dagger)^\dagger = A$.

Proposition 24.2.2. *If $A \in \mathscr{L}(\mathscr{H})$, it is*

$$\boxed{\mathscr{H} = \operatorname{Ran} A \oplus \operatorname{Ker} A^\dagger} \tag{24.0}$$

Proof. Since $\hat{A}$ is continuous, its range is a closed set. Then $\mathscr{H} = \mathrm{Ran}A \oplus (\mathrm{Ran}A)^\perp$. If $x \in \operatorname{Ker} A^\dagger$, it is $0 = (A^\dagger x|y) = (x|Ay)\ \forall y$. Then $x \perp \operatorname{Ran} A$, i.e. $x \in (\mathrm{Ran}A)^\perp$. The converse is also true, therefore $\operatorname{Ker} A^\dagger = (\mathrm{Ran}A)^\perp$. □

Definition 24.2.3. An operator $A \in \mathscr{L}(\mathscr{H})$ is **selfadjoint** if $A = A^\dagger$.

If the equation $\hat{A}x = \lambda x$ has nonzero solution $x \in \mathscr{H}$, then λ and x are respectively an eigenvalue and an eigenvector of A (it follows that $\overline{\lambda}$ and x are eigenvalue and eigenvector of $A^\dagger$). Moreover $|\lambda| \le \|A\|$.

Theorem 24.2.4. *The eigenvalues of a selfadjoint operator in $\mathscr{L}(\mathscr{H})$ are real, and the eigenvectors corresponding to different eigenvalues are orthogonal.*

Proof. Let x and x' be eigenvectors corresponding to eigenvalues $\lambda \neq \lambda'$. From $(Ax|x) = (x|Ax)$ one obtains $\overline{\lambda} = \lambda$. From $(Ax|x') = (x|Ax')$ one obtains $(\lambda - \lambda')(x|x') = 0$. □

E) Show that if A is *normal* (i.e. $AA^\dagger = A^\dagger A$), then eigenvectors correponding to different eigenvalues of A are orthogonal.
E) Show that if $A = A^\dagger$, and λ is an eigenvalue, $\inf_{\|x\|=1}(x|\hat{A}x) \le \lambda \le \sup_{\|x\|=1}(x|\hat{A}x)$.

24.2.1 Integral operators

An important class of linear operators in functional analyis are integral operators. They arise for example in the study of linear differential equations. Consider the inhomogeneous integral equation $f(x) = g(x) + \lambda \int_\Omega dt k(x,y) f(y)$, where λ is a parameter, g is given and f is the unknown function. In operator form it is: $f = g + \lambda \hat{K} f$, with

$$(\hat{K} f)(x) = \int_\Omega dy\, k(x,y)\, f(y)$$

The appropriate functional setting is suggested by the g and the *kernel* k.

Let $\Omega = [0,1]$; if $k \in \mathrm{L}^2(Q)$ (Q is the unit square) then $\hat{K}$ is bounded operator on $\mathrm{L}^2(0,1)$: by Fubini's theorem the function $t \to |h(t,s)|$ is measurable and belongs to $\mathrm{L}^2(0,1)$, by Schwarz's inequality $|(\hat{K}f)(x)| \le \|k(x,.)\|\|f\|$. Integration in x gives:

$$\|\hat{K}f\| \le \|k\|\,\|f\|$$

where $\|k\|^2 = \int_Q dxdy|k(x,y)|^2$. The adjoint is now studied[1]: $(h|\hat{K}f) = \int_0^1 dx\overline{h}(x)\left[\int_0^1 dyk(x,y)f(y)\right] = \int_0^1 dy\overline{\left[\int_0^1 dx\overline{k(x,y)}h(x)\right]}f(y)$. Then the adjoint operator is:

$$(\hat{K}^\dagger h)(x) = \int_0^1 dy\overline{k(y,x)}h(y)$$

The integral operator is self-adjoint if $k(x,y) = \overline{k(y,x)}$.
Let us enquire about the solution of the integral equation $(I - \lambda\hat{K})f = g$. A solution exists if $g \in \mathrm{Ran}(I - \lambda K)$ i.e. $g \perp \mathrm{Ker}(I - \overline{\lambda}K^\dagger)$. This means that g must be orthogonal to the eigenvectors $\hat{K}^\dagger u = \overline{\lambda}^{-1}u$.
In particular, if $|\lambda|\|K\| < 1$, it is both Ker $(1-\overline{\lambda}K^\dagger) = \{0\}$ and Ker $(1-\lambda K) = \{0\}$, and the operator $(1-\lambda\hat{K})^{-1}$ exists with domain $\mathscr{H}$. The operator can be expanded in a geometric series that converges for any g:

$$f = g + \lambda\hat{K}g + \lambda^2\hat{K}^2 g + \dots$$

The powers of the operator are integral operators with kernels $k_{n+1}(x,y) = \int_0^1 du\, k(x,u)\, k_n(u,y)$, $k_1 = k$.

Example. Consider the equation $f(x) = g(x) + \int_0^x f(y)dy$, a.e. $x \in [0,1]$, $g \in \mathrm{L}^2(0,1)$ (it corresponds to the Cauchy problem $f' = f + g'$ with $f(0) = g(0)$). The kernel $k(x,y) = \theta(x-y)$ has norm $1/\sqrt{2}$, therefore the iterative solution of the integral equation converges, with iterated kernels $k_{n+1}(x,y) = \theta(x-y)\,(x-y)^n/n!$:

$$f(x) = g(x) + \int_0^x dy\, e^{x-y}g(y)$$

Example. In $L^2(0,2\pi)$ consider the equation $f = \lambda\hat{K}f + g$ with

$$(\hat{K}f)(x) = \int_0^{2\pi} dy \sin(x+y)f(y) = A\sin x + B\cos x$$

where $A = (\cos|f)$ and $B = (\sin|f)$ (we restrict to real functions and parameter λ). The operator $\hat{K}$ has finite range and is self-adjoint. The problem is then basically two dimensional and the solution, if existent, is of the form:

$$f(x) = A\sin x + B\cos x + g(x)$$

[1] the steps are justified by Fubini's theorem for the exchange of order of integration: let $f(x,y)$ be measurable on $M\times N$, then $\int_M dm(\int_N dm|f|) < \infty$ iff $\int_N dm(\int_M dm|f|) < \infty$ and, if they are finite it is:

$$\int_M dm(\int_N dmf) = \int_N dm(\int_M dmf)$$

By taking the inner products with $\sin x$ and $\cos x$ we get:

$$\begin{bmatrix} 1 & -\lambda\pi \\ -\lambda\pi & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} (\cos|g) \\ (\sin|g) \end{bmatrix}$$

The solution exists for any g if $\lambda \neq \pm 1/\pi$:

$$f(x) = g(x) + \int_0^{2\pi} dy \frac{\sin(x-y) + \lambda\pi\cos(x-y)}{1-\lambda^2\pi^2} g(y)$$

If $\lambda = 1/\pi$, a solution exists if g is in the range of $(I - \frac{1}{\pi}\hat{K})$ i.e. g is orthogonal to Ker $(I - \frac{1}{\pi}\hat{K})$ (remember that $\hat{K}$ is self-adjoint). The Kernel set contains the solutions of the equation: $\hat{K}u = \pi u$: (up to a prefactor) $u(x) = \sin x + \cos x$. Then: $f(x) = A(\sin x + \cos x) + g(x)$ where A is an arbitrary constant.

24.2.2 Orthogonal Projections

Recall that, given a closed subspace M and a vector x, the projection theorem states that $x = p + w$, where $p \in M$ and $w \in M^\perp$, and the decomposition is unique. Therefore the projection operator on M, $\hat{P}(M) : x \to p$, is well defined.

Proposition 24.2.5. *The map $\hat{P}(M)$ is a linear selfadjoint operator, such that $\hat{P}(M)^2 = \hat{P}(M)$.*

Proof. For short we write $\hat{P}$ for $\hat{P}(M)$. Linearity: let $\hat{P}x = p$ and $\hat{P}x' = p'$. For any α and β it is $\alpha x + \beta x' = (\alpha p + \beta p') + (\alpha w + \beta w')$, where $\alpha p + \beta p' \in M$ and $\alpha w + \beta w' \in M^\perp$. As the decomposition is unique, necessarily it is $P(\alpha x + \beta x') = \alpha p + \beta p'$.
$(x - \hat{P}x|y - \hat{P}y) = (x - \hat{P}x|y)$ (because $x - \hat{P}x$ is orthogonal to M), and for analogous reason $(x - \hat{P}x|y - \hat{P}y) = (x|y - \hat{P}y)$; then $(\hat{P}x|y) = (x|\hat{P}y)$.
$\hat{P}^2 x = \hat{P}p = p = \hat{P}x$, then $\hat{P}^2 = \hat{P}$ ($\hat{P}$ is idempotent). □

Remarks
1) Since $\|x\|^2 = \|\hat{P}x + (x - \hat{P}x)\|^2 = \|\hat{P}x\|^2 + \|x - \hat{P}x\|^2$ it follows that $\|x\| \geq \|\hat{P}x\|$ with equality holding for $x \in M$. Then $\|\hat{P}\| = 1$. Being bounded, $\hat{P}$ is continuous.
2) It is Ran $\hat{P}(M) = M$ and Ker $\hat{P}(M) = M^\perp$.
3) A projector $\hat{P}(M)$ has only the eigenvalues 1 and 0, with eigenspaces M and $M^\perp$.

Definition 24.2.6. An orthogonal projection operator is a linear operator on $\mathscr{H}$ such that: $\hat{P}^2 = \hat{P}$, $\hat{P}^\dagger = \hat{P}$.

Examples: 1) In $L^2(\mathbb{R})$ the multiplication operator $f \to \chi_{[a,b]} f$, where $\chi_{[a,b]}$ is the characteristic function of a interval $[a, b]$, is an orthogonal projector. The invariant subspace M is given by functions that vanish a.e. for $x \notin [a, b]$.
2) The operators $(P_\pm f)(x) = \frac{1}{2}[f(x) \pm f(-x)]$ are orthogonal projectors on the orthogonal subspaces of (a.e.) even and odd functions.

Proposition 24.2.7. *Let $\hat{P}(M)$ and $\hat{P}(M')$ be projectors, then:*
1) $\hat{P}(M) + \hat{P}(M')$ is a projector iff M and M' are orthogonal. In this case $\hat{P}(M) + \hat{P}(M') = \hat{P}(M \oplus M')$.
2) $\hat{P}(M)\hat{P}(M')$ is a projector iff $P(M)$ and $P(M')$ commute.

24.2.3 Unitary operators

Definition 24.2.8. An operator V on $\mathscr{H}$ to itself is an **isometry** if the domain is the whole space and $\|Vx\| = \|x\|$ for all x. Moreover, if the range of V is the whole Hilbert space, the operator is **unitary**.

1) The conservation of the norm implies that $(Ux|Uy) = (x|y)$ for all x, y.
2) A unitary operator U has norm $\|U\| = 1$.
3) The adjoint operator coincides with the inverse operator, and are unitary operators: $(x|y) = (Ux|Uy) = (U^\dagger Ux|y)$ for all x, y implies $U^\dagger U = I$.

Two important families of unitary operators on $\mathrm{L}^2(\mathbb{R}^n)$ are the translations by $\vec{a} \in \mathbb{R}^n$ and multiplications by a phase factor with vector $\vec{k} \in \mathbb{R}^n$:

$$(\hat{U}_{\vec{a}} f)(\vec{x}) = f(\vec{x} - \vec{a}), \quad (\hat{V}_{\vec{k}} f)(\vec{x}) = e^{i\vec{k}\cdot\vec{x}} f(\vec{x}) \tag{24.0}$$

Each family is an abelian group, with product rule $\hat{U}_{\vec{a}}\hat{U}_{\vec{b}} = \hat{U}_{\vec{a}+\vec{b}}$, $\hat{V}_{\vec{k}}\hat{V}_{\vec{q}} = \hat{V}_{\vec{k}+\vec{q}}$.

Proposition 24.2.9 (Weyl's commutation relations)**.**

$$\hat{V}_{\vec{q}}\hat{U}_{\vec{a}} \; = \; e^{i\vec{q}\cdot\vec{a}}\hat{U}_{\vec{a}}\hat{V}_{\vec{q}} \tag{24.0}$$

Proof. $(\hat{V}_{\vec{q}}\hat{U}_{\vec{a}} f)(\vec{x}) = e^{i\vec{q}\cdot\vec{x}}(\hat{U}_{\vec{a}} f)(\vec{x}) = e^{i\vec{q}\cdot\vec{x}} f(\vec{x} - \vec{a}) = e^{i\vec{q}\cdot\vec{a}}(\hat{V}_{\vec{q}} f)(\vec{x} - \vec{a})$
$= e^{i\vec{q}\cdot\vec{a}}(\hat{U}_{\vec{a}}\hat{V}_{\vec{q}} f)(\vec{x})$ □

The translations (and phase multiplications) along a direction $\vec{a} = s\vec{n}$, $s \in \mathbb{R}$, are *strongly continuous one-parameter unitary groups* (continuity in the origin suffices, because of the group property):

Definition 24.2.10. A family of unitary operators $\hat{U}_s$, $s \in \mathbb{R}$, is a strongly continuous one-parameter unitary group if:
1) $\hat{U}_s\hat{U}_{s'} = \hat{U}_{s+s'}$
2) $\lim_{s\to 0} \|U_s x - x\| = 0 \quad \forall x \in \mathscr{H}$.

For such groups, the following fundamental theorem holds:

Theorem 24.2.11 (Stone)**.** *Let $\hat{U}_s$ be a strongly continuous one-parameter unitary group on a Hilbert space. Then there is a selfadjoint operator $\hat{H}$ (the* infinitesimal generator*) such that*

$$\hat{U}_s = e^{is\hat{H}}$$

The domain of $\hat{H}$ is the set of vectors x such that $\lim_{s\to 0} \frac{U_s x - x}{s}$ exists.

The generators of translations along the three directions are the *momentum* operators $\hat{P}_i = -i\hbar\partial_i$ on a suitable dense domain (a minus and a constant $\hbar$ are introduced for convenience)

$$\hat{U}_{\vec{a}} = e^{-\frac{i}{\hbar}\vec{a}\cdot\vec{\hat{P}}}$$

This can be checked on analytic functions, by Taylor expansion: $(\hat{U}_{\vec{a}}f)(\vec{x}) = f(\vec{x}-\vec{a}) = f(\vec{x}) - a_k(\partial_k f)(\vec{x}) + \frac{1}{2!}a_k a_\ell(\partial_k\partial_\ell f)(\vec{x}) + \ldots$ Since the group is Abelian, the generators form an abelian algebra: $[\hat{P}_i, \hat{P}_j] = 0$. The generators of phase multiplications are the *position* operators $(\hat{Q}_i f)(\vec{x}) = x_i f(\vec{x})$, and $[\hat{Q}_i, \hat{Q}_j] = 0$. The generators of the two groups form Heisenberg's algebra, with the extra Lie brackets: $[\hat{Q}_i, \hat{P}_j] = i\hbar\delta_{ij}$.

24.3 The Rotation group

24.3.1 space rotations, SO(3)

The space rotations are described by 3×3 matrices R such that $RR^t = I$ and $\det R = 1$; they form the group $SO(3)$. Rows or colums of R are orthogonal and normalized. A useful choice of parameters are the unit vector $\vec{n}$ that identifies the direction and the rotation angle φ, measured anticlockwise. Note that $R\vec{n} = \vec{n}$, and an eigenvalue is unity. The other two have product 1. If they are real the matrix is diagonal (1,1,1 gives the unit rotation, 1,-1,-1 in various order gives the 3 possible inversions of two axis). Therefore nondiagonal rotation matrices have eigenvalues $1, e^{\pm i\varphi}$ (it can be shown that φ is the angle of rotation around the invariant vector $\vec{n}$. Note that $\mathrm{tr}R = R_{11} + R_{22} + R_{33} = 1 + 2\cos\varphi$.)

For an infinitesimal angle the rotation is expanded near the unity of the group, $R = I + \delta\varphi\, A + \ldots$; the matrix A turns to be real antisymmetric. Since $A\vec{n} = 0$, A is obtained explicitly (up to a constant that is absorbed in $\delta\varphi$):

$$A = \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} = \vec{n}\cdot\vec{A} \tag{24.0}$$

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The infinitesimal rotation gives the well known expression $R\vec{x} = \vec{x} + \delta\varphi\vec{n}\times\vec{x} + \ldots$. The rotations with same invariant vector form a commutative subgroup:

$$R(\vec{n}\varphi_1)R(\vec{n}\varphi_2) = R(\vec{n}(\varphi_1+\varphi_2)) \tag{24.-1}$$

If one angle is infinitesimal: $R(\vec{n}\varphi)(I + \delta\varphi\,\vec{n}\cdot\vec{A} + \ldots) = R(\vec{n}(\varphi+\delta\varphi))$. Since the factors can be exchanged, one concludes that the matrices of the subgroup $R(\vec{n}\varphi)$ commute with $\vec{n}\cdot\vec{A}$. Moreover, by taking the limit $\delta\varphi$ to zero:

$$\frac{d}{d\varphi}R(\vec{n}\varphi) = (\vec{n}\cdot\vec{A})R(\vec{n}\varphi) \tag{24.-1}$$

with the initial condition $R(0) = I$. Since factors commute, the solution is

$$R(\vec{n}\varphi) = e^{\varphi\,\vec{n}\cdot\vec{A}} \tag{24.-1}$$

The matrix $\vec{n}\cdot\vec{A}$ is the *generator* of rotations along the direction $\vec{n}$. The three matrices A_i are the generators along the three coordinate directions, and are a *basis* for antisymmetric matrices. The antisymmetric matrices form a linear space that is closed under the operation $A, A' \to [A, A']$. The commutator is a Lie product[2], and the linear space with Lie product is the *Lie algebra* $so(3)$ of the $SO(3)$ group.
Because A_i are a basis, the Lie product of two basis matrices is expandable in the basis itself,

$$[A_i, A_j] = \epsilon_{ijk} A_k \tag{24.-1}$$

The coefficients (*structure constants*) are typical of the rotation group.
By means of Cayley-Hamilton equation, it is possible to expand the exponential into a finite sum of powers:

$$R(\vec{n}\varphi) = I + \sin\varphi(\vec{n}\cdot A) + (1-\cos\varphi)(\vec{n}\cdot\vec{A})^2 \tag{24.-1}$$

Each rotation of $SO(3)$ is identified by a vector $\vec{n}\varphi$ in the ball of radius π. A diameter corresponds to the subgroup of rotations with same rotation axis; points $\pm\vec{n}\pi$ (antipodal at the surface) give the same rotation and must be identified. The diameter is then a circle, and the manifold of parameters is a sphere with pairs of points at the surface identified: a bundle of circles through the origin (unit element of the group). The identification makes the manifold doubly connected: two rotations (two points in the manifold) are connected by two inequivalent paths (one cannot be deformed continuously into the other). For example: a segment and a path that touches the surface and continues from the antipode.

24.3.2 SU(2)

A more fundamental group is $SU(2)$ of unitary 2×2 complex matrices ($U^\dagger U = I$) with $\det U = 1$. Their explicit form is

$$U(\vec{n}\varphi) = e^{-\frac{i}{2}\varphi\vec{n}\cdot\vec{\sigma}} = \cos\frac{\varphi}{2} - i\vec{n}\cdot\vec{\sigma}\,\sin\frac{\varphi}{2} \tag{24.0}$$

where σ_i are the Pauli matrices. The matrices $\sigma_i/2$ are a basis for traceless Hermitian 2×2 matrices. With the Lie product $H, H' \to -i[H, H']$ the space is the Lie algebra $su(2)$, and the structure constants are the same of $su(3)$:

$$\boxed{-i\left[\frac{\sigma_i}{2}, \frac{\sigma_j}{2}\right] = \epsilon_{ijk}\frac{\sigma_k}{2}} \tag{24.0}$$

[2]In a linear space X, a Lie product is a bilinear map $*: X\times X \to X$ such that $x*x = 0$, $x*y = -y*x$, $(x*y)*z + (y*z)*x + (z*x)*y = 0$ (Jacobi property).

The exponential map takes the Lie algebra to the Lie group: exp : $su(2) \to SU(2)$. Despite the diversity of matrices in $SO(3)$ and $SU(2)$, their exponential representation is formally identical: only a replacement of the basis matrices changes one into another, the structure constants being the same.
In $SU(2)$ the parameter space is the ball with radius 2π. However, all the "surface" points correspond to the single matrix of inversion: $U(\pm\vec{n}\,2\pi) = -1$. Therefore, the manifold is a bundle of circles (the one parameter subgroups) with points ± 1 in common, and it is simply connected. This makes the $SU(2)$ unitary group more fundamental than $SO(3)$
E) Show that $U^\dagger \vec{\sigma} U = R\vec{\sigma}$, $R \in SO(3)$. Therefore both $\pm U$ correspond to the same R matrix.

24.3.3 Representations

From the point of view of physics, space rotations are a subgroup both of the Galilei and the Lorentz groups of symmetries for physical laws. An observer O is specified by an orthonormal frame $\mathbf{i}_k$, which can be identified materially by rigid bars of unit lenght for measuring positions positioned at a point (the origin). In O a physical point P receives coordinates x_k, meaning that the sequence of rigid shifts $x_k \mathbf{i}_k$ from the origin reaches the point P.
A rotated observer O' is an orthonormal frame $\mathbf{i'}_k$ with same origin and orientation. The point P has coordinates x'_k linked by a rotation matrix: $x'_k = R_{kj} x_j$. Suppose that O and O' measure in every point some quantity (a field). In the simplest case the quantity would be a single number (a scalar) that is a property of the point. They measure two fields f and f', but the field values at P must be the same:

$$f(x) = f'(x'), \quad \text{i.e.} \quad \boxed{f'(x) = f(R^{-1}x)} \tag{24.0}$$

If they measure a vector field (for example a force field), they measure a physical arrow at each point. Since they use rotated frames, the components of the arrow at a point are different:

$$\sum_k V_k(x)\mathbf{i}_k = \sum_k V'_k(x')\mathbf{i'}_k, \quad \Rightarrow \quad \boxed{V'_k(x) = R_{kj} V_j(R^{-1}x)} \tag{24.0}$$

The two laws describe the transformation of a scalar field and a vector field under rotations. A spinor 1/2 field is a two component complex field that transforms with a $SU(2)$ rotation:

$$\boxed{\psi'(x, a) = U(R)_{ab}\psi(R^{-1}x, b)} \tag{24.1}$$

If the scalar, spinor, vector or whatever fields F, F' measured by O and O' are thought of as points in the same functional space, the rotation of coordinates R induces a map $U_R : F \to F'$. Since fields can be linearly combined, the map is asked to be linear. If two rotations R and R' are done in the order, the observer O'' is linked to O by $x'' = R'Rx$, and thus $F''(x) = U_{R'R}F$ i.e. maps form a **linear representation** of the rotation group:

$$U_{R'R} = U_{R'}U_R \tag{24.1}$$

Scalar field Think of a scalar field f as a point in $\mathrm{L}^2(\mathbb{R}^3)$. A rotation induces the linear map $f' = U_R f$ defined by $(\hat{U}_R f)(x) =: f(R^{-1}x)$. The operators U_R are unitary because Lebesgue's measure is rotation invariant, then $\hat{U}(I) = 0$, $\hat{U}_{R^{-1}} = \hat{U}_R^\dagger$.
The rotations with fixed direction n correspond to a commutative subgroup parametrized by the angle: $U_{\vec{n}}(\varphi)U_{\vec{n}}(\varphi') = U_{\vec{n}}(\varphi + \varphi')$. They are a strongly continuous one-parameter group, by Stone's theorem there is a self-adjoint (unbounded) generator such that

$$\hat{U}_n(\varphi) = e^{-\frac{i}{\hbar}\varphi \hat{L}(\vec{n})}$$

The selfadjoint operator $\hat{L}(\vec{n})$ is the generator of the one parameter group, and corresponds to the projection along $\vec{n}$ of the *orbital angular momentum.* A simpler approach is to assume that f is analytic and consider an infinitesimal rotation:

$$U(\vec{n}\delta\varphi)f(\vec{x}) = f(\vec{x} - \delta\varphi\vec{n} \times \vec{x} + \ldots) = f(x) - \delta\varphi(\vec{n} \times x)_k \frac{\partial f}{\partial x_k}(x) + \ldots$$

then:

$$U(\vec{n}\delta\varphi) = I - \frac{i}{\hbar}\delta\varphi\vec{n} \cdot \vec{L} + \ldots, \qquad \boxed{\vec{L} = \vec{x} \times (-i\hbar\vec{\nabla})} \tag{24.0}$$

The components are operators $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ that satisfy the Lie algebra

$$\boxed{-i[\hat{L}_i, \hat{L}_j] = \hbar\,\epsilon_{ijk}\hat{L}_k} \tag{24.1}$$

For a 1/2 **spinor field**, the expansion near unity gives

$$(U(\vec{n}\delta\varphi)\psi)(\vec{x}, a) = (\delta_{ab} - i\delta\varphi\,\vec{n} \cdot \frac{\vec{\sigma}_{ab}}{2} + \ldots)(I - \frac{i}{\hbar}\delta\varphi\,\vec{n} \cdot \vec{L} + \ldots)\psi(\vec{x}, b)$$

i.e. $U(\vec{n}\,\delta\varphi) \approx I - \frac{i}{\hbar}\,\delta\varphi\vec{n} \cdot \vec{J}$. The generator is the vector sum of spin and orbital angular momentum operators (total angular momentum), that act separately on the spin variables and on the position variables:

$$\boxed{\vec{J} = \frac{\hbar}{2}\vec{\sigma} + \vec{L}} \tag{24.1}$$

and again the algebra is $-i[\hat{J}_i, \hat{J}_j] = \hbar\,\epsilon_{ijk}\hat{J}_k$.